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VECTOR ε-SADDLE POINTS IN A DIFFERENTIAL GAME DESCRIBED BY A HYPERBOLIC SYSTEM

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ABSTRACT. An antagonistic differential game of hyperbolic type with a vector pay-off function is considered in the present paper. It is proved that there exists an ε-Slater saddle point, $\forall\varepsilon \in \mathbf{R}^N_{\geq}$ for this game. Sufficient conditions in order that a certain situation of program strategies is an ε-Slater (Pareto) saddle point are given. By means of an example it is shown that these sufficient conditions are not valid if at least one of the strategies is non-program. This example is also an illustration of the fact that two ε-Slater saddle points will not be interchangeable and equivalent if at least one of the strategies of these points is non-program.

Introduction. The main purpose in Section 1 is to obtain Theorems of existence of a saddle point (an ε-Slater saddle point) for the game (1) with a scalar (vector) pay-off function (Theorem 2 and Theorem 3). Theorem 1 is an essential result which enables us to use the method for the parabolic case, see [11, 12]. Theorem 1 is proved for $\sigma_1 = 1$ in [3,5] and for $\sigma_1 = 0$ – in [5] under some additional regularity properties of the coefficients in (4). But in the present paper these coefficients do not have such a regularity and this is the reason that the solution of (2)-(4) belongs to a space, larger than $L_2(G)$. Therefore the Dirichlet boundary-value conditions for problem (2)-(4) require additional considerations which are given in Theorem 1.

Section 2 comprises an example showing that the sufficient conditions of Lemma 4 and Corollary 4 are not valid for non-program strategies.

The following multicriterial antagonistic differential game with a vector pay-off function is considered:

$$\langle \Xi, \{\mathcal{U}, \mathcal{V}\}, \{\rho_i(h(T))\}_{i\in\mathbf{N}}\rangle, \tag{1}$$

where $\mathbf{N} = 1, \ldots, N$, $N \geq 1$ is the number of criteria.

The controlled system Ξ is described by the boundary-value problem of hyperbolic type

$$\partial^2 y/\partial t^2 = Ay + b_1u_1 + c_1v_1 + f_1 \quad \text{in } G = (t_0, T) \times \Omega, \tag{2}$$

$$y\Big|_{t=t_0} = y_0, \quad \partial y/\partial t\Big|_{t=t_0} = y_1 \text{ in } \Omega \tag{3}$$

$$\sigma_1 \partial y/\partial\nu_A + \sigma_2 y = b_2u_2 + c_2v_2 + f_2 \text{ in } \Sigma = (t_0, T) \times \Gamma, \tag{4}$$

where $\sigma_i \in \{0, 1\}$, $i = 1, 2$, $\sigma_1 + \sigma_2 \geq 1$.

First we are going to consider the problem (2)-(4) formally. Then the initial and boundary-value conditions will be specified.

It is supposed that the coefficients of equation (2)-(4) satisfy the conditions:

$$y_0 = y_0(x) \in L_2(\Omega), \quad y_1 = y_1(x) \in (H_2^1(\Omega))^*,$$

$$f_1 = f_1(x, t) \in L_2(G), \quad f_2 = \sum_{j=1}^{m} f_{2j}^{(1)}(t) f_{2j}^{(2)}(x),$$

where $f_{2j}^{(1)}(t) \in L_\infty(t_0, T)$, $f_{2j}^{(2)}(x) \in L_2(\Gamma)$, $j = 1, \ldots, m$; $H_p^s(\Omega) = W_p^s(\Omega)$, $L_2(\Omega) = H_2^0(\Omega)$, $H_0^1(\Omega) = \overset{\circ}{W}{}_2^1(\Omega)$ etc., [6,8,9] and $H^{r,s}(G) = H_{x,t}^{r,s}(G)$ etc., see [10], are the respective Sobolev spaces, H^* is the dual functional space of H (for example $H_2^{-1}(\Omega) = (H_0^1(\Omega))^*$, $H^{-r,-s}(G) = (H_{0,0}^{r,s}(G))^*$, $r \geq 0$, $s \geq 0$ etc.). The functions $b_1 = b_1(x, t)$ and $c_1 = c_1(x, t)$ ($b_2 = b_2(x)$ and $c_2 = c_2(x)$) are measurable, bounded in $G(\Gamma)$ and take values in $\mathbf{R}^1$ and $\mathbf{R}^{m_1}$ ($\mathbf{R}^{r_2}$ and R^{m_2}) respectively; $\Omega \neq \emptyset$ is a bounded and open set in $\mathbf{R}^n$ with a boundary $\Gamma = \partial\Omega$. The operator A is of the form:

$$A[\cdot] = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial[\cdot]}{\partial x_j}\right) - a(x)[\cdot],$$

where $a_{ij}(x) = a_{ji}(x)$, $a(x) \geq a_0 = \text{const} > 0$, $\partial a_{ij}(x)/\partial x_k$, $i, j, k = 1, \ldots, n$ are functions which are measurable (in the Lebesgue sense), bounded in Ω and there exist constants $\alpha > 0$ and $\beta > 0$ such that for each $x \in \Omega$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$, the following inequalities are valid:

$$\alpha \sum_{j=1}^{n} \xi_j^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \beta \sum_{j=1}^{n} \xi_j^2;$$

$\partial[\cdot]/d\nu_A = \sum_{i,j=1}^{n} a_{ij}(x) \dfrac{\partial[\cdot]}{\partial x_j} \cos(\nu, x_i)$ is the conormal derivative, corresponding to the self-adjoint elliptic operator A of second order and ν is the exterior normal to Γ; the set Ω and its boundary Γ satisfy Conditions 1), 2) and $\mathcal{R}$ from [6, p. 212, 222].

Next, the sets of strategies will be described. The following sets $P(t) = P_1(t) \times P_2(t)$ ($Q(t) = Q_1(t) \times Q_2(t)$), $t \in [t_0, T]$, $0 \leq t_0 < T$ where $P_1(t) \subset L_2(\Omega; \mathbf{R}^{r_1})$ and $P_2(t) \subset R^{r_2}(Q_1(t) \subset L_2(\Omega; \mathbf{R}^{m_1})$ and $Q_2(t) \subset \mathbf{R}^{m_2})$ are given. These sets are convex, closed (in the respective spaces), measurable and uniformly bounded with respect to t, $\forall t \in [t_0, T]$. The vector-functions $u = (u_1, u_2) \in P(t)$ and $v = (v_1, v_2) \in Q(t)$ are called program strategies.

The present paper deals with the formalization of a differential game described by a hyperbolic system. The solution of the initial boundary-value problem is treated as in [10] and the controlled process obtained is considered for another space. The respective objects are linked by one and the same Fourier series.

1. Saddle points and vector ε-saddle points. Let $\mathcal{H} = L_2(\Omega) \times (H_2^1(\Omega))^*$ for $\sigma_1 = 1$ and $\mathcal{H} = H_2^{-1}(\Omega) \times (H_{2,0}^2(\Omega))^*$, (where $H_{2,0}^2(\Omega) \overset{\text{def}}{=} H_0^1(\Omega) \cap H_2^2(\Omega)$) for $\sigma_1 = 0$.

Let us consider the set $\Phi = \{\varphi \in H_2^1(G) | \partial^2\varphi/\partial t^2 = A\varphi + g$ in G, $\varphi(x,T) = (\partial\varphi/\partial t)(x,T) = 0$ in Ω, $\sigma_1 \partial\varphi/\partial\nu_A + \sigma_2\varphi = 0$ in $\Sigma = (t_0, T) \times \Gamma$, where $g = g(x,t)$ takes all the possible values of H, $H = L_2(G)$ for $\sigma_1 = 1$ and $H = H_{0,0}^{0,1}(G) = H_0^1([t_0, T], L_2(\Omega))$ for $s_1 = 0\}$. Then Φ can be equipped with the structure of a Hilbert space, where $||\varphi||_\Phi = ||g||_H$ and the operator $\varphi \to \partial^2\varphi/\partial t^2 - A\varphi$ is an isomorphism $\Phi \to H$, see [8, p. 301].

Now the problem (2)-(4) will be specified. From conditions (2)-(4) (after the formal application of Green formula and integration by parts) the following equation is obtained

$$
\begin{aligned}
\int_G y(\frac{\partial^2\varphi}{\partial t^2} - A\varphi)dxdt &= \int_G \varphi(b_1u_1 + c_1v_1 + f_1)dxdt - \int_\Omega y_0(x)\frac{\partial\varphi}{\partial t}(x,t_0)dx \\
&+ \int_\Omega y_1(x)\varphi(x,t_0)dx + \int_\Sigma (b_2u_2 + c_2v_2 + f_2)F(\varphi)d\Gamma dt, \forall\varphi \in \Phi,
\end{aligned}
\tag{5}
$$

where

$$
F(\varphi) = \begin{cases} -\partial\varphi/\partial\nu_A & \text{for } \sigma_1 = 0 \\ \varphi & \text{for } \sigma_1 = 1. \end{cases}
$$

Lemma 1. *There exists a unique function $y \in L_2(G)$ for $\sigma_1 = 1$ ($y \in H^{0,-1}(G)$ for $\sigma_1 = 0$), satisfying* (5).

The proof of Lemma 1 for $\sigma_1 = 1$ is given in [8, p. 328, Lemma 7.1] and for $\sigma_1 = 0$ – in [10, p. 116, Theorem 4.1]. Note that the assumptions made in the introduction imply that the conditions given in [8, 10] are satisfied. For example, for $s_1 = 0$, the operator A satisfies the conditions in [10, p.99-100, (1.6), (1.7), (1.9)]; the function $g = b_2u_2 + c_2v_2 + f_2 \in L_2(\Sigma)$ satisfies the condition in [10, p.115, (4.14)], etc. The proof is completed.

Thus, following [3] and [10],the solution of problem (2)-(4) will be the function y of Lemma 1.

Further, in Theorem 1 we shall prove by Fourier method that in the case $\sigma_1 = 0$ the solution $y(x,t)$ is continuous with respect to $tH^{-1}(\Omega)$-valued distribution.

As in [3] and [4], we give the following definition.

Definition. *Let $t_0 \leq t_1 \leq t_2 \leq T$ and $P(t_1, t_2]$ $(Q(t_1, t_2])$ be a set of restrictions of program strategies from $P(t) = P(t_0, T]$ $(Q(t) = Q(t_0, T])$ in $(t_1, t_2] \times \Omega$; h is an arbitrary chosen element of $\mathcal{H}$.*

If an ordered triplet (t_1, t_2, h) corresponds to a unique measurable function of $P(t_1, t_2]$ $(Q(t_1, t_2])$, then such a mapping will be called a positional strategy

$$U : (t_1, t_2, h) \to u(t) \in P(t_1, t_2](V : (t_1, t_2, h) \to v(t) \in Q(t_1, t_2]),$$

see [3,4].

The sets of positional strategies related to $P(t)(Q(t))$ are denoted by $\mathcal{U}(\mathcal{V})$.

Let $\Delta \in \boldsymbol{\Delta}$ be an arbitrary partition of the interval $[t_0, T]$ by the points $t_0 = \tau_1 < \tau_2 < \ldots < \tau_{m(\Delta)} = T$ and let us define $\delta(\Delta) = \max\left\{ (\tau_{j+1} - \tau_j)\Big|_{j=0,1,\ldots,m(\Delta)-1} \right\}$, see [3]. The function

$$h_\Delta[t] = h_\Delta[t; p_0, U, V] = (y_\Delta[t; p_0, U, V], y'_\Delta[t; p_0, U, V])$$

$$= (y_\Delta[x, t; p_0, U, V], y'_\Delta[x, t; p_0, U, V]) = (y_\Delta[x, t], y'_\Delta[x, t]), \quad t_0 \leq t \leq T,$$

$$(p_0 = \{t_0, y_0, y_1\}, y'_\Delta[x, t] = (\partial y/\partial t)_\Delta[x, t])$$

is defined as follows. In the interval $(\tau_j, \tau_{j+1}]$, $j = 0, 1, \ldots, m(\Delta) - 1$ the function $y_\Delta[x, t]$ is the solution of (2) and (4) with

$$u = u_{(j)}(t) = U(\tau_j, \tau_{j+1}, h_\Delta[t_j]) \in P(\tau_j, \tau_{j+1}],$$

$$v = v_{(j)}(t) = V(\tau_j, \tau_{j+1}, h_\Delta[t_j]) \in Q(\tau_j, \tau_{j+1}].$$

The function $h_\Delta[t; p_0, U, V]$ satisfies the initial conditions of (3), where $t \in (\tau_0, \tau_1]$ and for each of the consequent intervals $(\tau_j, \tau_{j+1}]$, $j = 1, \ldots, m(\Delta) - 1$, the initial conditions are defined by the preceding interval, i.e.

$$(6) \qquad h_\Delta[t; p_0, U, V]\Big|_{t=\tau_j} = h_\Delta[\tau_j; \tau_{j-1}, h_\Delta[\tau_{j-1}], U, V].$$

Thus the function $y_\Delta[x, t]$ for $t \in (\tau_j, \tau_{j+1}]$ is presented by the Fourier series of the type (11), (12), where (y_0, y_1) is replaced by (6) and $u = (u_1, u_2)$, $v = (v_1, v_2)$ are taken as above and $y'_\Delta[x, t] = (\partial y/\partial t)_\Delta[x, t]$.

Definition. *The function $h_\Delta[t; p_0, U, V]$ thus defined is called a step motion which is caused by the positional strategies U and V, the partition $\Delta \in \boldsymbol{\Delta}$ and the initial position $p_0 = \{t_0, y_0, y_1\}$,* [3, 4].

The following set is considered:

$$D(p_0) = \left\{ h_\Delta[\cdot] = h_\Delta[\cdot; p_0, U, V]\Big|_{U\in\mathcal{U}, V\in\mathcal{V}, \Delta\in\Delta} \right\}.$$

Obviously

$$D(p_0) = \left\{ h(\cdot) = h(\cdot; p_0, u, v)\Big|_{u\in P(t), v\in Q(t)} \right\},$$

where $h(t) = (y(t), y'(t))$ and $y(t)$ is the solution of the system (2)-(4) for the given functions $u(t) \in P(t)$, $v(t) \in Q(t)$ and $y'(t) = (\partial y/\partial t)(x,t)$, $\forall t \in [t_0, T]$. The following assertion will be proved by using [5]:

Theorem 1. *For each choice of the initial position* $p_0 \in [0,T]\times\mathcal{H}$, $u(t) \in P(t)$, $v(t) \in Q(t)$, *there exists a corresponding solution of* (2)-(4) *such that*

a) $h(t) = (y(t), y'(t)) \in \mathcal{H}$, $\forall t \in [t_0, T]$,

b) the set $D(p_0)$ *is a compact subset in* $C([t_0,T],\mathcal{H})$,

c) the set $D(T; p_0) = D(p_0) \cap \{t = T\}$ *is a compact subset in* $\mathcal{H}$.

Proof. The assertions of Theorem 1 are proved in [3, 5] for $\sigma_1 = 1$. Therefore let us consider the case for $\sigma_1 = 0$. We shall solve the problem (2)-(4) by the Fourier method. To this end we shall prove that the Fourier series is convergent in the space $H^{0,-1}(G)$ and satisfies (5). Further, we prove that this Fourier series is convergent in $C([t_0,T], H^{-1}(\Omega))$ and that it satisfies a), b), c).

From the conditions imposed on the operator A the spectral problem $A\omega = -\lambda\omega$ in Ω, $\omega = 0$ in Γ is solvable in $H_0^1(\Omega)$ for countably many eigenvalues $\lambda = \lambda_j$, $j = 1, 2, \ldots$. Each of them has a finite rate frequency and they can be arranged into an increasing sequence $0 < \lambda_1 \le \lambda_2 \le \ldots \lambda_j \le \ldots, \lambda_j \to \infty$ when $j \to \infty$, (by taking into account their rate frequency) [6]. The corresponding eigenfunctions ω_j form an orthogonal basis in $L_2(\Omega)$, i.e. $\langle\omega_i, \omega_j\rangle = 0$ for $i \ne j$ and $\|\omega_j\| = 1$, $i, j = 1, 2, \ldots$.

First, the following boundary-value problem will be considered

$$(7) \qquad z_1'' = Az_1 \text{ in } G,\ z_1(t_0) = z_1'(t_0) = 0 \text{ in } \Omega,\ z_1 = gw_1 \text{ in } \Sigma,\ z_1 \in L_2(G),$$

where $g = g(x) \in L_2(\Gamma)$, $w_1 = w_1(t) \in H_2^1(t_0, T)$, $w_1(t_0) = 0$, $z'' = \partial^2 z/\partial t^2$.

From [5] it follows that the problem (7) has a unique solution which is given by the Fourier series

$$z_1 = \sum_{j=1}^{\infty} z_{1j}\omega_j \in C([t_0,T], L_2(\Omega)),$$

where

$$z_{1j} = -\lambda_j^{-1/2} \int_{t_0}^{t} w_1(\tau)\langle g, \partial w_j/\partial\nu_A\rangle_\Gamma \sin\sqrt{\lambda_j}(t-\tau)d\tau$$

and

$$(8)\qquad \begin{aligned}\partial z_1/\partial t &= \sum_{j=1}^{\infty} z'_{1j}\omega_j \\ &= -\sum_{j=1}^{\infty}[\lambda_j^{-1/2}\int_{t_0}^{t} w_1(\tau)\langle g, \partial w_j/\partial \nu_A\rangle_\Gamma \sin\sqrt{\lambda_j}(t-\tau)d\tau]\ \omega_j(x) \in H^{0,-1}(G),\end{aligned}$$

($w(t) = w'_1(t)$), obtained after integration by parts.

On the one hand, the boundary-value problem of the type (7) with the boundary condition $z = gw$ in Σ will be considered, where $w(t) = w'_1(t)$ can be an arbitrary function of $L_2(t_0, T)$. Here $gw \in L_2(\Sigma)$ and from [10, p. 116, Theorem 4.1], the solution $z = z(x,t) = \sum_{j=1}^{\infty} z_j\omega_j$ of such a boundary-value problem can be obtained by taking into account that $z \in H^{0,-1}(G)$ satisfies the following equation of type (5):

$$(9)\qquad \int_G z(\frac{\partial^2\varphi}{\partial t^2} - A\varphi)dxdt = -\int_\Sigma w(t)g(x)\frac{\partial\varphi}{\partial\nu_A}d\Gamma dt, \quad \forall\varphi \in \Phi.$$

Putting in (9)

$$(10)\qquad \varphi(x,t) = \psi(t)\omega_j(x),$$

where $\psi(t) \in H^2(t_0,T)$, $\psi(T) = \psi'(T) = 0$, (see also [8, p. 329]), it follows that $z_j = z_j(t)$ satisfies the conditions

$$z''_j + \lambda_j z_j = -w(t)\langle g, \partial\omega_j/\partial\nu_A\rangle_\Gamma, \quad z_j(t_0) = z'_j(t_0) = 0, \quad j = 1,2,\ldots,$$

hence

$$z_j = -\lambda_j^{-1/2}\int_{t_0}^{t} w(\tau)\langle g, \partial w_j/\partial\nu_A\rangle_\Gamma \sin\sqrt{\lambda_j}(t-\tau)d\tau.$$

After comparing the obtained Fourier series $z = \sum_{j=1}^{\infty} z_j\omega_j$ to (8), we conclude that $z = \sum_{j=1}^{\infty} z_j\omega_j = \partial z_1/\partial t$. Since the functions (10) form a basis of Φ, (here $\Phi \subset H^{2,2}(G)$, [10, p. 114]), it can be proved that the function $z = \partial z_1/\partial t$, given by (8), satisfies (9), i.e. z is the solution of the considered boundary-value problem of type (7) with $w(t) = w'_1(t)$, where $w(t)$ is an arbitrary function of $L_2(t_0,T)$.

On the other hand, the problem

$$z''_2 = Az_2 + f \text{ in } G,\ z_2(t_0) = y_0,\ z'_2(t_0) = y_1 \text{ in } \Omega,\ z_2 = 0 \text{ in } \Sigma,$$

has a unique solution $z_2 \in C([t_0,T],L_2(\Omega))$, [9, p. 320-327]. Thus, the existence of a solution of the problem (2)-(4) is proved and this solution can be presented by the Fourier series

$$y(x,t)=\sum_{j=1}^{\infty} y_j(t)\omega_j(x), \tag{11}$$

where

$$\begin{aligned} y_j(t) &= \langle y_0,\omega_j\rangle \cos\sqrt{\lambda_j}(t-t_0)+\langle y_1,\omega_j\rangle \lambda_j^{-1/2}\sin\sqrt{\lambda_j}(t-t_0) \\ &+\lambda_j^{-1/2}\int_{t_0}^{t}\langle f_1+b_1u_1+c_1v_1,\omega_j\rangle_{L_2(\Omega)}\sin\sqrt{\lambda_j}(t-\tau)d\tau \\ &-\lambda_j^{-1/2}\int_{t_0}^{t}\sum_{k=1}^{m} f_{2k}^{(1)}(\tau)\langle f_{2k}^{(2)}(x),\partial\omega_j/\partial\nu_A\rangle_\Gamma \sin\sqrt{\lambda_j}(t-\tau)d\tau \\ &-\lambda_j^{-1/2}\int_{t_0}^{t}\langle u_2(\tau)b_2(x),\partial\omega_j/\partial\nu_A\rangle_\Gamma \sin\sqrt{\lambda_j}(t-\tau)d\tau \\ &-\lambda_j^{-1/2}\int_{t_0}^{t}\langle u_2(\tau)c_2(x),\partial\omega_j/\partial\nu_A\rangle_\Gamma \sin\sqrt{\lambda_j}(t-\tau)d\tau, \end{aligned} \tag{12}$$

see also [5]. It is sufficient to prove that the series $\sum_{j=1}^{\infty}\lambda_j^{-1}(y_j(t))^2$ and $\sum_{j=1}^{\infty}\lambda_j^{-2}(dy_j/dt)^2$ are uniformly convergent, where $t\in[t_0,T], u(t)\in P(t)$, $v(t)\in Q(t)$ (similar method is used in [5]). Let us consider the first of the above series. From the representation of $y_j(t)$ this series is majorated by a sum of several terms. Let us for instance consider one of them having the form

$$\sum_{j=1}^{\infty}\lambda_j^{-1}\left(\lambda_j^{-1/2}\sum_{k=1}^{m_2}\int_{t_0}^{t} v_{2k}(\tau)\sin\sqrt{\lambda_j}(t-\tau)d\tau\langle c_{2k}(x),\partial\omega_j/\partial\nu_A\rangle_\Gamma\right)^2$$

$$\le\sum_{j=1}^{\infty}\left(\sum_{k=1}^{m_2}\left|\int_{t_0}^{t} v_{2k}(\tau)\sin\sqrt{\lambda_j}(t-\tau)d\tau\right|^2\sum_{k=1}^{m_2}\langle c_{2k}(x),\partial\omega_j/\partial\nu_A\rangle_\Gamma^2/\lambda_j^2\right)$$

where $c_2(\cdot)=(c_{21}(\cdot),\ldots,c_{2m_2}(\cdot))$, $v_2(\cdot)=(v_{21}(\cdot),\ldots,v_{2m_2}(\cdot))$.

The latter is uniformly convergent in $t\in[t_0,T]$ and $v_2\in Q_2(t)$, since

$$\sum_{k=1}^{m_2}\left|\int_{t_0}^{t} v_{2k}(\tau)\sin\sqrt{\lambda_j}(t-\tau)d\tau\right|^2\le \text{const}\cdot\sum_{k=1}^{m_2}\|v_{2k}\|^2_{L_2(t_0,T)}\le\text{const}$$

and according to [5, Lemma 2.2] the series

$$\sum_{j=1}^{\infty}\sum_{k=1}^{m_2}\langle c_{2k}(x), \partial\omega_j/\partial\nu_A\rangle_\Gamma^2/\lambda_j^2,$$

is convergent. Similarly the remaining terms corresponding to the boundary function of (4) are considered.

The uniform convergence of the series $\sum_{j=1}^{\infty}\lambda_j^{-2}(y_j'(t))^2$ is obtained by using the same method. Let us point out that from this convergence it follows that

$$\sum_{j=1}^{m} y_j'(t)\omega_j(x) \to \sum_{j=1}^{\infty} y_j'(t)\omega_j(x),$$

when $m \to \infty$ as continuous functionals in the space $H_{2,0}^2(\Omega)$. The theorem is proved.

Remark 1. Let $\sigma_1 = 0$ and let the assumptions of Theorem 1 hold. It follows from [5, Theorem 2.2] that the set $D(p_0)$ is a compact subset in $H^{0,-1}(G) \times H^{0,-2}(G)$. The proof is obtained using the proof of Theorem 1. For example a representation of the type $z = \partial z_1/\partial t$, where $z_1 \in C([t_0, T], L_2(\Omega))$ (see (8)) can be used for the terms of $y(x,t)$, including the boundary function $b_2u_2 + c_2v_2 + f_2$.

Remark 2. Let (y_0, y_1) be an arbitrary function of $\mathcal{H} = H_2^{-1}(\Omega) \times (H_{2,0}^2(\Omega))^*$ and let $y = y(x,t)$ be defined by (11) and (12). The same method can be used to prove that $h = (y, y') \in C([t_0, T], \mathcal{H})$.

Theorem 1, Remark 1 and Remark 2 are sufficient to prove the existence of step motions.

The result of game (1) is evaluated by criteria, given by the functionals ρ_i in $\mathcal{H}$, $i \in \mathbf{N}$; $\rho(h(T)) = (\rho_1(h(T)), \ldots, \rho_N(h(T)))$ is called a vector pay-off function of game (1). It is supposed that the functionals ρ_i are strong continuous in $\mathcal{H}$. The first player choosing the strategy $U \in \mathcal{U}$ strives to smaller possible values of all criteria $\rho_i(h(T))$, $i \in \mathbf{N}$; the second using a strategy $V \in \mathcal{V}$, strives to their maximization. Each player chooses a strategy of his own which is independent of the other player's strategy.

First, consider the case $N = 1$, i.e. the game (1) is with a scalar pay-off function $\rho_i(h(T))$:

$$\langle \Xi, \{\mathcal{U}, \mathcal{V}\}, \rho_1(h(T))\rangle. \tag{13}$$

Definition 1. *The situation $(U^\varepsilon, V^\varepsilon) \in \mathcal{U} \times \mathcal{V}$ is called ε-saddle point for game* (13) *if there exists a constant $\delta_0 > 0$ such that*

$$\forall h_{\Delta^{(1)}}[\cdot] \in h_{\Delta^{(1)}}[\cdot; p_0, U^\varepsilon], \quad \forall h_{\Delta^{(2)}}[\cdot] \in h_{\Delta^{(2)}}[\cdot; p_0, U^\varepsilon, V^\varepsilon],$$

$$\forall h_{\Delta^{(3)}}[\cdot] \in h_{\Delta^{(3)}}[\cdot;\rho_0, V^\varepsilon] \quad with \quad \delta(\Delta^{(m)}) \le \delta_0, \quad (m = 1,2,3),$$

the following inequalities hold:

$$\rho_1(h_{\Delta^{(1)}}[T]) - \varepsilon \le \rho_1(h_{\Delta^{(2)}}[T]) \le \rho_1(h_{\Delta^{(3)}}[T]) + \varepsilon. \tag{14}$$

Here

$$h_{\Delta^{(1)}}[\cdot;\rho_0, U^\varepsilon] = \{h_{\Delta^{(1)}}[\cdot;\rho_0, U^\varepsilon, v] \mid v \in Q(t)\}$$
$$(h_{\Delta^{(3)}}[\cdot;\rho_0, V^\varepsilon] = \{h_{\Delta^{(3)}}[\cdot;\rho_0, u, V^\varepsilon] \mid u \in P(t)\})$$

is the corresponding bundle of step motions caused by the strategy U^ε (V^ε), the partition $\Delta^{(1)}$ ($\Delta^{(3)}$) and the initial position $\rho_0 = \{t_0, y_0, y_1\}$.

The following assertions hold.

Theorem 2. *There exists an ε-saddle point for each choice of the initial position $\rho_0 \in [0,T] \times \mathcal{H}$ and each $\varepsilon > 0$ in the game* (13).

We have to point out that similar assertions are obtained in [11, 1, 2]. Next Theorem 2 will be proved by using primarily Theorem 1 and the proof of the analogous assertion in [11] without any details.

Proof of Theorem 2. Consider the set $M_1(c) = \{h \in \mathcal{H} | \rho_1(h) \le c\}$. From Theorem 1, the set $M_1(c) \cap D(T;\rho_0)$ is compact in $\mathcal{H}$ and for it the Theorem of the Alternative holds (see [3] – for $\sigma_1 = 1$ and [4] – for $\sigma_1 = 0$). Let us note that for $\sigma_1 = 0$ all the conditions of [4, Theorem 2.1] are satisfied; this can be proved following [4, Example 3.1]: In our case $X = \mathcal{H}$ and the respective system of ordinary differential equations (see Example 3.1 of [4]) is of the form

$$y_j'' + \lambda_j y_j = \langle f_1 + b_1 u_1 + c_1 v_1, \omega_j\rangle_{L_2(\Omega)} - \langle f_2 + u_2(t) b_2(x) + v_2(t) c_2(x), \partial\omega_j/\partial v_A\rangle_\Gamma ,$$

$$y_j(t_1) = x_i^{1j}, \quad y_j'(t_1) = x_j^{2i}, \quad j = 1, \dots, i,$$

where $y(t_1) = x^{(1)}$, $y'(t_1) = x^{(2)}$, $x_i^{1j} = \langle x^{(1)}, \omega_j\rangle$, $x_i^{2j} = \langle x^{(2)}, \omega_j\rangle$, $y(t)$ is the solution of (2)-(4), λ_j, ω_j are defined in the proof of Theorem 1, $x = (x^{(1)}, x^{(2)})$, $Y(t_1, x, t_2, u, v) = h(t_2; t_1, x, u, v)$. The operators A_i, $A_i^*(t)$, the sets M_i, N_i etc., are defined as in [4, Example 3.1]. Condition 1 and Condition 2 of [4] are obtained by using Theorem 1 and its proof. The other conditions of [4, Theorem 2.1] are proved as it is shown in [4, Example 3.1]. Thus the conditions of [4, Theorem 2.1] are verified.

For each initial position $\rho_0 = \{t_0, y_0, y_1\} \in [0,T] \times \mathcal{H}$ let us consider the set of the numbers c for each of which there exists a corresponding strategy $U \in \mathcal{U}$, realizing an ε-approach towards $M_1(c)$. The set of these numbers is denoted by $\mathbb{C}_1$. It can be proved that $C_1 = [c_0^1, \infty)$ for some number c_0^1.

Let the strategy $U^0 \in \mathcal{U}$ be a solution of an ε-approach problem towards $M_1(c_0^1)$. This means that for each $\varepsilon > 0$, there exists a number $\delta(\varepsilon) > 0$, such that for each step motion $h_\Delta[\cdot] \in h_\Delta[\cdot;\rho_0,U^0]$ with $\delta(\Delta) \le \delta(\varepsilon)$, the following condition is satisfied: $\rho_1(h_\Delta[T]) \le c_0^1 + \varepsilon$. Take the exact upper limit of this inequality when $\delta(\Delta) \to 0$. We get:

$$\lim_{\delta\to 0} \sup_{h_\Delta[\cdot]\in h_\Delta[\cdot;\rho_0,U^0],\delta(\Delta)\le\delta} \rho_1(h_\Delta[T]) \le c_0^1. \tag{15}$$

From Theorem 1 it follows that such a limit exists and is bounded. It will be shown that there is an equality in (15) as a matter of fact. Suppose that there exists a strategy $U^* \in \mathcal{U}$ such that

$$\lim_{\delta\to 0} \sup_{h_\Delta[\cdot]\in h_\Delta[\cdot;\rho_0,U^*],\delta(\Delta)\le\delta} \rho_1(h_\Delta[T]) = \widehat{c}_0^1 < c_0^1, \quad (\widehat{c}_0^1 \in \mathbf{R}). \tag{16}$$

The relation (16) means that $\widehat{c}_0^1 \in \mathbf{C}_1$ and at the same time, $\widehat{c}_0^1 < c_0^1 = \min \mathbf{C}_1$. The obtained contradiction shows that for each strategy $U \in \mathcal{U}$,

$$\begin{aligned}
&\lim_{\delta\to 0} \sup_{h_\Delta[\cdot]\in h_\Delta[\cdot;\rho_0,U],\delta(\Delta)\le\delta} \rho_1(h_\Delta[T]) \ge c_0^1 \\
&= \lim_{\delta\to 0} \sup_{h_\Delta[\cdot]\in h_\Delta[\cdot;\rho_0,U^0],\delta(\Delta)\le\delta} \rho_1(h_\Delta[T]) \\
&= \inf_{U\in\mathcal{U}} \lim_{\delta\to 0} \sup_{h_\Delta[\cdot]\in h_\Delta[\cdot;\rho_0,U],\delta(\Delta)\le\delta} \rho_1(h_\Delta[T]) = c_0^1,
\end{aligned}$$

i.e. $U^0 \in \mathcal{U}$ is the minimax strategy of game (13). It is proved similarly that there exists a strategy $V^0 \in \mathcal{V}$ for which the following relation holds:

$$\begin{aligned}
&\sup_{V\in\mathcal{V}} \lim_{\delta\to 0} \inf_{h_\Delta[\cdot]\in h_\Delta[\cdot;\rho_0,V],\delta(\Delta)\le\delta} \rho_1(h_\Delta[T]) \\
&= \lim_{\delta\to 0} \inf_{h_\Delta[\cdot]\in h_\Delta[\cdot;\rho_0,V^0],\delta(\Delta)\le\delta} \rho_1(h_\Delta[T]) = c_0^2. \quad (c_0^2 \in \mathbf{R}),
\end{aligned}$$

i.e. $V^0 \in \mathcal{V}$ is a maximin strategy.

To prove Theorem 2 it is sufficient to show that $c_0^1 = c_0^2$. First it is supposed that $c_0^2 < c_0^1$. Then, there exists a number c_* such that $c_0^2 < c_* < c_0^1$, i.e. $c_* \notin [c_0^1, \infty) = \mathbf{C}_1$ and according to the Theorem of the Alternative, the evasion problem from the set $M_1(c_*)$ is solvable. Then there exists a strategy $V_* \in \mathcal{V}$ and numbers $\varepsilon_* > 0$, $\delta_* > 0$, such that for each step motion $h_\Delta[\cdot] \in h_\Delta[\cdot;\rho_0,V_*]$ with $\delta(\Delta) \le \delta_*$, the following inequality holds: $\rho_1(h_\Delta[T]) \ge c_* + \varepsilon_*$, where it is assumed that $\varepsilon_* < c_0^1 - c_*$. Then

$$\lim_{\delta\to 0} \inf_{h_\Delta[\cdot]\in h_\Delta[\cdot;\rho_0,V_*],\delta(\Delta)\le\delta} \rho_1(h_\Delta[T]) \ge c_* + \varepsilon_*$$

and

$$c_0^2 = \sup_{V\in\mathcal{V}} \lim_{\delta\to 0} \inf_{h_\Delta[\cdot]\in h_\Delta[\cdot;\rho_0,V_*],\delta(\Delta)\le\delta} \rho_1(h_\Delta[T])$$

$$\geq \lim_{\delta\to 0} \inf_{h_\Delta[\cdot]\in h_\Delta[\cdot;\rho_0,V_*],\delta(\Delta)\leq\delta} \rho_1(h_\Delta[T]) \geq c_* + \varepsilon_* > c_0^2 + \varepsilon_*,$$

i.e. a contradiction is obtained.

Next we have to show that the inequality $c_0^2 > c_0^1$ is not satisfied. Suppose the contrary, i.e.

$$\lim_{\delta\to 0} \inf_{h_\Delta[\cdot]\in h_\Delta[\cdot;\rho_0,V^0],\delta(\Delta)\leq\delta} \rho_1(h_\Delta[T]) = c_0^2 > c_0^1.$$

Let $\gamma = (c_0^2 - c_0^1)/3$. From the last inequality it follows that there exists such a $\delta(\gamma) > 0$ that for each step motion $h_\Delta[\cdot] \in h_\Delta[\cdot;\rho_0,V^0]$ with $\delta(\Delta) \leq \delta(\gamma)$, the following inequality is satisfied: $\rho_1(h_\Delta[T]) \geq c_0^2 - \gamma$. From this inequality it follows that the evasion problem is solvable for the number $\tilde{c} = c_0^1 + \gamma$ and the corresponding set $M_1(\tilde{c})$, since $c_0^2 - \gamma = c_0^1 + 2\gamma > \tilde{c}$. But since $\tilde{c} > c_0^1$ and $\mathbf{C}_1 = [c_0^1, +\infty)$, then the ε-approach problem is solvable for the set $M_1(\tilde{c})$, which contradicts the Theorem of the Alternative. This contradiction proves $c_0^1 = c_0^2$. Thus the proof is completed.

Lemma 2. *Let the situation $(U^\varepsilon, V^\varepsilon) \in \mathcal{U} \times \mathcal{V}$ be an ε-saddle point for game (13). Then there exists a constant $d_0 > 0$ such that $\forall h_{\Delta^{(1)}}[\cdot] \in h_{\Delta(1)}[\cdot;\rho_0,U^\varepsilon]$ and $\forall h_{\Delta^{(2)}}[\cdot] \in h_{\Delta(2)}[\cdot;\rho_0,V^\varepsilon]$ with $\delta(\Delta^{(m)} \leq \delta_0$, $m = 1,2$, the following inequalities are valid:*

$$\rho_1(h_{\Delta^{(1)}}[T]) - \widehat{\varepsilon} \leq \lim_{\delta\to 0} \inf_{h_\Delta[\cdot],\delta(\Delta)\leq\delta} \rho_1(h_\Delta[T;\rho_0,U^\varepsilon,V^\varepsilon])$$

(17)

$$\leq \lim_{\delta\to 0} \inf_{h_\Delta[\cdot],\delta(\Delta)\leq\delta} \rho_1(h_\Delta[T;\rho_0,U^\varepsilon,V^\varepsilon]) \leq \rho_1(h_{\Delta^{(2)}})[T]) + \widehat{\varepsilon},$$

for each $\widehat{\varepsilon} \geq \varepsilon$, where ε is defined in (14).

Conversely from (17) it follows that the situation $(U^\varepsilon, V^\varepsilon)$ is an ε-saddle point with $\varepsilon > \widehat{\varepsilon}$.

Now let us consider the case when $N > 1$. First some standard notations will be introduced

$$\mathbf{R}^N_> = \{\rho = (\rho_1,\ldots,\rho_N) \in \mathbf{R}^N | \rho_i > 0, \forall i \in \mathbf{N}\},$$
$$\mathbf{R}^N_{\geqq} = \{\rho = (\rho_1,\ldots,\rho_N) \in \mathbf{R}^N | \rho_1 \geq 0, \forall i \in \mathbf{N}\},$$
$$\mathbf{R}^N_{\geq} = \{\rho = (\rho_1,\ldots,\rho_N) \in \mathbf{R}^N | \rho_i \geq 0, \forall i \in \mathbf{N}, \rho \neq 0_N\},$$

where 0_N is the zero-vector in $\mathbf{R}^N$, $\rho^{(1)} > \rho^{(2)} \iff \rho^{(1)} - \rho^{(2)} \in \mathbf{R}^N_> \iff \rho^{(2)} < \rho^{(1)}$, $\rho^{(1)} \not> \rho^{(2)} \iff \rho^{(1)} - \rho^{(2)} \notin \mathbf{R}^N_>$. The other relations are introduced. For example $\rho^{(1)} \geq \rho^{(2)}$ if and only if the relation $\rho^{(1)} \geq \rho^{(2)}$ is not satisfied, if and only if $\exists i_0 \in \mathbf{N}$: $\rho^{(1)}_{i_0} < \rho^{(2)}_{i_0}$ or $\rho^{(1)} = \rho^{(2)}$.

Besides,

$$\underset{\delta\to 0}{\underline{\mathrm{LIM}}}\ \rho(h_\Delta[T;\rho_0,U,V]) = (\lim_{\delta\to 0} \inf_{h_\Delta[\cdot],\delta(\Delta)\leq\delta} \rho_1(h_\Delta[T;\rho_0,U,V]),\ \ldots,$$

$$\lim_{\delta\to 0} \inf_{h_\Delta[\cdot],\delta(\Delta)\le\delta} \rho_N(h_\Delta[T;\rho_0,U,V])),$$

$$\overline{\mathrm{LIM}}_{\delta\to 0}\ \rho(h_\Delta[T;\rho_0,U,V]) = (\lim_{\delta\to 0} \sup_{h_\Delta[\cdot],\delta(\Delta)\le\delta} \rho_1(h_\Delta[T;\rho_0,U,V]), \ \ldots,$$

$$\lim_{\delta\to 0} \sup_{h_\Delta[\cdot],\delta(\Delta)\le\delta} \rho_N(h_\Delta[T;\rho_0,U,V])),$$

Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N) \in \mathbf{R}^N_>$ be a fixed vector.

Definition 2. *The situation $(U^\varepsilon, V^\varepsilon) \in \mathcal{U}\times\mathcal{V}$ is called an ε-Slater saddle point for game* (1) *if there exists a constant $\delta(\varepsilon) > 0$ such that $\forall h_{\Delta^{(1)}}[\cdot] \in h_{\Delta^{(1)}}[\cdot;\rho_0,U^\varepsilon]$ and $\forall h_{\Delta^{(2)}}[\cdot] \in h_{\Delta^{(2)}}[\cdot;\rho_0,V^\varepsilon]$ with $\delta(\Delta^{(m)}) \le \delta(\varepsilon)$, $m = 1,2$, the following vector inequalities are valid:*

$$\rho(h_{\Delta^{(1)}}[T]) - \varepsilon \not> \underline{\mathrm{LIM}}_{\delta\to 0}\ \rho(h_\Delta[T;\rho_0,U^\varepsilon,V^\varepsilon])$$
(18)
$$\rho(h_{\Delta^{(2)}}[T]) - \varepsilon \not< \overline{\mathrm{LIM}}_{\delta\to 0}\ \rho(h_\Delta[T;\rho_0,U^\varepsilon,V^\varepsilon]).$$

If in the relations (18), *the signs $\not>$ and $\not<$ are replaced respectively by $\not\geq$ and $\not\leq$, then the situation $(U^\varepsilon, V^\varepsilon)$ is called an ε-Pareto saddle point for game* (1).

The given definition for an ε-Slater (Pareto) saddle point includes the concept of an ε-saddle point for game (13) with a scalar pay-off function (Definition 1) as a particular case.

The following assertion is obtained from Lemma 2:

Corollary 2. *The ε-Slater and the ε-Pareto saddle points are $\widehat{\varepsilon}$-saddle points, $\forall\widehat{\varepsilon} > \varepsilon$ in game* (13) *with a scalar pay-off function.*

It is easy to prove the following assertion:

Lemma 3. *Let the situation $(U^{(j)}, V^{(j)}) \in \mathcal{U}\times\mathcal{V}$ be an ε_j-saddle point for the differential game with a scalar pay-off function $\langle \Xi, \{\mathcal{U},\mathcal{V}\}, \rho_j(h(T))\rangle$ for some constant $\varepsilon_j > 0$, $j \in \mathbf{N}$. Then, this situation is an ε-Slater saddle point for game* (1), *$\forall\varepsilon = (\varepsilon_1, \ldots, \widehat{\varepsilon}_j, \ldots, \varepsilon_N)$, where $\varepsilon_i \ge 0$, $\forall i \in \mathbf{N}$ and $\widehat{\varepsilon}_j > \varepsilon_j$.* From Theorem 2 and Lemma 3 we have [12]:

Theorem 3. *There exists an ε-Slater saddle point in the differential game* (1) *for each choice of $\varepsilon \in \mathbf{R}^N_\geq$.*

2. Sufficient conditions. Example. Now sufficient conditions for existence of an ε-Slater saddle point will be given. For this purpose the following differential game with scalar pay-off function is considered:

$$\langle \Xi, \{\mathcal{U},\mathcal{V}\}, \rho_\beta(h(T))\rangle, \tag{19}$$

where $\rho_\beta(h(\cdot)) = \sum_{i\in \mathbf{N}} \beta_i \rho_i(h(\cdot))$ and $\beta = (\beta_1, \dots, \beta_N) \in \mathbf{R}^N_{\geq}$.

By analogy with the differential game described by a system of ordinary differential equations and partial differential equations of parabolic type the following assertions hold.

Lemma 4. *For the situation $(U^*, V^*) \in \mathcal{U} \times \mathcal{V}$ it is supposed that*

1) *(U^*, V^*) consists of program strategies $U^* \div u^*(\cdot) = \{u(t), t_0 \leq t \leq T\}$ and $V^* \div v^*(\cdot) = \{v(t), t_0 \leq t \leq T\}$,*

2) *(U^*, V^*) is a γ-saddle point ($\gamma > 0$) for game* (19), *where $\beta \in \mathbf{R}^N_{\geq}$.*

Then, for each vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N) \in R^N_{>}$ with $\sum_{i\in\mathbf{N}} \beta_i\varepsilon_i \geq \gamma$ the situation (U^*, V^*) is an ε-Slater saddle point for game (1).

Lemma 4 is proved by using the proof of [7, Assertion 9.1] and taking into account that

$$\overline{\mathrm{LIM}}_{\delta\to 0}\ \rho(h_\Delta[T;\rho_0, U^*, V^*]) = \underline{\mathrm{LIM}}_{\delta\to 0}\ \rho(h_\Delta[T;\rho_0, U^*, V^*]) = \rho(h(T;\rho_0, u^*(\cdot), v^*(\cdot))$$

since U^* and V^* are program strategies.

Corollary 4. *Let us consider the situation $(U^*, V^*) \in \mathcal{U} \times \mathcal{V}$ which has the following properties:*

1) *It consists of program strategies $U^* \div u^*(\cdot)$, $V^* \div v^*(\cdot)$;*

2) *It is a γ-saddle point for game* (19), $\forall \gamma > 0$.

Then, the situation (U^*, V^*) is an ε-Slater saddle point for game (1), $\forall \varepsilon \in \mathbf{R}^N_{>}$.

Lemma 4 and Corollary 4 are valid only if U^* and V^* are program strategies. The example given at the end of this paper shows that Lemma 4 and Corollary 4 are not true for the positional strategies (U^*, V^*).

The following lemma is proved by analogy with Lemma 4.

Lemma 5. *For the situation $(U^*, V^*) \in \mathcal{U} \times \mathcal{V}$ it is supposed that:*

1) *it consists of program strategies;*

2) *it is a γ-saddle point ($\gamma > 0$) for game* (19), *where $\beta \in R^N_{>}$.*

Then, for every vector $\varepsilon \in \mathbf{R}^N_{>}$ such that $\sum_{i\in\mathbf{N}} \beta_i\varepsilon_i \geq \gamma$, the situation (U^*, V^*) is an ε-Pareto saddle point for game (1).

Two situations $(U^{(1)}, V^{(1)}) \in \mathcal{U} \times \mathcal{V}$ $(U^{(2)}, V^{(2)}) \in \mathcal{U} \times \mathcal{V}$ are ε-Slater saddle points for each $\varepsilon \in \mathbf{R}^N_{>}$ are called

1. equivalent, if

$$\underline{\mathrm{LIM}}_{\delta\to 0}\ \rho(h_\Delta[T;\rho_0, U^{(1)}, V^{(1)}]) = \underline{\mathrm{LIM}}_{\delta\to 0}\ \rho(h_\Delta[T;\rho_0, U^{(2)}, V^{(2)}])$$

and

$$\overline{\mathrm{LIM}}_{\delta\to 0}\ \rho(h_\Delta[T;\rho_0,U^{(1)},V^{(1)}]) = \overline{\mathrm{LIM}}_{\delta\to 0}\ \rho(h_\Delta[T;\rho_0,U^{(2)},V^{(2)}]);$$

2. interchangeable, if $(U^{(1)},V^{(2)})$ and $(U^{(2)},V^{(1)})$ are ε-Slater saddle points, for each $\varepsilon\in\mathbf{R}^N_>$.

For game (13) with a scalar pay-off function it is proved, that all saddle points are equivalent and interchangeable (see [7, Lemma 1.5]). The ε-saddle points have similar properties.

In general when $N>1$, the ε-Slater saddle points are not interchangeable and are not equivalent which is shown by the following example.

Example. It is supposed that the controlled system Ξ for game (1) is described by the following boundary-value problem

$$\begin{aligned}&\partial^2 y/\partial t^2=\partial^2 y/\partial x^2 \quad \text{in}\ \ G=(0,1)\times(0,\pi)\\ &y(x,0)=(\partial y/\partial t)(x,0)=0\ \ \text{for}\ \ x\in\Omega=(0,\pi)\\ &-(\partial y/\partial x)(0,t)=u(t)+v(t),\ \ (\partial y/\partial x)(\pi,t)=0\ \ \text{for}\ \ t\in(0,1).\end{aligned}\tag{20}$$

Program and positional strategies will be used, where $P_2(t)=Q_2(t)=[0,1]$ and $P_1(t)=Q_1(t)=\emptyset$. The set of strategies of the first (second) player is denoted by $\mathcal{U}(\mathcal{V})$ as well.

The vector pay-off function has two components

$$\rho(h(T))=(\rho_1(y(1)),\rho_2(y(1)))$$

and it is of the form

$$\rho_1(y(1))=\int_0^\pi y(x,1)dx,\quad \rho_2(y(1))=-\int_0^\pi y(x,1)dx.$$

This differential game will be denoted by

$$\langle\Xi,\{\mathcal{U},\mathcal{V}\},\{\rho_1(y(1)),\rho_2(y(2))\}\rangle\tag{21}$$

further on. Here $N=2$ and $\rho_2=-\rho_1$. Then, from Definition 2, each situation $(U^*,V^*)\in\mathcal{U}\times\mathcal{V}$ for which the condition

$$\underline{\mathrm{LIM}}_{\delta\to 0}\ \rho(y_\Delta[1;0,0,0,U^*,V^*]) = \overline{\mathrm{LIM}}_{\delta\to 0}\ \rho(y_\Delta[1;0,0,0,U^*,V^*])\tag{22}$$

is satisfied will be an ε-Slater saddle point, $\forall\varepsilon\in\mathbf{R}^2_>$. In particular, this assertion is valid for the case when U^* and V^* are program strategies.

Consider the program strategies

$$\begin{aligned}&U^{(0)}\div u^{(0)}(t)\equiv 0,\quad V^{(0)}\div v^{(0)}(t)\equiv 0,\quad \forall t\in[0,1],\\ &U^{(1)}\div u^{(1)}(t)\equiv 1,\quad V^{(1)}\div v^{(1)}(t)\equiv 1,\quad \forall t\in[0,1],\end{aligned}$$

Each of the situations $U^{(0)}, V^{(0)})$ and $U^{(1)}, V^{(1)})$ is an ε-Slater saddle point for game (21) $\forall\varepsilon \in \mathbf{R}^2_>$, because the condition (22) is satisfied for them.

From [5], the solution $y(t)$ of (20) for fixed functions $u(t)$ and $v(t)$ is of the form

$$y(t) = \pi^{-1}\int_0^t\int_0^\tau [u(\xi)+v(\xi)]d\xi d\tau$$

$$+\sum_{j=1}^{\infty} j^{-1}\omega_j(0)\int_0^t [u(\tau)+v(\tau)]\sin j(t-\tau)d\tau\omega_j(x),$$

where $\omega_j = \sqrt{2/\pi}\cos jx$. Since $\int_0^\pi \omega_j(x)dx = 0$, we obtain

$$\rho_1(y(1)) = \int_0^\pi \pi^{-1}\int_0^1\int_0^1 [u(\xi)+v(\xi)]d\xi d\tau dx = \int_0^1\int_0^1 [u(\xi)+v(\xi)]d\xi d\tau.$$

Thus, the following assertion is proved:

Lemma 6. *The functional $\rho_1(y(1)) = \rho_1(u,v)$ is strictly monotonously increasing with respect to u and v. More exactly, if $u_1(t)+v_1(t) \le u_2(t)+v_2(t)$, $\forall t \in [0,1]$ and $u_1+v_1 \ne u_2+v_2$ as functions in $L_2(0,1)$, then $\rho_1(u_1,v_1) < \rho_1(u_2,v_2)$. In particular,* $\min\rho_1(u,v) = 0$ *and it is attained for $u(t)=v(t)=0$, similarly* $\max\rho_1(u,v) = 2c > 0$, *($c = \text{const} > 0$) and it is attained for $u(t) = v(t) = 1$.*

From Lemma 6,

$$\rho_1(h_\Delta[1;0,0,0,U^{(1)},V^{(1)}]) = 2c > 0 = \rho_1(h_\Delta[1;0,0,0,U^{(0)},V^{(0)}]),$$

where the value of the functional $\rho_1(h_\Delta[1;0,0,0,U^{(j)},V^{(j)}])$, $(j=0,1)$ does not depend on the partition $\Delta \in \mathbf{\Delta}$, it is one and the same for all $\Delta \in \mathbf{\Delta}$. Thus the program situations $(U^{(0)}, V^{(0)})$ and $(U^{(1)}, V^{(1)}$ are not equivalent.

Next let us proceed by constructing the situation $(U^{(2)}, V^{(2)}$, for positional strategies $U^{(2)}$ and $V^{(2)}$. Let us remind that a positional strategy is a mapping, for which to every ordered triplet $(t_1, t_2, h(t_1)) \in [0,1]\times[0,1]\times\mathcal{H}$, $(\forall t_1, t_2 \in [0,1] : t_1 < t_2)$ there corresponds a function $u \in P(t_1,t_2]$ $(v \in Q(t_1,t_2])$ [4], (here $\mathcal{H} = L_2(0,\pi)\times(H^1_2(0,\pi))^*$). Let the set $S \subset [0,1]$, $(0 \in S, 1 \in S)$ be such that the sets S and $[0,1]\setminus S$ are dense in the interval $[0,1]$. The strategies $U^{(2)}$ and $V^{(2)}$ are defined as follows: if for the triplet (t_1,t_2,h), t_1 and t_2 belong to S and $h(t_1) = (0,0)$, then $U^{(2)} \div u^{(2)} = 0$ $V^{(2)} \div v^{(2)} = 1$; otherwise $U^{(2)} \div u^{(2)} = 1$ and $V^{(2)} \div v^{(2)} = 0$.

The constructed situation $(U^{(2)}, V^{(2)})$ is an ε-Slater saddle point, $\forall\varepsilon \in \mathbf{R}^2_>$, since for every partition $\Delta \in \mathbf{\Delta}$, $u^{(2)}(t) + v^{(2)}(t) = 1$, $\forall t \in [0,1]$, where $u^{(2)}(t)$ and $v^{(2)}(t)$ correspond to the strategies $U^{(2)}$ and $V^{(2)}$ and hence

$$\underset{\delta\to 0}{\underline{\mathrm{LIM}}}\ \rho(y_\Delta[1;0,0,0,U^{(2)},V^{(2)}]) = \underset{\delta\to 0}{\overline{\mathrm{LIM}}}\ \rho(y_\Delta[1;0,0,0,U^{(2)},V^{(2)}]) = (c,-c).$$

It will be shown that the ε-Slater saddle points $(U^{(0)}, V^{(0)})$ and $(U^{(2)}, V^{(2)})$ are not interchangeable. More exactly, it will be proved that the situation $(U^{(2)}, V^{(0)})$ is not an ε-Slater saddle point for game (21) for $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbf{R}^2_>$ and $\varepsilon_1 + e_2 < c$, where the number $c > 0$ is defined in Lemma 6.

Suppose the contrary, i.e. that for $(U^{(2)}, V^{(0)})$, the relations (18) are satisfied. The first of them is valid if and only if, there exists a constant $\delta(\varepsilon) > 0$ such that for $\forall h_\Delta[\cdot] \in h_\Delta[\cdot; 0, 0, 0, U^{(2)}]$ for $\delta(\Delta) \le \delta(\varepsilon)$ at least one of the following two inequalities

$$\rho_1(y_\Delta[1]) - \varepsilon_1 \le \lim_{\delta\to 0} \inf_{y_{\bar\Delta}[\cdot], \delta(\bar\Delta) \le \delta} \rho_1(y_{\bar\Delta}[1; 0, 0, 0, U^{(2)}, V^{(0)}]) \tag{23}$$

$$\rho_1(y_\Delta[1]) + \varepsilon_2 \ge \lim_{\delta\to 0} \inf_{y_{\bar\Delta}[\cdot], \delta(\bar\Delta) \le \delta} \rho_1(y_{\bar\Delta}[1; 0, 0, 0, U^{(2)}, V^{(0)}]). \tag{24}$$

is valid. Indeed, if the first relation (18) holds, but the inequality (23) is not satisfied, then

$$\rho_2(y_\Delta[1]) - \varepsilon_2 \le \lim_{\delta\to 0} \inf_{y_{\bar\Delta}[\cdot], \delta(\bar\Delta) \le \delta} \rho_2(y_{\bar\Delta}[1; 0, 0, 0, U^{(2)}, V^{(0)}]).$$

Multiplying the last inequality by -1 and taking into account that

$$\rho_2(y_\Delta[1]) = -\rho_1(y_\Delta[1]),$$

(24) is obtained.

The exact lower limit in (23) is reached in such a sequence of partitions $\{\Delta^{(k)}\}_1^\infty \subset \mathbf{\Delta}$ for which all points of the partitions $\tau_j^{(k)} \in [0, 1]$, $j = 0, 1, \ldots, m(\Delta)$, corresponding to $\Delta^{(k)}$, $k = 1, 2, \ldots$ belong to S and the exact lower limit is equal to 0. The exact upper limit in (24) is obtained by using such sequences of partitions $\{\bar\Delta^{(k)}\}_1^\infty \subset \mathbf{\Delta}$, for which for every partition $\bar\Delta^{(k)}$, $k = 1, 2, \ldots$, the numbers $\bar\tau_j^{(k)} \in (0, 1)$ do not belong to S and the exact upper limit is equal to c. Thus, the inequalities (23) and (24) take the form

$$\rho_1(h_\Delta[1]) - \varepsilon_1 \le 0 \quad\text{and}\quad \rho_1(h_\Delta[1]) + \varepsilon_2 \ge c, \tag{25}$$

where $h_\Delta[\cdot] \in h_\Delta[\cdot; 0, 0, 0, U^{(2)}]$ is an arbitrary element and $\delta(\Delta) \le \delta(\varepsilon)$.

Now consider the step motion

$$h_{\Delta_{s^\circ}}[\cdot] = h_{\Delta_{s^\circ}}[\cdot; 0, 0, 0, U^{(2)}, V^{(0)}], \quad s^\circ \in [0, 1].$$

The partition Δ_{s° is defined as follows: if the numbers of the partition $\bar\tau_j \in [0, s^\circ]$, $j = 0, 1, \cdot, \rho < m(\Delta) - 1$, then $\bar\tau_j \in S$, and if the numbers $\bar\tau_j \in (s^\circ, 1)$, $j = \rho + 1, \ldots, m(\Delta) - 1$, then $\bar\tau_j \notin S$. Moreover for every constant $\delta(\varepsilon) > 0$, the partition Δ_{s° can be chosen so that $\delta(\Delta_{s^\circ}) < \delta(\varepsilon)$. Furthermore, $\forall h_{\Delta_{s^\circ}}[\cdot] = h_{\Delta_{s^\circ}}[\cdot; 0, 0, 0, U^{(2)}, V^{(0)}]$.

$0 \leq \rho_1(h_{\Delta_{s^\circ}}[1]) \leq c$, while the minimum of $\rho_1(h_{\Delta_{s^\circ}}[1])$ is reached for $s^\circ = 1$ and the maximum for $s^\circ = 0$. The latter is obtained from Lemma 6. We take into account that

$$u^{(2)}(t) + v^{(0)}(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq \hat{s}^\circ \\ 1 & \text{for } \hat{s}^\circ < t \leq 1 \end{cases},$$

($\hat{s}^\circ \in (s^\circ - \delta(\Delta_{s^\circ}), s^\circ]$) and that the solution of (20) depends continuously of the control realizations "u" and "v". Hence, the solution of (20) depends continuously on $\hat{s}^\circ$. Thus it follows (choosing $\delta(\Delta_{s^\circ})$ sufficiently small) that the set $\{\rho_1(y_{\Delta_{s^\circ}}[1; 0, 0, 0, U^{(2)}, V^{(0)}])$, $\forall s^\circ \in [0, 1]$, $\forall \Delta_{s^\circ} : \delta(\Delta_{s^\circ}) \leq \delta(\varepsilon)\}$ is dense in the interval $[0, c]$. Therefore, for each a and b such that $0 \leq a < b \leq c$, there exist $\bar{c} \in (a, b)$ and $\Delta^{(\bar{c})} \in \mathbf{\Delta}$ with $\delta(\Delta^{(\bar{c})}) < \delta(\varepsilon)$ so that the following equality is valid:

$$p_1(y_{\Delta^{(\varepsilon)}}[1; 0, 0, 0, U^{(2)}, V^{(0)}]) = \bar{c}. \tag{26}$$

Let $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbf{R}^2_>$ and $e_1 + \varepsilon_2 < c$. Then, the number $\bar{c}$ can be chosen so that $\bar{c} \in (\varepsilon_1, c - \varepsilon_2)$ and for the step motion $h_{\Delta^{(\varepsilon)}}[\cdot] = (y_{\Delta^{(\varepsilon)}}[\cdot], y'_{\Delta^{(\varepsilon)}}[\cdot])$, corresponding to (26), none of the inequalities (25) is satisfied. This shows that the relations (18) are not valid, i.e. the situation $(U^{(2)}, V^{(0)})$ is not an ε-Slater saddle point, $\forall \varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbf{R}^2_> : \varepsilon_1 + \varepsilon_2 < c$. Therefore the e-Slater saddle points for game (21) $(U^{(2)}, V^{(2)})$ and $(U^{(0)}, V^{(0)})$ are not interchangeable.

In game (19) constructed for the considered differential game (21) put $\beta_1 = \beta_2 = 0{,}5$. Then the scalar pay-off function $\rho_\beta(y(1)) = 0$. From Definition 1, each situation $(U, V) \in \mathcal{U} \times \mathcal{V}$ is γ-saddle point for a game defined in (19) for $\beta_1 = \beta_2 = 0{,}5$, $\forall \gamma > 0$. Moreover, for the program situations (U^*, V^*), $U^* \div u(t)$, $V^* \div v(t)$, the assertions of Lemma 4 and Corollary 4 are valid, i.e. they are ε-Slater saddle points for game (21), $\forall \varepsilon \in \mathbf{R}^2_>$. At the same time, as shown above, the positional situation $(U^{(2)}, V^{(0)})$ is not an ε-Slater saddle point for game (21) if $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbf{R}^2_>$ and $\varepsilon_1 + \varepsilon_2 < c$, where the number $c > 0$ is defined in Lemma 6.

Thus, for the situation $(U^{(2)}, V^{(0)}) \in \mathcal{U} \times \mathcal{V}$, in which $U^{(2)}$is positional and $V^{(0)}$ is a program strategy, Lemma 4 and Corollary 4 are not valid due to the fact that strategy $U^{(0)}$ is positional.

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