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ON SPACES WITH POINT–COUNTABLE K–NETWORKS AND THEIR MAPPINGS

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Abstract. A collection γ of subsets of a topological space X is point–countable if card $(\{\xi \in \gamma | \xi \ni x\}) \leq \aleph_0$ for every point $x \in X$. Spaces with point–countable networks were studied by a number of authors (see [4] for bibliography). The spaces with point–countable k-networks are closely related to quotient images of metric spaces. Thus it is interesting to know what spaces are the images of spaces with such networks under certain quotient maps. Corollary 6.3 shows that if X is a k–space with a point–countable k–network, then so does $f(X)$ where f is a closed map. On the other hand it follows from Proposition 10 that $p(X)$ need not have a point–countable k–network in case p is an open map with compact fibers. We also introduce another type of networks as a technical tool and study some of their properties.

A collection γ of subsets of a topological space X is called k–network if whenever $K \subseteq U$, K compact and U open in X, then there is a finite $\gamma' \subseteq \gamma$ such that $K \subseteq \cup\gamma' \subseteq U$. Point–countable k–networks are of special interest. For example, it is shown in [4] that k–space with point-countable closed (i.e. consisting of closed sets) k–networks is an image of metric space under a quotient s–map (recall that s–map is a map with separable fibers). Since spaces with point - countable k–networks are closely related to quotient images of metric spaces (see [3], [4], [8], [9]) it is worth knowing what kinds of spaces could be images of spaces with such networks under quotient maps from particular classes. Thus it is shown in([4], theorem 7.1(e)) that perfect maps preserve point–countable k–networks. Corollary 6.3 generalizes that results in case of k–spaces. In this case a point–countable k–network is preserved by closed maps. On the other hand open compact maps may not preserve such a property as shown in Proposition 10.

We use standard notations for basic operations and spaces. By $(A)^{\circ}_{X}$ $(\overline{(A)_X})$ we denote the interior (closure) of A in space X. By $A(\tau)$ we denote the one–point compactification of a discrete space of cardinality τ. By S_τ we denote the space obtained by identifying the limit points of τ convergent sequences. $\mathbf{R}$ denotes the set of all real numbers and c denotes the cardinality of $\mathbf{R}$. Less common notations are $f(\gamma) = \{f(\xi)|\xi \in \gamma\}$ and $\gamma|_Y = \{\xi \cap Y|\xi \in \gamma\}$ where $f: X \to Y$ is a map and γ is a family of subsets of $X, Y \subseteq X$.

All spaces are assumed to be Hausdorff, all maps to be continuous and onto.

Definition 1. *Let γ be a collection of subsets of space X. γ is said to be an s–network if whenever $A \neq \overline{A} \subseteq X$ there exists $x \in X$ such that if $x \in U \subset X$, U open then $x \in \xi \subseteq U$, card$(U \cap \xi) \geq \aleph_0$ for some $\xi \in \gamma$.*

Definition 2. *Let γ be a subspace of X, γ be a network for Y. Let us say that γ is an exterior s–network for Y (an exterior k–network for Y) in X if whenever $A \neq \overline{A} \subseteq X$, $A \subseteq Y$, then there is $x \in X$ such that if $x \in U$, U open in X, then $\xi \subseteq U$,* card$(\xi \cap A) \geq \aleph_0$ *for some $\xi \in \gamma$ (whenever $K \subseteq U$ with K compact and U open in X, then $K \cap Y \subseteq \cup\gamma' \subseteq U$ for some finite $\gamma' \subseteq \gamma$.*

We will use „*pces*–network“ instead of „point countable exterior s–network“.

Proposition 1. *If γ is an s–network and f is a quotient map, then $f(\gamma)$ is an s–network.*

Lemma 2. *If γ is an exterior s–network (exterior k–network) for Y in X, and $Y = X$, then γ is an s–network (k–network) for X.*

Recall that a space X is strongly $\aleph_0$ – monolite if $\overline{F}$ has a countable base for any countable $F \subseteq X$. Analogously call a space X near–monolite if for any countable $F \subseteq X$, F has a *pces*–network.

The following proposition shows some connections between notions introduced.

Proposition 3. (1) *every point countable s–network is a k–network;*

(2) *if X is a k–space, and γ is a k–network for X, then γ is an s–network;*

(3) *every pces–network is an exterior k–network;*

(4) *in k–spaces every exterior k–network is an exterior s–network.*

Proof. (3) Let δ be *pces*–network for Y, $Y \subseteq X$; we shall prove that δ is an exterior k–network for Y. Suppose $K \subseteq U$ with K compact and U open in X and there is no finite $\delta' \subseteq \delta$ such that $Y \cap K \subseteq \cup\delta' \subseteq U$. For each $x \in K$ let $\{\xi \in \delta|x \in \xi\} = \{\xi_i(x)|i \in \omega\}$. Inductively choose $x_n \in K$ such that $x_n \notin \bigcup\{\xi_m(x_k)|m, k < n\}$. Since K is compact the set $S = \{x_n|n \in \omega\}$ has a cluster point x', so $A = S \setminus \{x'\}$ is not closed.

Thus there exist $x \in X$ and $\xi \in \delta$ such that $x \in \xi \subseteq U$ and $\mathrm{card}(\xi \cap A) \geq \aleph_0$. Then $\xi = \xi_n(x_k)$ for some n and k. However, $x_n \notin \xi_m(x_k)$ for $n > \max\{k, m\}$ contradicting $\mathrm{card}(\xi \cap A) \geq \aleph_0$.

(4) Now let γ be an exterior k–network for Y, $Y \subseteq X$. We shall prove that γ is an exterior s–network for Y. Let $A \neq \overline{A} \subseteq X$ and $A \subseteq Y$. Since X is a k–space there exists a compact $K \subseteq X$ such that $B = K \cap A$ is not closed. Let x be an arbitrary cluster point for B and let $x \in U$, U is open in X. Then $x \in V \subseteq U$, $K' = \overline{(V \cap K)_K} \subseteq U$ for some open V in X. Since γ is an exterior k–network, $K' \subseteq \cup\gamma' \subseteq U$ for some finite $\gamma' \subseteq \gamma$. Hence $\mathrm{card}(\xi \cap A) \geq \aleph_0$ for some $\xi \in \gamma'$. Thus γ is an exterior s–network.

Proofs of (1) and (2) can be obtained from the proofs of (3) and (4) supposing $X = Y$ and using Lemma 2.

Proposition 4. *Let X be a countably compact space. Suppose $K \subseteq X$ has pces–network γ in X. Then $\overline{K}$ is a metrizable compact space.*

Proof. Let $\gamma' \subseteq \gamma$ be the subfamily consisting of such $\xi \in \gamma$ that there exists a finite subcollection $\gamma'' \subseteq \gamma$ such that $\gamma'' \cup \{\xi\}$ is a minimal cover of K. By virtue of Miščhenko lemma [7], γ' is countable. Let us prove that for any $x \in \overline{K}$ and any $U \ni x$ open in X there is a finite $\gamma'' \subseteq \gamma'$ such that $\cup\gamma'' \subseteq U$, $\overline{(\cup\gamma'')^{\circ}_{K}} \ni x$.

Let $\delta \subseteq \gamma$ consist of all $\xi \in \gamma$ such that $x \notin \overline{\xi}$ or $\xi \in U$. We will use notation $\xi_i(y)$ as in Proposition 1. Inductively choose $S = \{x_n | n \in \omega\} \subseteq K$ as in Proposition 1. Since X is countably compact the set S has a cluster point x', so $S \setminus \{x'\}$ is not closed. Since γ is *pces*–network, there is a point x'' such that for any $V \ni x''$ there is $\xi \in \gamma$, $\xi \subseteq V$ such that $\mathrm{card}(\xi \cap S) \geq \aleph_0$. If $x'' = x$ let $U = V$. If $x'' \neq x$ choose $U' \ni x$ and $V \ni x''$ open in X such that $U' \cap V = \emptyset$. In any case $\xi \in \delta$ and $\mathrm{card}(\xi \cap S) \geq \aleph_0$ contradicting the way the x_i's were chosen. Hence there is a minimal finite cover $\delta' \subseteq \delta$ of K. Let $\gamma'' = \{\xi \in \delta' | \xi \subseteq U\}$. Then $\gamma'' \subseteq \gamma'$, $x \notin C = \bigcup_{\xi \in \delta' \setminus \gamma''} \overline{\xi}$. C is a closed set, and $K \setminus C \subseteq \cup\gamma'$, so γ' has the required property. Let ψ consist of all finite intersections of closures of elements of γ'; ψ is countable. We will prove that ψ is a network for $\overline{K}$. Let $x \in \overline{K}$, $x \in U$ open in X, $\{\xi_i | i \in \omega\} = \{\xi \in \gamma' | \overline{\xi_i} \ni x\}$. There is $n \in \omega$ such that $\bigcap_{i<n} \overline{\xi_i} \subseteq U$. Suppose not. Then the family $\Xi = \{\overline{\xi_i} | i \in \omega\} \cup \{\overline{K} \setminus U\}$ is countable, nested collection of closed sets because $\Xi \neq \emptyset$ as was proved before. Since X is countably compact there is x' such that $x' \in \overline{K} \setminus U$ and $x' \in \overline{\xi_i}$ for all $i \in \omega$. Since $x \neq x'$ there are $U' \ni x$ and $V \ni x'$ open in X such that $U' \cap V = \emptyset$ and it follows from what has been proved above that there is $\xi \in \gamma$ such that $\overline{\xi} \ni x$ and $\xi \subseteq U'$. But then $\xi = \xi_i$ for some $i \in \omega$ and $x' \notin \overline{\xi}$, a contradiction.

Corollary 4.1. *A countably compact space with a point-countable s–network is a metrizable compact space.*

Corollary 4.2. *A countably compact space X is nearmonolite iff it is strongly $\aleph_0$–monolite.*

Remark. It may be worth pointing out that if $K \subseteq X$ is relatively countably compact and has *pces*–network, then K is metrizable and moreover if X is regular, then $\overline{K}$ is also metrizable and compact.

Let γ be an arbitrary collection of subsets of space X. Let us call a property P of γ a hereditary–bijective property (or simply *hb*–property) if for any $Y \subseteq X$ $\gamma|_Y$ has the property P and for any continuous bijection $f: X \to Z$ $f(\gamma)$ has the property P.

Lemma 5. *The following properties are hb–properties:*

(1) *to be point–countable (finite);*

(2) *to be σ–point–finite;*

(3) *to have cardinality less than τ;*

(4) *the cardinality of any element is less than τ;*

(5) *to be a cover (network).*

Proposition 6. *Let $f: X \to Y$ be a closed map onto Y, $F \subseteq X$ has an exterior s–network γ with an hb–property P. Then $f(F)$ has an exterior s–network with property P.*

Proof. Consider the set $B = f(F) \subseteq Y$ for any $y \in B$ choose $x_y \in F$ such that $f(x_y) = y$. Let $G = \{x_y | y \in B\} \subseteq F$. One can easily check that $\gamma|_G$ is an exterior network for G. We will prove that $f(\gamma|_G)$ is an exterior s–network for $f(F)$. Let $A \neq \overline{A} \subseteq f(F)$. Since f is closed, $A' = f^{-1}(A) \cap G$ is not closed in X. Let $U' \ni f(x)$ be any open neighborhood of $f(x)$, $U = f^{-1}(U')$ and let $\xi \in \gamma$ be an element of γ with the property demanded by Definition 2. One can easily check that $\xi \cap G = \xi' \in \gamma|_G$, $f(\xi') \subseteq U'$, $\mathrm{card}(\xi' \cap A') \geq \aleph_0$. Hence $f(\gamma|_G)$ is an exterior network. Now $f|_G$ is a bijection, P is an *hb*–property, so $f(\gamma|_G)$ has the property P.

Corollary 6.1. *Let $f: X \to Y$ be a closed map onto Y, $F \subseteq X$ have a pces–network. Then $f(F)$ has a pces–network.*

Corollary 6.2. *An image of a nearmonolite space under a closed map is nearmonolite.*

Corollary 6.3. *Let X be a k–space with a point–countable k–network, $f: X \to Y$ be a closed map onto Y. Then Y has a point–countable k–network.*

Corollary 6.4. *Let X be a k–space with a σ–point–finite k–network, $f: X \to Y$ be a closed map onto Y. Then Y has a σ–point–finite k–network.*

The well-known Lashnev theorem [5] says that every closed map $f : M \to Y$, where M is metrizable is „perfect mod Q“, where $Q \subseteq Y$ is a σ-discrete subspace of Y. The following example shows that „metrizable“ here cannot be weakened to „paracompact with a point-countable base and a closed point-countable k-network“. Thus Lashnev theorem is not valid in the class of all k-spaces with point-countable k-networks.

Example 1. Let us recall the construction of Michael line, ML. The space ML is obtained from $\mathbb{R}$ by isolating the irrational points. It is easily seen that ML is not a σ-discrete space. Now consider the set $A = ML \times \omega$. Define the typical neighborhood of the point $(x, 0)$ to be of the form:

$O_A(x, 0) = (O(x) \times \omega) \setminus (\{x\} \times (\omega \setminus \{0\}))$, where $x \in O(x) \subseteq ML$ is open and all points (x, n) $x \in ML$, $n > 0$ are isolated. It is easy to check that the obvious map $\pi_{ML} : A \to ML$ is closed and not compact at each point $x \in ML$. One can see that A has a point-countable base and a closed point-countable k-network.

Although Lašnev theorem is not true for k-spaces with point-countable k-networks we can obtain the following particular case of it (we omit its proof).

Proposition 7. *Let X be a regular k-space with a point-countable k-network or a k-space with a point-countable closed k-network. Let $f : X \to K$ be a closed map onto a paracompact feathery space K. Then f is „perfect mod Q“, where $Q \subseteq K$ is a σ-discrete set.*

We omit also the proofs of the following two propositions since we are not interested in them at present.

Proposition 8. *Let X be a Frechet - Urysohn space with a point-countable k-network γ. Then for any convergent sequence $S = \{x_n | n \in \omega\}$, $x_n \to x$ as $n \to \infty$ there is a sequence $SU = \{U_n \ni x_n | n \in \omega\}$ of open sets such that $U_n \to x$ as $n \to \infty$.*

Recall that a collection $\{U_\alpha | \alpha \in A, U_\alpha \subseteq X, U_\alpha \neq \emptyset\}$ of open sets is a π-base for X if for any open U in X there is $\alpha \in A$ such that $U_\alpha \subseteq U$.

Proposition 9. *Let X be a Fréchet-Urysohn space with a point-countable k-network. Then X has a point-countable π-base.*

Corollary 9.1. *An image X of a space with a point-countable base under a closed map has a point-countable π-base.*

P r o o f. It is easily seen that X is a Fréchet-Ursohn space. By Corollary 6.3 X has a point-countable k-network. Using Proposition 9 we obtain that X has a point-countable π-base.

The following Proposition shows the property of a space to have a point–countable k–network is not preserved by open mappings.

Proposition 10. *Let X be a sequential space and $card(X) \leq c$. Then there are M – a metrizable space, $p : M \to p(M)$ – a quotient two–to–one map and $f : p(M) \to X$ – an open map onto X such that $f^{-1}(x)$ is compact for every $x \in X$.*

Proof. Assume without loss of generality that X is not a discrete space. Let $Sq = \{1/n | n \in \mathbf{N}\} \cup \{0\}$ be the standard converging sequence. Since card$(X) \leq c$, the set of all embeddings $\phi : Sq \to X$ has cardinality equal to c. Let $\{\phi_\alpha | \alpha \in A\}$, $card(A) \leq c$ be all such embeddings. Let $[0,1] = \bigcup_{\alpha \in A} Q_\alpha$ where $Q_{\alpha'} \cap Q_{\alpha''} = \emptyset$ if $\alpha' \neq \alpha''$ and $Q_\alpha = \{x^n_\alpha | n \in \omega\}$ and $\overline{Q}_\alpha = I = [0,1]$. Let $M = \oplus_{\alpha \in A}(Sq_\alpha \times \omega) \cup (\oplus_{x \in X} I_x)$ where $Sq_\alpha = Sq$. M is metrizable since M is a sum of metrizable spaces. Let a relation $\approx$ be defined as follows: $x \approx x'$ if $x, x' \in M$, $x \in Sq_\alpha \times \{n\}$, $x' = x^n_\alpha \in I_{\varphi_\alpha(x)}$. Now let $x \sim x'$ if either $x \approx x'$ or $x' \approx x$. It is easy to check that the obvious map $p : M \to M/\sim$ is two–to–one map. Let $f(y) = x \in X$ if $y \in p(I_x)$. We shall prove that f is an open map. Let $U \subseteq M/\sim$ be an open subset of $M/\sim$. Suppose $f(U) = F \subseteq X$ and $X \setminus F$ is not closed. Then there is a sequence $\{x_n | n \in \omega\} \subseteq X \setminus F$ and $x \in F$ such that $x_n \to x$ as $n \to \infty$. Let $\phi(1/n) = x_n$ for any $n \in \omega \setminus \{0\}$ and $\phi(0) = x$. Then ϕ is an embedding and $\phi = \phi_\alpha$ for some $\alpha \in A$. Since $I_x \cap p^{-1}(p(I_x) \cap U)$ is open in I_x there is $n \in \omega$ such that $x^n_\alpha \in U$. Then $Sq_\alpha \times \{n\} \cap p^{-1}(p(Sq_\alpha \times \{n\}) \cap U)$ is open in $Sq_\alpha \times \{n\}$ and thus there is $k \in \omega$ such that there is $t = (1/k, n) \in Sq_\alpha \times \{n\}$ such that $p(t) \in U \cap p(Sq_\alpha \times \{n\})$. Then $f(p(Sq_\alpha \times \{n\})) \ni x_k$ contradicting the way x_n's were chosen.

If in Proposition 10 we take for X a sequential space without a point-countable k–network, then we obtain an example of open s–map $f : Y \to X$ onto X where Y is a finite–to–one quotient image of a metric space, thus providing the negative answer to questions (10.3) and (10.4) in [4]. Unfortunately Y may be a nonregular space. The following is an example of a regular space X such that X is a two–to–one quotient image of metric space and there exists an open map $p : X \to Y$ with compact fibres where Y has no point–countable k–network.

Example 2. Let $T = \{T_{nk} | n, k \in \omega, k < 2^n\}$ be the collection of closed intervals, produced by the standard construction of the Cantor set K. let $X = K \oplus \oplus_{n,k \in \omega} T_{nk}$. Define the topology on X as follows. All points $x \in T_{nk}$, n, $k \in \omega$ have standard Euclidean neighborhoods, as points of closed interval $T_{nk} \subset \mathbf{R}$. Define the typical neighborhood of $t \in K$ to be of the form:

$$O_p(t) = \oplus_{n \in \omega \setminus \{0,\dots,p\}} \{U_{nk} | t \in U_{nk} - \text{open in } T_{nk}\} \cup \{t\}.$$

It is easy to check that X is a sequential space which is two–to–one quotient image of a metrizable space. Let $p : X \to K \oplus (\bigcup_{n,k\in\omega}(n,k)) = Y$, $p(x) = x$ for $x \in K$ and $p(x) = (n,k)$ for $x \in T_{nk}$. Let Y have the topology induced by p. One can easily check that $p : X \to Y$ is an open map and $p^{-1}(y)$ is a compact subspace of X for any $y \in Y$. Consider the family $O = \{p(O_0(x))|x \in K\}$ where $O_0(x) = \{x\} \cup \oplus\{T_{nk}|n,k \in \omega,\ x \in T_{nk}\}$. One can easily check that $\text{card}(O) = c$, every element of O is an open compact subset of Y, and O is an almost disjoint family, i.e. for any $\xi_1, \xi_2 \in O$, $\xi_1 \cap \xi_2$ is finite. Suppose Y has point–countable k–network γ. Then there is $\xi_\alpha \in \gamma$ such that $\xi_\alpha \subseteq \alpha$, $\text{card}(\xi_\alpha \cap \alpha) \geq \aleph_0$ for any $\alpha \in O$. Since O is an almost disjoint family $\xi_{\alpha'} \setminus K \neq \xi_{\alpha''} \setminus K$ for $\alpha' \neq \alpha''$. Thus the family $\{\xi_\alpha \setminus K|\alpha \in O\}$ is a non–countable point–countable collection of subsets of the countable set $Y \setminus K$, a contradiction.

The following example shows that not all sequential spaces can be obtained in the way Proposition 10 suggests (i.e. $\text{card}(X) \leq c$ is essential there).

We omit the proofs of the following two examples.

Example 3. The space $A((2^c)^+)$ cannot be an image of a k–space with a point–countable k–network under a quotient map.

Although Proposition 3 closely ties point–countable k–networks and point–countable s–networks there are some differences between these notions. In Example 4 we discuss the preservation of the property under product operation.

It is well–known ([4]) that a product of two spaces having point–countable k–networks has a point–countable k–networks also and it is easy to check that if $X \times Y$ has a point–countable k–networks, then there exist γ_1 and γ_2 – point–countable k–networks of X and Y respectively such that $\gamma = \{\xi_1 \times \xi_2|\xi_1 \in \gamma_1,\ \xi_2 \in \gamma_2\}$ is a k–network for $X \times Y$. As the following example shows the similar proposition is not true for space with point–countable s–networks.

Example 4. The space $S_{C^+} \times S_\omega$ has no point–countable s–network. Assuming CH the space $S_C \times S_\omega$ has a point–countable s–network, and for any γ_1, γ_2 – point–countable s–networks of S_C and S_ω respectively the family $\{\xi_1 \times \xi_2|\xi_1 \in \gamma_1, \xi_2 \in \gamma_2\}$ is not an s–network.

The following simple proposition gives a sufficient (but not necessary) condition for $X \times Y$ to have a point–countable s–network.

Proposition 11. *If $X \times Y$ is a k–space, X and Y have point–countable s–networks, then so does $X \times Y$.*

It is easy to check that the space $S_\omega \times Q$ where Q denotes all rationals has a countable s–network and it is not a k–space.

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