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SRINIVASA RAMANUJAN AND GENERALIZED BASIC HYPERGEOMETRIC FUNCTIONS

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Dedicated to Academician Ljubomir Iliev on the Occasion of his Eightieth Birthday

ABSTRACT. In this paper we shall first consider a number of interesting formulas of Srinivasa Ramanujan (1887-1920) and others involving products of certain classes of generalized hypergeometric functions and point out their connections with some general series identities given in the recent literature. We shall then investigate various basic (or $q-$) extensions of Ramanujan's formulas (to hold true for products of certain classes of basic hypergeometric functions) and indicate their connections with some general q-series identities also available in the literature.

1. Introduction, Definitions, and Preliminaries. In terms of the Pochhammer symbol $(\lambda)_n$ given by

$$(1)\qquad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n = 1,2,3,\ldots, \end{cases}$$

let ${}_rF_s$ denote a generalized hypergeometric function with r numerator and s denominator parameters, defined by

$$(2)\qquad {}_rF_s\left[\begin{matrix}\alpha_1,\ldots,\alpha_r; \\ \beta_1,\cdots,\beta_s;\end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_s)_n}\frac{z^n}{n!}$$

$$r, s = 0, 1, 2, \ldots;$$

$$r \leq s \text{ and } |z| < \infty \text{ (or } r = s+1 \text{ and } |z| < 1).$$

Among well-known special cases of ${}_rF_s$, we have the Gaussian hypergeometric function for $r = 2$ and $s = 1$, (Kummer's) confluent hypergeometric function for $r = 1$ and $s = 1$, and the Clausenian hypergeometric function for $r = 3$ and $s = 2$.

whose special cases (when $\beta = \alpha$ and $\beta = 1 - \alpha$) were given by **C. T. Preece (1924)**.

S. Ramanujan (Notebooks [8]):

$$(11) \qquad {}_1F_1\left[\begin{matrix}\lambda; \\ \mu;\end{matrix}\; z\right] {}_1F_1\left[\begin{matrix}\lambda; \\ \mu;\end{matrix}\; -z\right] = {}_2F_3\left[\begin{matrix}\lambda, \mu-\lambda; \\ \mu, \Delta(2;\mu);\end{matrix}\; \tfrac{1}{4}z^2\right],$$

which is Entry 24 (Chapter 13) of Volume I [or Entry 18 (Chapter 11) of Volume II];

$$(12) \qquad {}_0F_2\left[\begin{matrix}\text{---}; \\ \rho, \sigma;\end{matrix}\; z\right] {}_0F_2\left[\begin{matrix}\text{---}; \\ \rho, \sigma;\end{matrix}\; -z\right] = {}_3F_8\left[\begin{matrix}\Delta(3;\rho+\sigma-1); \\ \rho, \sigma, \Delta(2;\rho), \Delta(2;\sigma), \Delta(2;\rho+\sigma-1);\end{matrix}\; -\tfrac{27}{64}z^2\right],$$

which is Entry 22 (Chapter 13) of Volume I [or Entry 16 (Chapter 11) of Volume II]. (See also Hardy [4].)

P. Henrici (1987) [5]:

$$(13) \qquad {}_0F_1\left[\begin{matrix}\text{---}; \\ 6\rho;\end{matrix}\; z\right] {}_0F_1\left[\begin{matrix}\text{---}; \\ 6\rho;\end{matrix}\; \omega z\right] {}_0F_1\left[\begin{matrix}\text{---}; \\ 6\rho;\end{matrix}\; \omega^2 z\right]$$

$$= {}_2F_7\left[\begin{matrix}3\rho-\tfrac{1}{4}, 3\rho+\tfrac{1}{4}; \\ 6\rho, \Delta(3;6\rho), \Delta(3;12\rho-1);\end{matrix}\; (\tfrac{4z}{9})^3\right],$$

where $\omega = \exp(2\pi i/3)$, $i = \sqrt{-1}$.

Each of the above formulas [(4) through (12)] has been extended to hold true for various double series with bounded coefficients (see Srivastava and Karlsson [12, Chapters 1 and 9]). It is not difficult to prove an analogous triple-series extension of Henrici's formula (13). (See, for details, Karlsson and Srivastava [7].) In particular, by applying Dixon's theorem for summing a well-poised hypergeometric ${}_3F_2$ series with argument 1, Buschman and Srivastava [1] gave the aforementioned double-series extensions of Ramanujan's formulas (11) and (12). More generally, they applied Whipple's quadratic transformation:

$$(14) \qquad {}_3F_2\left[\begin{matrix}\alpha, \beta, \gamma; \\ 1+\alpha-\beta, 1+\alpha-\gamma;\end{matrix}\; z\right]$$

$$= (1-z)^{-\alpha}\, {}_3F_2\left[\begin{matrix}\Delta(2;\alpha), 1+\alpha-\beta-\gamma; \\ 1+\alpha-\beta, 1+\alpha-\gamma;\end{matrix}\; -\tfrac{4z}{(1-z)^2}\right]$$

to derive certain double-series transformations which eventually imply Ramanujan's formulas (11) and (12).

2. Basic (or $q-$) Extensions. Since basic (or $q-$) extensions of each of the results of Buschman and Srivastava [1] has been given already by Srivastava and Jain [11], it is not difficult to derive various extensions of Ramanujan's formulas (11) and (12) to hold true for basic (or $q-$) hypergeometric functions defined by

$$(15)\qquad {}_r\Phi_s\left[\begin{matrix}\alpha_1,\ldots,\alpha_r;\\ \beta_1,\ldots,\beta_s;\end{matrix}\; q,z\right]=\sum_{n=0}^{\infty}(-1)^{(1-r+s)n}q^{(1-r+s)n(n-1)/2}$$

$$\cdot\frac{(\alpha_1;q)_n\cdots(\alpha_r;q)_n}{(\beta_1;q)_n\cdots(\beta_s;q)_n}\frac{z^n}{(q;q)_n},$$

$$|q|<1;\quad r,s=0,1,2,\ldots;$$

$$r\leq s \text{ and } |z|<\infty \quad (\text{or } r=s+1 \text{ and } |z|<1),$$

where, by analogy with the definition (1),

$$(16)\qquad (\lambda;q)_n=\begin{cases}1, & \text{if } n=0,\\ (1-\lambda)(1-\lambda q)\cdots(1-\lambda q^{n-1}), & \text{if } n=1,2,3,\ldots,\end{cases}$$

so that

$$(17)\qquad \lim_{q\to 1}\left\{\frac{(q^\lambda;q)_n}{(q^\mu;q)_n}\right\}=\frac{(\lambda)_n}{(\mu)_n}$$

$$(n=0,1,2,\ldots;\quad \mu\neq 0,-1,-2,\ldots).$$

Indeed, making use of the following q-extension of a terminating version of (14):

$$(18)\qquad {}_3\Phi_2\left[\begin{matrix}\alpha,\beta,\gamma;\\ \alpha q/\beta,\alpha q/\gamma;\end{matrix}\; q,\frac{\alpha q z}{\beta\gamma}\right]$$

$$={}_1\Phi_0\left[\begin{matrix}\alpha;\\ -;\end{matrix}\; q,z\right]\,{}_5\Phi_4\left[\begin{matrix}\sqrt{\alpha},-\sqrt{\alpha},\sqrt{\alpha q},-\sqrt{\alpha q},\alpha q/\beta\gamma;\\ \alpha q/\beta,\alpha q/\gamma,\alpha z,q/z;\end{matrix}\; q,q\right]$$

$$\alpha=q^{-n}(n=0,1,2,\ldots),$$

given independently by Sears [9] and Carlitz [2], we can prove directly various q-extensions of Ramanujan's formulas (11) and (12). For example, we thus find that (cf. Srivastava [10])

$$(19)\qquad {}_1\Phi_1\left[\begin{matrix}\lambda;\\ \mu;\end{matrix}\; q,-z\right]\,{}_2\Phi_1\left[\begin{matrix}\lambda,0;\\ \mu;\end{matrix}\; q,\zeta\right]$$

$$=\sum_{m,n=0}^{\infty}q^{n(n-1)}\frac{(\lambda;q)_{m+n}(-q^n z/\zeta;q)_m(\mu/\lambda;q)_n}{(\mu;q)_{m+2n}(\mu;q)_n}\frac{\zeta^m}{(q;q)_m}\frac{(-\lambda z\zeta)^n}{(q;q)_n},$$

which, for $\zeta = -z/q$, would provide a q-extension of Ramanujan's formula (11);

$$(20)\qquad {}_1\Phi_2\left[\begin{matrix} \text{---}; \\ \rho, \sigma; \end{matrix}\; q, z\right]\; {}_2\Phi_2\left[\begin{matrix} 0, 0; \\ \rho, \sigma; \end{matrix}\; q, -\tfrac{\zeta}{q}\right] = \sum_{m,n=0}^{\infty} q^{mn+m(m-3)/2+n(3n-5)/2}$$

$$\cdot \frac{(\rho\sigma q^{m+2n-1}; q)_n(-q^{n+1}z/\zeta; q)_m}{(\rho; q)_{m+2n}(\sigma; q)_{m+2n}(\rho; q)_n(\sigma; q)_n} \frac{\zeta^m}{(q; q)_m} \frac{(z\zeta)^n}{(q; q)_n},$$

which, for $\zeta = -z$, would provide a q-extensions of Ramanujan's formula (12). As a matter of fact, a q-extension of Dixon's theorem will lead us similarly to the result:

$$(21)\qquad {}_2\Phi_1\left[\begin{matrix} \lambda, -\mu/\lambda; \\ \mu; \end{matrix}\; q, z\right]\; {}_2\Phi_1\left[\begin{matrix} \lambda, -\mu/\lambda; \\ \mu; \end{matrix}\; q, -z\right]$$

$$= {}_4\Phi_3\left[\begin{matrix} \lambda^2, -\mu, -\mu q, \mu^2/\lambda^2; \\ \mu^2, \mu, \mu q; \end{matrix}\; q^2, z^2\right],$$

which is a more interesting q-extension of Ramanujan's **first** formula (11) than that provided by the product in (19) with $\zeta = -z/q$. A q-extension of Ramanujan's **second** formula (12), more interesting than that provided by the product in (20) with $\zeta = -z$, does not seem to have been found as yet (cf. Strivastava [10]).

3. Remarks and Observations. In *Geometric Function Theory*, which is the study of the relationship between the *analytic* properties of a given function $f(z)$ and the *geometric* properties of the image domain

$$(22)\qquad \mathcal{D} = f(\mathcal{U}),$$

where

$$(23)\qquad \mathcal{U} := \{z : z \in \mathbf{C} \text{ and } |z| < 1\},$$

it is an extremely difficult problem to find a (useful) set of conditions on the coefficients of the power-series expansion of $f(z)$ that are both *necessary* and *sufficient* for the function $f(z)$ to be class $\mathcal{S}$ of normalized univalent (and analytic) functions in $\mathcal{U}$. One of several *partial* results in connection with this problem is provided by de Branges' theorem (cf., e.q., Iliev [6]), which asserts the truth of the **Milin conjecture** of 1971 (and hence also of the **Robertson conjecture** of 1936 and the *celebrated* **Biberbach conjecture** of 1916.)

The key ingredients in de Branges' proof of the Milin conjecture include (among other known results) the *special* cases ($\alpha = 2, 4, 6, \ldots$) of the following *non-negativity* result for the Clausenian hypergeometric function:

$$(24)\qquad {}_3F_2\left[\begin{matrix} -n, \alpha + n + 2, \frac{1}{2}(\alpha + 1); \\ \alpha + 1, \frac{1}{2}(\alpha + 3); \end{matrix}\; x\right] \geq 0$$

$$(0 \leq x < 1; \quad \alpha > -2; \quad n = 0, 1, 2, \ldots).$$

The *known* proofs of the non-negativity result (24), used in Geometric Function Theory (as observed above), are based rather heavily upon Clausen's identity (4) for hypergeometric functions (cf., e.q., Gasper [3] who also considered q-extensions of these results).

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