Serdica

Bulgariacae mathematicae publicationes

Сердика

Българско математическо списание

SERDICA — Bulgaricae mathematicae publicationes
19 (1993) 234-242

ON COMMUTATIVITY OF RINGS WITH CONSTRAINTS ON SUBSETS

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ABSTRACT. Let R be a ring with center $Z(R)$, and let $A(R)$ be an appropriate subset of R. In this paper, it is shown that R is commutative if and only if for every $x, y \in R$, there exist integers $k = k(x,y) \geq 0$, $m = m(x,y) > 1$, and $n = n(x,y) \geq 0$ such that $[x, x^n y - y^m x^k] = 0$ and for each $x \in R$ either $x \in Z(R)$, or there exits a polynomial $f(t) \in Z[t]$ such that $x - x^2 f(x) \in A(R)$, where $A(R)$ is a nil commutative subset of R. If R is a left or right s-unital ring, then the following are equivalent: (i) R is commutative; (ii) For every $x, y \in R$, there exist integers $k = k(x,y) \geq 0$, $m = m(x,y) > 1$, $n = n(x,y) \geq 0$ such that $[x, x^n y - y^m x^k] = 0$ and for each $x \in R$ either $x \in Z(r)$ or there exits a polynomial $f(t) \in Z[t]$ such that $x - x^2 f(x) \in A(R)$, where $A(R)$ is a nil subset of R; (iii) For each $y \in R$, there exists an integer $m = m(y) > 1$ such that $[x, x^n y - y^m x^k] = 0 = [x, x^n y^m - y^{m^2} x^k]$ for all $x \in R$, where $k \geq 0$ and $n \geq 0$ are fixed non-negative integers. Our results generalize some well-known commutativity theorems.

Introduction. There is a number of conditions each of which implies the commutativity of certain rings. The equivalence of few such conditions to that of commutativity of rings was established by Tominaga and Yaqub [17]. The list of these equivalent conditions was further extended by these authors in [18].

The major purpose of this paper is to use the work of Tominaga and Yaqub [17], Ashraf et al. [7] and Abujabal [4], for rings satisfying more general polynomial identities. In fact, several commutativity theorems can be obtained as corollaries of our results, for instance, [1, Theorem], [2, Theorem], [3, Theorem], [6, Theorem], [12, Theorem], [14, Theorem], [15, Theorem] and [16, Theorem].

Throughout this paper, R is an associative ring not necessarily with unity 1. Let $Z(R)$ denote the center of $R, N(R)$ the set of all nilpotent elements of $R, C(R)$ the commutator ideal of $R, A(R)$ a non-empty subset of R, and $V_R(A(R))$ the centralizer of a subset $A(R)$ of R. $Z[t]$ stands for the totality of polynomials in t with coefficients in Z, the ring of integers. For any x, y in R, we set as usual $[x, y] = xy - yx$.

In the present paper, we consider the following properties:

$(I - A(R))$: For each $x \in R$, there exists a polynomial $f(\lambda)$ in $Z[\lambda]$ such that $x - x^2 f(x) \in A(R)$.

$(I' - A(R))$: For each $x \in R$, either $x \in Z(R)$, or there exists a polynomial $f(\lambda)$ in $Z[\lambda]$ such that $x - x^2 f(x) \in A(R)$.

$(II - A(R))$: For every $a \in A(R)$ and $x \in R$, $[[a, x], x] = 0$.

(III): For every $x, y \in R$, there exist integers $k = k(x, y) \geq 0$, $m = m(x, y) > 1$ and $n = n(x, y) \geq 0$ such that $[x, x^n y - y^m x^k] = 0$.

$(III)'$: For every $x, y \in R$, there exist integers $k = k(x, y) \geq 1$ and $m = m(x, y) > 1$ such that $[x, xy - y^m x^k] = 0$.

$(III)''$: For every $x, y \in R$, there exist integers $m = m(x, y) > 1$ and $n = n(x, y) \geq 1$ such that $[x, x^n y - y^m x] = 0$.

(IV): For each $y \in R$, there exists an integer $m = m(y) > 1$ such that $[x, x^n y - y^m x^k] = [x, x^n y^m - y^{m^2} x^k] = 0$ for all $x \in R$, where $k \geq 0$ and $n \geq 0$ are fixed integers.

$(IV)'$: For each $y \in R$, there exists an integer $m = m(y) > 1$ such that $[x, xy - y^m x^k] = [x, xy^m - y^{m^2} x^k] = 0$ for all $x \in R$, where $k \geq 1$ is a fixed positive integer.

$(IV)''$: For each $y \in R$, there exists an integer $m = m(y) > 1$ such that $[x, x^n y - y^m x] = [x, x^n y^m - y^{m^2} x^k] = 0$ for all $x \in R$, where $n \geq 1$ is a fixed positive integer.

(V): For every $x, y \in R$, there exist fixed integers $k \geq 0$, $m > 1$ and $n \geq 0$ such that $[x^n y - y^m x^k, x] = 0$.

The major purpose of this paper is to study the equivalence of the above listed properties with reference to the commutativity of the ring under consideration.

2. Preliminary Results. In preparation for the proofs of our results, we first collect a number of well-known concepts and results.

Definition 1. A ring R is called left (resp. right) s-unital if $x \in Rx$ (resp. $x \in xR$) for each x in R. Further, R is called s-unital if it is both left and right s-unital, that is $x \in xR \cap Rx$ for all x in R.

Definition 2. If R is s-unital (resp. left or right), then for any finite subset F of R there exists an element e in R such that $ex = xe = e$ (resp. $ex = x$ or $xe = x$) for all x in F. Such an element e is called the pseudo (resp. pseudo left or pseudo right) identity of F in R.

Definition 3. A ring R is said to be normal if every idempotent element in R is in $Z(R)$.

Lemma 1 ([13, Lemma 3]). *Let R be a ring such that $[x, [x, y]] = 0$ for all x and y in R. Then $[x^k, y] = kx^{k-1}[x, y]$ for any positive integer k.*

Lemma 2 ([5, Lemma 2]). *Let R be a ring with unity 1, and let x and y be* *elements of R. If* $kx^m[x,y]=0$ *and* $k(x+1)^m[x,y]=0$ *for some integers* $m\geq 1$ *and* $k\geq 1$, *then necessarily* $k[x,y]=0$.

Lemma 3 ([17, Lemma 1]).

(i) *Let* Φ *be a ring homomorphism of* R *onto* R^*. *If* R *satisfies* $(I'-A(R))$, $(I-A(R))$ *or* $(II-A(R))$, *then* R^* *satisfies* $(I-\Phi(A(R)))$, $(I'-\Phi(A(R)))$ *or* $(II-\Phi(A(R)))$ *respectively.*

(ii) *If* $A(R)$ *is commutative and* R *satisfies* $(I'-A(R))$, *then* $N(R)$ *is commutative nil ideal of* R *containing* $C(R)$ *and is contained in* $V_R(A(R))$. *In particular,* $(N(R))^2\subseteq Z(R)$.

(iii) *If there exists a commutative subset* $A(R)$ *of* $N(R)$ *for which* R *satisfies* $(I'-A(R))$ *and* $(II-A(R))$, *then* R *is commutative.*

Lemma 4 ([18,Lemma]). *Let* R *be a left (resp. right) s-unital ring. If for each pair of elements* x *and* y *in* R *there exists a positive integer* $k=k(x,y)$ *and an element* $e=e(x,y)$ *of* R *such that* $x^ke=x^k$ *and* $y^ke=y^k$ *(resp.* $ex^k=x^k$ *and* $ey^k=y^k$*), then* R *is an s-unital ring.*

Lemma QK ([15,Lemma 3]). *Let* R *be a ring with unity, and let* k *and* m *be natural number. If* $(1-y^k)x=0$, *then* $(1-y^{km})x=0$ *for all* $x,y\in R$.

Theorem K ([11,Theorem]). *Let* f *be a polynomial in* n *non–commuting indeterminates* $x_1,x_2,\ldots,x_n$ *with relatively prime integral coefficients. Then the following are equivalent:*

(a) *Every ring satisfying the polynomial identity* $f=0$ *has a nil commutator ideal.*

(b) *Every semi-prime ring satisfying* $f=0$ *is commutative.*

(c) *For every prime* $p,(GF(p))_2$, *the ring of* 2×2 *matrices over the Galois field* $GF(p)$, *fails to satisfy* $f=0$.

Now, let P be a ring property. If P is inherited by every subring and every homomorphic image, then P is called an h -property. More weakly, if P is inherited by every finitely generated subring and every natural homomorphic image modulo the annihilator of a central element, then P is called an H-property. A ring property P such that a ring R has the property P if only if all its finitely generated subrings have p, is called an F-property.

Proposition 1 ([10, Proposition 1]). *Let* P *be an* H*-property, and let* P' *be an* F*-property. If every ring* R *with unity 1 having the property* P *has the property* P', *then every s-unital ring having* P *has* P'.

The following theorems are due to Herstein.

Theorem H_1 ([9, Theorem 3]). *If* R *is a ring with center* $Z(R)$ *such that for every* $a\in R$ *there exists a polynomial* $p_a(t)$ *such that* $a-a^2p_a(t)\in Z(R)$, *then* R *is commutative.*

Theorem H_2 ([8, Theorem 19]). *Let R be a ring and let $n = n(x) > 1$ be an integer depending on x. If $x^n - x \in Z(R)$ for all $x \in R$, then R is commutative.*

Main Results. We obtained the following results.

Theorem 1. *The following statements are equivalent:*
(i) *R is commutative.*
(ii) *R satisfies* (III) *and $(I' - A(R))$ for a commutative subset $A(R)$ of $N(R)$.*

Theorem 2. *Let R be a left or right s-unital ring. Then the following statements are equivalent:*
(i) *R is commutative.*
(ii) *R satisfies* (III) *and there exists a subset $A(R)$ of $N(R)$ for which R satisfies $(I' - A(R))$.*
(iii) *R satisfies* (IV).

The following lemmas are essential in proving our theorems.

Lemma 5. *Let $k = k(x,y) \geq 1$, $m = m(x,y) > 1$ and $n = n(x,y) \geq 1$. If R is associative ring satisfying $[x^n y - y^m x^k, x] = 0$ for all $x, y \in R$, then R is normal.*

Proof. Let e be an idempotent element in R and let $x \in R$. Then there exist integers $n = n(e, e + ex(1-e)) \geq 1$, $m = m(e, e + ex(1-e)) > 1$ and $k = k(e, e + ex(1-e)) \geq 1$ such that for $x = e$ and $y = e + ex(1-e)$, we have $[e, e^n(e, e + ex(1-e)) - (e + ex(1-e))^m e^k] = 0$. So, $e^{n+1}(e + ex(1-e)) - e(e + ex(1-e))^m e^k - e^n(e + ex(1-e))e + (e + ex(1-e))^m e^{k+1} = 0$. As $e^k = e$ for all $k \geq 1$, we get $e^{n+1}(e+ex(1-e)) - e^n(e+ex(1-e))e = 0$, and $e^{n+2} + e^{n+2}x(1-e) - e^{n+1} - e^{n+1}x(1-e)e = 0$. Thus $ex(1-e)) - ex(e - e^2) = 0$. Hence, $ex(1-e) = 0$. Similarly, we can prove that $(1-e)xe = 0$. Therefore, $ex = xe$ for all $x \in R$. Thus R is normal.

Lemma 6. *Let R be a ring with unity 1 satisfying* (III). *Then $N(R) \subseteq Z(R)$.*

Proof. Let $a \in N(R)$ and $x \in R$. Then, we may assume that there are integers $m_1 = m(x,a) > 1$, $n_1 = n(x,a) \geq 0$ and $k_1 = k(x,a) \geq 1$ such that $x^{n_1}[x,a] = [x, a^{m_1}]x^{k_1}$ for all $x \in R$. Now, consider $m_2 = m(x, a^{m_1}) > 1$, $n_2 = n(x, a^{m_1}) \geq 1$ and $k_2 = k(x, a^{m_1}) \geq 1$. Then as above we can write $x^{n_2}[x, a^{m_1}] = [x, (a^{m_1})^{m_2}]x^{k_2} = [x, a^{m_1 m_2}]x^{k_2}$ for all $x \in R$. So $x^{n_1+n_2}[x,a] = [x, a^{m_1 m_2}]x^{k_1+k_2}$ for all $x \in R$. Thus for any positive integer t, we have $x^{n_1+n_2+\cdots+n_t}[x,a] = [x, a^{m_1 m_2 \cdots m_t}]x^{k_1+k_2+\cdots+k_t}$ for all $x \in R$. As a is nilpotent, $a^{m_1 m_2 \cdots m_t} = 0$ for sufficiently large t. Hence, $x^{n_1+n_2+\cdots+n_t}[x,a] = 0$ for all $x \in R$. Let $n'(x) = n_1 + n_2 + \cdots + n_t$. So $x^{n'(x)}[x,a] = 0$ for all $x \in R$. Set $n' = \max\{n'(x), n'(x+1)\}$. Thus $x^{n'}[x,a] = 0 = (x+1)^{n'}[x,a]$ for all $x \in R$ which by Lemma 2 yields $[x,a] = 0$ for all $x \in R$. Hence $a \in Z(R)$.

From Theorem H_2 we have the following.

Theorem 3. *Let R be a ring with unity 1 satisfying* (III) *and $(I' - A(R))$ for a subset $A(R)$ of $N(R)$. Then R is commutative.*

Lemma 7. *Let R be a ring with unity 1 satisfying* (IV). *Then* $C(R) \subseteq Z(R)$.

Proof. Let n and k be fixed positive integers. Then for any $y \in R$ there exists an integer $m = m(y) > 1$ such that the polynomial identities in (IV) can be rewritten as

(1) $$x^n[x, y] = [x, y^m]x^k \text{ for all } x \in R,$$

and

(2) $$x^n[x, y^m] = [x, y^{m^2}]x^k \text{ for all } x \in R.$$

Now, we replacing x by $x + 1$ in (1), we get

$$(x+1)^n[x, y]x^k = [x, y^m](1+x)^k x^k = x^n[x, y](1+x)^k \text{ for all } x, y \in R.$$

So, by Theorem K, we observe that $C(R) \subseteq N(R)$, since $x = e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $y = e_{12} + e_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ fail to satisfy the identity

$$(x+1)^n[x, y]x^k - x^n[x, y](1+x)^k = 0.$$

in $(GF(p))_2$. Hence, by Lemma 6, $C(R) \subseteq N(R) \subseteq Z(R)$.

Lemma 8. *Let R be an associative ring with unity 1 satisfying* (IV). *Then R is commutative.*

Proof. For $n = 0 = k$, we get $[x, y] = [x, y^m]$ for all $x, y \in R$. So the commutativity of R follows from Theorem H_2. Again for $n = 1$ and $k = 1$ we have $x[x, y] = [x, y^m]x$ for all $x, y \in R$. Replacing x by $x + 1$, we obtain $[x, y - y^m] = 0$ for all $x, y \in R$. So, R is commutativity of R.

Now, we suppose that $n > 1$ and $k > 1$. If $k = n$, then $x^n[x, y] = [x, y^m]x^n$, and by Lemma 7, we get $x^n[x, y] = x^n[x, y^m]$. Therefore, $x^n[x, y - y^m] = 0$ and $(x+1)^n[x, y - y^m]$ for all $x, y \in R$. By Lemma 1, we have $[x, y - y^m] = 0$ for all $x, y \in R$. Therefore, R is commutative by Theorem H_2. Without loss of generality, we suppose that $n > k$. Let $t = 2^{n+1} - 2^{k+1}$. Then $t > 0$, for $n > k$. By using (1), we see that

$$tx^n[x, y] = (2^{n+1} - 2^{k+1})x^n[x, y] = 2^{n+1}x^n[x, y] - 2^{k+1}x^n[x, y]$$

$$= (2x)^n[(2x), y] - [(2x), y^m](2x)^k = 0.$$

Hence by Lemma 2, $t[x, y] = 0$. Again, Lemma 1 and Lemma 7 together imply that $[x^t, y] = tx^{t-1}[x, y] = 0$ for all x and y in R. So, $x^t \in Z(R)$ for all $x \in R$.

Further, using (1), (2) and the fact that $C(R) \subseteq Z(R)$ by Lemma 7, we see that

$$(1 - y^{(m-1)^2})[x, y]x^{2n-k} = [x, y]x^{2n-k} - y^{(m-1)^2}[x, y]x^{2n-k} = [x, y^m]x^n - y^{(m-1)^2}[x, y^m]x^n$$

$$= x^n[x, y^m] - my^{m-1}y^{(m-1)^2}[x, y]x^n = x^n[x, y^m] - my^{m(m-1)}x^n[x, y]$$

$$= x^n[x, y^m] - my^{m(m-1)}[x, y^m]x^k = x^n[x, y^m] - [x, y^{m^2}]x^k = 0.$$

This implies that $(1 - y^{(m-1)^2})[x, y]x^{2n-k} = 0$ for all $x, y \in R$. Replacing x by $x + 1$ and using Lemma 2, we get

$$(1 - y^{(m-1)^2})[x, y] = 0 \text{ for all } x, y \in R. \tag{3}$$

Since, $x^t \in Z(R)$, for all $x \in R$, then by (3), we get

$$[x, y - y^{t(m-1)^2+1}] = (1 - y^{t(m-1)^2})[x, y] = 0 \text{ for all } x, y \in R.$$

Thus $y - y^{t(m-1)^2+1} \in Z(R)$, for $m = m(y) > 1$. Hence, R is commutative by Theorem H_2.

Now, we are in a position to prove our results.

Proof of Theorem 1. It is straightforward to see that a commutative ring R satisfies the condition given in the theorem.

Now, if R has unity 1, then the result follows from Theorem 3. So we suppose that R does not contain unity 1. In view of Lemma 3 (i), R can be assumed to be a subdirectly irreducible ring without unity 1. Let $x \in R \setminus Z(R)$ be an arbitrary element. By hypothesis $(I' - A(R))$, there exists an element $y \in < x >$, the subring generated by x, and a positive integer m such that $x^m = x^{m+1}y$. Clearly, $e = x^m y^m$ is idempotent with $x^m = x^m e$, and also e is a central element by Lemma 5. Since R has no identity, $e = 0$. Again by Lemma 3 (ii), x is in the commutative ideal $N(R)$ and $[x, [x, a]] = 0$ for all $a \in A(R)$. Hence R is commutative by Lemma 3 (iii). This completes the proof.

Proof of Theorem 2. Every commutative left or right s-unital ring satisfies (ii) and (iii).

If R satisfies (ii), then we claim that R is s-unital ring. Let R be a right s-unital ring, and let $x, y \in R$. Then there exists an element $e \in R$ such that $xe = x$ and $ye = y$. Also, there are integers $m = m(x, y) > 1$, $n = n(x, y) \geq 0$ and $k = k(x, y) \geq 0$ such that $e^m x^{n+k+1} = [x, x^n e - e^m x^k] = x^{n+k+1}$. Similarly, if $m' = m'(y, e) > 1$, $n' = n'(y, e) \geq 0$ and $k' = k'(y, e) \geq 0$ are integers, then we have

$$e^{m'} y^{n'+k'+1} = y^{n'+k'+1}.$$

Hence,

$$e^m x^{n+n'+k+k'+2} = x^{n+k+n'+k'+2},$$

and

$$e^{m'} y^{n+k+n'+k'+2} = y^{n+k+n'+k'+2}.$$

So

$$e^{mm'} x^{n+n'+k+k'+2} = x^{n+n'+k+k'+2}$$

and

$$e^{mm'} y^{n+n'+k+k'+2} = y^{n+n'+k+k'+2}.$$

Then by Lemma 4, R is an s-unital ring.

Now, suppose that R is a left s-unital. Let x and y be arbitrary elements of R. Then we can find an element $e \in R$ such that $ex = x$ and $ey = y$. Further, there are integers $m = m(x,e) > 1$, $n = n(x,e) \geq 0$ and $k = k(x,e) \geq 0$ such that $x^{n+1}e = [x, x^n e - e^m x^k] + x^2 = x^{n+1}$. Similarly, $y^{n+1}e = y^{n+1}$. So by Lemma 4, R is s-unital.

According to Proposition 1, we may assume that R has unity 1. Hence, R is commutative by Theorem 3. Thus (ii) implies (i).

In case R satisfies (iii), then as argued above, we may assume that R has unity 1. Hence again by Lemma 8, R is commutative.

Corollary 1 ([7, Theorem 1]). *A ring R is commutative if and only if R satisfies* (III)$'$ *and* (I'-A(R)) *for a commutative subset $A(R)$ of $N(R)$.*

Corollary 2 ([17, Theorem 1]). *A ring R is commutative if and only if R satisfies* (III)$''$ *and* (I'-A(R)) *for a commutative subset $A(R)$ of $N(R)$.*

Corollary 3 ([7, Theorem 2]). *If R is a left or right s-unital ring, then the following statements are equivalent:*

(i) *R is commutative.*

(ii) *R satisfies* (III)$'$ *and there exists a subset $A(R)$ of $N(R)$ for which R satisfies $(I' - A(R))$.*

(iii) *R satisfies* (IV)$'$.

Corollary 4 ([17, Theorem 2]). *If R is a left or right s-unital ring, then the following statements are equivalent:*

(i) *R is commutative.*

(ii) *R satisfies* (III)$'$ *and there exists a subset $A(R)$ of $N(R)$ for which R satisfies $(I' - A(R))$.*

(iii) *R satisfies* (IV)$''$.

Corollary 5 ([4, Theorem 1]). *If R is a left or right s-unital ring satisfying* (V), *then R is commutative.*

Corollary 6 ([4, Theorem]). *Let m, n be fixed non-negative integers. Suppose that R satisfies the polynomial identity $x^n[x,y] = [x,y^m]$ for all $x, y \in R$.*

(a) *If R is left s-unital, then R is commutative except the case $(m,n) = (1,0)$.*

(b) *If R is right s-unital, then R is commutative except the case $(m,n) = (1,0)$; and also $m = 0$, $n > 0$.*

Example 1. *Let R be an algebra over $GF(2)$ of dimension 4 with $\{1, a, b, c\}$ as a basis which also satisfies the multiplication rule.*

$a^2 = 1 + a, ab = c, ca = b, ac = ba = b + c,$ and $bc = cb = b^2 = c^2 = 0$.

Then R becomes a non-commutative ring whose nilpotent elements commute among themselves. Let $A(R) = N(R)$ which is a commutative subset of R. Then for any $x \in R$, we see that $x^2 - x^4 = x^2 - x^2(x^2) \in N(R)$. Thus R satisfies $(I' - A(R))$. So, R fails to be commutative if it does not satisfy (III).

Example 2. *Theorem 1 need not be true if we drop the condition that $A(R)$ is commutative. For this, consider*

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in GF(2) \right\}.$$

Then R is a nilpotent ring of index 3 and also $N(R) = R$. Further, R satisfies (III). *However, with $A(R) = N(R)$, R also satisfies $(I' - A(R))$. But R is not commutative.*

Remark. Example 2 also shows that Theorem 2 can not be extended for arbitrary rings.

Example 3. *This example shows that both conditions* (III) *and* $(I' - A(R))$ *in Theorem 2 (ii) are essential for the ring R with unity 1 to be commutative. Let*

$$R = \left\{ aI + S : S = \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \text{ and } I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a, b, c, d \in GF(2) \right\}.$$

Then, it is easy to check that $N(R) = \{S\}$, and R does not satisfy (III). Let $A(R) = N(R)$. Then for all $x \in R$, we have $x - x^2 f(x) \in A(R)$. However, R is not commutative.

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Received 30.07.1991