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FIRST ORDER STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Local and global solutions of the Cauchy problem for first order stochastic partial differential equations are investigated. The theorems are proved by the characteristics and the successive approximations methods.

1. Introduction. Partial differential equations with random coefficients and related problems were proposed by many mathematicians and physicists in connection with analysing random phenomena. Among these, random transportation equations were studied extensively by many authors, e.g. Keller [7], Frisch [4], Ogawa [8] and Funaki [3].

In this paper, we consider the partial differential equation of first order with a random coefficient

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x;\omega) \quad &+\{a(t,x)+b(t,x)\dot{w}(t;\omega)\}\frac{\partial u}{\partial x}(t,x;\omega) \\ &= f(t,x,u(t,x;\omega))+g(t,x,u(t,x;\omega))\dot{w}(t;\omega), \quad t\geq 0 \end{aligned} \tag{1}$$

with initial data

$$u(0,x;\omega)=\varphi(x), \tag{2}$$

where $\dot{w}(t,\omega)$, $t\geq 0$ is the white noise process.

Generalized (weak) solutions of a linear partial differential equation with the white noise as a coefficient (if $b(t,x)=1$, $f(t,x,z)=c(t,x)z+d(t,x)$ and $g(t,x,z)=0$) have been considered by Ogawa [8] for one space variable and by Funaki [3] (if $f(t,x,z)=c(t,x)z+d(t,x)$ and $g(t,x,z)=0$) for many space variables. The particular case of equation (1) (if $b(t,x)=0$) is also the equation studied by Gikhman and Miestechkina [6].

Let $\mathbf{R}^m$ denote the m-dimensional Euclidean space with the norm $|\cdot|$, and $\mathbf{R}_+ = [0,+\infty)$. Set $|\gamma| = (\operatorname{tr}\gamma\gamma^*)^{1/2}$ if γ is a matrix, and $D_T = [0,T] \times \mathbf{R}^m$. Let $w(t)(t \geq 0)$ be a d-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and $\mathcal{F}_t$ the smallest σ-field generated by $\{w(s), 0 \leq s \leq t\}$ and the set of all $\mathbf{P}$-null sets.

Now, we consider the stochastic integral equation

$$u(t,x) = \varphi(y(0;t,x)) + \int_0^t f(s, y(s;t,x), u(s, y(s;t,x)))\, ds$$

$$+ \int_0^t \mathbf{E}\{g(s, y(s;t,x), u(s, y(s;t,x))) \mid \mathcal{F}_s\}\, dw(s),$$

where $y(s;t,x)$ is a solution of the following stochastic integral equation

$$y(s) = x + \int_s^t a(\tau, y(\tau))\, d\tau + \int_s^t b(\tau, y(\tau))\, dw(\tau). \tag{4}$$

The second integrals in equations (3) and (4) are understood in the sense of Ito.

We can notice the close analogy between our consideration and the common theory of partial differential equations of first order. In this sense we call the stochastic process $\{y(s;t,x), s \leq t\}$ the characteristic line through (t,x) of equation (1).

Recently, Gikhman [5] has studied the existence and the uniqueness of solutions of equation (3) under the condition that the drift coefficient f and the diffusion coefficient g satisfy a Lipschitz condition with respect to the last variable (see also [6]). In the present paper, applying the characteristics and successive approximations methods, we prove the local and global existence and uniqueness theorems for the solutions of (3), without assuming a Lipschitz condition in the last argument for f and g.

2. The auxiliary results. Let $B(T)$ be the set of all functions $z : [0,T]\times\Omega \to \mathbf{R}^m$ satisfying the following conditions:

(a) $z(t,\cdot) : \Omega \to \mathbf{R}^m$ is measurable for each fixed $t \in [0,T]$;

(b) $z(\cdot,\omega) : [0,T] \to \mathbf{R}^m$ is continuous for a.e. fixed $\omega \in \Omega$.

Consider $B(T)$ with the norm $\| z \|_{B(T)} = \{E(\sup_{0\leq t\leq T} |z(t,\omega)|^2)\}^{1/2}$. It is easy to prove that $B(T)$ is a Banach space [9].

Let $L^2(\Omega, \mathbf{R}^m)$ be the set of all $\mathbf{R}^m$-valued, mean square integrable functions on Ω and let $S(x,r)$ be the closed ball of center x with radius r in $L^2(\Omega, \mathbf{R}^m)$, that is, $S(x,r) = \{ y \in L^2(\Omega, \mathbf{R}^m) : \mathbf{E}|y - x| \leq r\}$.

Assumption H_1. *Suppose that:*

1^0 *The vector-valued function* $a : D_T \to \mathbf{R}^m$ *and the matrix-valued function*

$b : D_T \to \mathcal{L}(\mathbf{R}^d, \mathbf{R}^m)$ are measurable and $a(t,\cdot) : \mathbf{R}^m \to \mathbf{R}^m, b(t,\cdot) : \mathbf{R}^m \to \mathcal{L}(\mathbf{R}^d, \mathbf{R}^m)$ are continuous for each fixed $t \in [0,T]$, where $\mathcal{L}(\mathbf{R}^d, \mathbf{R}^m)$ is the space of all linear maps from $\mathbf{R}^d$ into $\mathbf{R}^m$;

2^0 *There is a function $H : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+, H(\cdot, z) : \mathbf{R}_+ \to \mathbf{R}_+$ which is locally integrable for each fixed $z \in \mathbf{R}_+$ and $H(t,\cdot) : \mathbf{R}_+ \to \mathbf{R}_+$ is continuous, non-decreasing for each fixed $t \in \mathbf{R}_+$ and such that*

$$\mathbf{E}|a(t,z)|^2 + \mathbf{E}|b(t,z)|^2 \leq H(t, \mathbf{E}|z|^2),$$

for all $t \in [0,T]$ and all $z \in S(x,r)$;

3^0 *There is a function $G : [0,T] \times [0,4r] \to \mathbf{R}_+, G(\cdot, z) : [0,T] \to \mathbf{R}_+$ which is locally integrable for each fixed $z \in [0,4r]$ and $G(t,\cdot) : [0,4r] \to \mathbf{R}_+$ is non-decreasing, continuous for each fixed $t \in [0,T]$ and such that $G(t,0) = 0$ and*

$$\mathbf{E}|a(t,z) - a(t,\bar{z})|^2 + \mathbf{E}|b(t,z) - b(t,\bar{z})|^2 \leq G(t, \mathbf{E}|z - \bar{z}|^2),$$

for all $t \in [0,T]$ and all $z, \bar{z} \in S(x,r)$;

4^0 *$v(t) \equiv 0$ on $[0,T_1]$ is the unique nonnegative continuous function v such that $v(0) = 0$ and*

$$v(t) \leq A \int_0^t G(s, v(s))\, ds,$$

for all $t \in [0,T_1]$, where $A = 2(1+T)$ and $0 < T_1 \leq T$.

Lemma 1 [10]. *Let Assumption H_1 be satisfied. Then there is a unique local solution $y(s) = y(s;t,x)$ of (4).*

Remark 1. Note that, since $y(s) = y(s;t,x)$ is the unique solution of (4), y satisfies the following group property [5], [3]:

$$y(s;\tau, y(\tau,t,x)) = y(s;t,x), \tag{5}$$

for $\tau \in [s,t], (t,x) \in D_{T_1} = [0,T_1] \times \mathbf{R}^m$.

3. The local existence of solutions. Let B be the set of all functions $v : D_T \times \Omega \to \mathbf{R}^n$ measurable, $\mathcal{F}_t$-adapted, with the norm $\| v \|_B = \{\mathbf{E}(\sup_{t,x} |v(t,x,\omega)|^2)\}^{1/2}$. B is a Banach space. Let $\tilde{S} = S(\varphi(y(0)), \varrho)$ be the closed ball of centre $\varphi(y(0))$ with radius ϱ in $L^2(\Omega, \mathbf{R}^n)$, that is, $\tilde{S} = \{v \in L^2(\Omega, \mathbf{R}^n) : \mathbf{E}|v - \varphi(y(0))|^2 \leq \varrho\}$.

Assumption H_2 . *Suppose that:*

1^0 *The vector-valued function $f : D_T \times \mathbf{R}^n \to \mathbf{R}^n$ and the matrix-valued function $g : D_T \times \mathbf{R}^n \to \mathcal{L}(\mathbf{R}^d, \mathbf{R}^n)$ are measurable and $f(t,\cdot) : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}^n, g(t,\cdot) : \mathbf{R}^m \times \mathbf{R}^n \to \mathcal{L}(\mathbf{R}^d, \mathbf{R}^n)$ are continuous for each fixed $t \in [0,T]$;*

2^0 *There is a function* $K : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+, K(\cdot, z) : \mathbf{R}_+ \to \mathbf{R}_+$ *which is locally integrable for each fixed* $z \in \mathbf{R}_+$, *and* $K(t, \cdot) : \mathbf{R}_+ \to \mathbf{R}_+$ *is continuous, non-decreasing for each fixed* $t \in \mathbf{R}_+$ *and such that*

$$\mathbf{E}|f(t,x,z)|^2 + \mathbf{E}|g(t,x,z)|^2 \le K(t, \mathbf{E}|z|^2), \tag{6}$$

for all $t \in [0, T]$ *and all* $z \in \tilde{S}$;

3^0 *The vector-valued function* $\varphi : \mathbf{R}^m \to \mathbf{R}^n$ *is continuous.*

Now, we define the sequence of successive approximations $\{u_n(t,x)\}$ as follows:

$$\begin{aligned} u_{n+1}(t,x) &= \varphi(y(0;t,x)) + \textstyle\int_0^t f[u_n](s;t,x)\,ds \\ &\quad + \textstyle\int_0^t \mathbf{E}\{g[u_n](s;t,x) \mid \mathcal{F}_s\}\,dw(s), \\ u_0(t,x) &= \varphi(y(0;t,x)), \end{aligned} \tag{7}$$

where

$$f[u_n](s;t,x) = f(s, y(s;t,x), u_n(s, y(s;t,x)))$$

$$g[u_n](s;t,x) = g(s, y(s;t,x), u_n(s, y(s;t,x)))$$

Lemma 2. *Let assumption* H_2 *be satisfied and* $\varphi(y(0;t,x))$ *be independent of the Brownian motion* $w(s), s \ge 0$, *and* $\sup_{t,x} \mathbf{E}|\varphi(y(0;t,x))|^2 < \infty$. *Then, there is a time* $\bar{T}$ *such that* $0 < \bar{T} < T$ *and the sequence of functions* $\{\mathbf{E}|u_n(t,x)|^2\}, (t,x) \in D_{\bar{T}}$ *is uniformly bounded.*

Proof. It is easy to show that the integrals on the right-hand side of (7) are well defined [5]. Now, let us note, by condition 2^0 of H_2 and Caratheodory's theorem [2], that the differential equation

$$\frac{dz}{dt} = 3(1+T)K(t, z(t)) \tag{8}$$

has a local solution with any initial value $z_0 \ge 0$. Let $z(t) = z(t; 0, z_0)$ be the local solution of (8) with $z_0 > 3\sup_{t,x} \mathbf{E}|\varphi(y(0;t,x))|^2$. Now, we shall show that

$$\mathbf{E}|u_n(t,x)|^2 < z(t), \tag{9}$$

$$\mathbf{E}|u_n(t,x) - \varphi(y(0;t,x))|^2 \le \varrho, \tag{10}$$

$(t,x) \in D_{\bar{T}}$, $n = 1, 2, \ldots$

Utilizing Doob's and Schwarz's inequalities, conditions (5),(6), and the property of conditional expectations [1], we get

$$\begin{aligned}\mathbf{E}|u_1(t,x)|^2 &\le 3\mathbf{E}|\varphi(y(0;t,x))|^2 + |\int_0^t f[u_0](s;t,x)\,ds|^2 \\ &+ |\int_0^t \mathbf{E}\{g[u_0](s;t,x)\,|\,\mathcal{F}_s\}\,dw(s)| \\ &\le 3\sup_{t,x}\mathbf{E}|\varphi(y(0;t,x))|^2 + 3(1+T)\int_0^t K(s,\sup_{t,x}\mathbf{E}|\varphi(y(0;t,x))|^2)\,ds,\end{aligned}$$

for all $(t,x)\in D_T$, from which, by condition 2^0 of H_2, we obtain the existence of a time T_0 such that $0<T_0<T$ and

$$z(t)-\mathbf{E}|u_1(t,x)|^2 > 3(1+T)\int_0^t [K(s,z(s)) - K(s,\sup_{t,x}\mathbf{E}|\varphi(y(0;t,x))|^2)\,]\,ds \ge 0$$

for all $(t,x)\in D_{T_0}$ since $z_0 > 3\sup_{t,x}\mathbf{E}|\varphi(y(0;t,x))|^2$ and $z(t)$ is the local solution of (8) starting with the initial point z_o. Next, since the function $z(t)$ is continuous on $[0,T_o]$, set $q_0=\max\{z(t) : t\in[0,T_0]\}<\infty$.Thus, $K(s,z(s))\le K(s,q_0)$ for each $s\in[0,T_0]$ by condition 2^0 of H_2. Hence, the function $K(s,z(s))$ is integrable on $[0,T_0]$. Therefore, we get a time $\bar{T}$ such that $0<\bar{T}<T_0$ and

$$\begin{aligned}\mathbf{E}|u_1(t,x)-\varphi(y(0;t,x))|^2 &\le 2\mathbf{E}|\int_0^t f[u_0](s;t,x)\,ds|^2 \\ &+ 2\mathbf{E}|\int_0^t \mathbf{E}\{g[u_0](s;t,x)\,|\,\mathcal{F}_s\}\,dw(s)|^2 \\ &\le 2(1+T)\int_0^t K(s,\sup_{t,x}\mathbf{E}|\varphi(y(0;t,x))|^2)\,ds \\ &\le 2(1+T)\int_0^t K(s,z(s))\,ds \le \varrho,\end{aligned}$$

for all $(t,x)\in D_{\bar{T}}$.

Assume that (9) and (10) hold for $n=k$. Then, we have for $n=k+1$,

$$\begin{aligned}\mathbf{E}|u_{k+1}(t,x)|^2 &\le 3\mathbf{E}|\varphi(y(0;t,x))|^2 + 3T\int_0^t \mathbf{E}|f[u_k](s;t,x)|^2\,ds \\ &+ 3\int_0^t \mathbf{E}|g[u_k](s;t,x)|^2\,ds \\ &\le 3\sup_{t,x}\mathbf{E}|\varphi(y(0;t,x))|^2 + 3(1+T)\int_0^t K(s,\mathbf{E}|u_k(s,y(s;t,x))|^2)\,ds,\end{aligned}$$

for all $(t,x) \in D_{\tilde{T}}$, which implies, since $z_0 > 3\sup_{t,x} \mathbf{E}|\varphi(y(0;t,x))|^2$, that

$$\begin{aligned}\mathbf{E}|u_{k+1}(t,x)|^2 &- 3(1+T)\int_0^t K(s,\mathbf{E}|u_k(s,y(s;t,x))|^2)\,ds \\ &< z(t) - 3(1+T)\int_0^t K(s,z(s))\,ds.\end{aligned}$$

Hence, by the inductive assumption,

$$z(t) - \mathbf{E}|u_{k+1}(t,x)|^2 > 3(1+T)\int_0^t [K(s,z(s)) - K(s,\mathbf{E}|u_k(s,y(s;t,x))|^2]\,ds \geq 0,$$

for all $(t,x) \in D_{\tilde{T}}$. Next,

$$\begin{aligned}\mathbf{E}|u_{k+1}(t,x) - \varphi(y(0;t,x))|^2 &\leq 2\mathbf{E}|\int_0^t f[u_k](s;t,x)\,ds|^2 \\ &+ 2\mathbf{E}|\int_0^t \mathbf{E}\{g[u_k](s;t,x)\,|\,\mathcal{F}_s\}dw(s)|^2 \\ &\leq 2(1+T)\int_0^t K(s,\mathbf{E}|u_k(s,y(s;t,x))|^2)\,ds \\ &\leq 2(1+T)\int_0^t K(s,z(s))\,ds \leq \varrho,\end{aligned}$$

for all $(t,x) \in D_{\tilde{T}}$. Thus, by induction, (9) and (10) hold for all n. Since $z(t)$ is continuous on $[0,\tilde{T}]$, there exists a real number $C > 0$ such that $\mathbf{E}|u_n(t,x)|^2 < C$ for all $(t,x) \in D_{\tilde{T}}$ and every integer $n \geq 0$. This completes the proof of the lemma. □

Assumption H_3. *Suppose that:*

1^0 *There is a function* $M : [0,T] \times [0,4\varrho] \to \mathbf{R}_+$, $M(\cdot,z) : [0,T] \to \mathbf{R}_+$ *which is locally integrable for each fixed* $z \in [0,4\varrho]$ *and* $M(t,\cdot) : [0,4\varrho] \to \mathbf{R}_+$ *is non-decreasing, continuous for each fixed* $t \in [0,T]$ *such that* $M(t,0) = 0$ *and*

$$\mathbf{E}|f(t,x,z) - f(t,x,\bar{z})|^2 + \mathbf{E}|g(t,x,z) - g(t,x,\bar{z})|^2 \leq M(t,\mathbf{E}|z-\bar{z}|^2), \tag{11}$$

for all $t \in [0,T]$ *and all* $z, \bar{z} \in \tilde{S}$;

2^0 $z(t) \equiv 0$ *on* $[0,\tilde{T}]$ *is the unique, non-negative, continuous function* z *such that* $z(0) = 0$ *and*

$$z(t) \leq A\int_0^t M(s,z(s))\,ds$$

for $t \in [0,\tilde{T}]$, *where* $A = 2(1+T)$ *and* $0 < \tilde{T} \leq T$.

Remark 2. If the function $M(t,z)$ is concave with respect to z for each fixed $t \geq 0$, then the inequality (11) can be replaced by the following

$$|f(t,x,z)-f(t,x,\bar{z})|^2+|g(t,x,z)-g(t,x,\bar{z})|^2 \leq M(t,|z-\bar{z}|^2)$$

for all $(t,x,z),(t,x,\bar{z}) \in \mathbf{R}_+ \times \mathbf{R}^m \times \mathbf{R}^n$.

Indeed, if $[M(t,z)]^2/z$ is concave with respect to z for each fixed $t \geq 0$, then by Jensen's inequality we get

$$\begin{aligned} \mathbf{E}\, M(t,|z-\bar{z}|^2) &\leq \mathbf{E}\{[M(t,|z-\bar{z}|^2)/|z-\bar{z}|][|z-\bar{z}|]\} \\ &\leq \{\mathbf{E}[M(t,|z-\bar{z}|^2)]^2/|z-\bar{z}|^2\}^{1/2}\{\mathbf{E}|z-\bar{z}|^2\}^{1/2} \\ &\leq \{[M(t,\mathbf{E}|z-\bar{z}|^2)]^2/\mathbf{E}|z-\bar{z}|^2\}^{1/2} \\ &= M(t,\mathbf{E}|z-\bar{z}|^2), \end{aligned}$$

$z,\bar{z} \in \tilde{S}$, which imples that

$$\begin{aligned} \mathbf{E}|f(t,x,z)-f(t,x,\bar{z})|^2 &+ \mathbf{E}|g(t,x,z)-g(t,x,\bar{z})|^2 \\ &\leq \mathbf{E}\, M(t,|z-\bar{z}|^2) \leq M(t,\mathbf{E}|z-\bar{z}|^2), \end{aligned}$$

for all $z,\bar{z} \in \tilde{S}$.

Theorem 1. *Let assumptions H_1, H_2 and H_3 be satisfied, and $\varphi(y(0;t,x))$ be independent of the Brownian motion $w(s), s \geq 0$, and $\sup_{t,x} \mathbf{E}|\varphi(y(0;t,x))|^2 < \infty$. Then, the sequence $\{u_n(t,x)\}$ defined by* (7) *converges to the unique local solution of* (3).

Proof. Let $\bar{T}$ be the time which is obtained in Lemma 2 with $0 < \bar{T} < T$. Now we define the sequence of functions $v_k : [0,\bar{T}] \to \mathbf{R}_+$ by

$$v_k(t) = \sup\{\sup_x v_{mn}(t,x) \,:\, m \geq n \geq k\},$$

where $v_{mn}(t,x) = \mathbf{E}|u_m(t,x)-u_n(t,x)|^2$.

Since the sequence $\{\mathbf{E}|u_n(t,x)|^2\}, (t,x) \in D_{\bar{T}}$ is uniformly bounded by Lemma 2, we have for some positive P

$$v_{mn}(t,x) \leq 2\mathbf{E}(|u_m(t,x)|^2+|u_n(t,x)|^2) < P \tag{12}$$

for all $(t,x) \in D_{\bar{T}}$. Next, we have that

$$\begin{aligned} |v_{mn}(t,x) &- v_{mn}(s,x)| \\ &\leq \mathbf{E}(|u_m(t,x)-u_n(t,x)|+|u_m(s,x)-u_n(s,x)|) \\ &\quad \times(|u_m(t,x)-u_m(s,x)|+|u_n(t,x)-u_n(s,x)|) \\ &\leq (\mathbf{E}(|u_m(t,x)-u_n(t,x)|+|u_m(s,x)-u_n(s,x)|)^2)^{1/2} \\ &\quad \times(\mathbf{E}(2|u_m(t,x)-u_m(s,x)|^2+2|u_n(t,x)-u_n(s,x)|^2))^{1/2}. \end{aligned}$$

On the other hand we get

$$\begin{aligned}\mathbf{E}|u_n(t,x) &- u_n(s,x)|^2 \\ &\le 2T\,\mathbf{E}\int_s^t |f[u_{n-1}](\tau;t,x)|^2\,d\tau \\ &\quad +2\mathbf{E}\int_s^t |\mathbf{E}\{g[u_{n-1}](\tau;t,x)\,|\,\mathcal{F}_\tau\}|^2\,d\tau \\ &\le 2(1+T)\int_s^t K(\tau,\mathbf{E}|u_{n-1}(\tau,y(\tau;t,x))|^2)\,d\tau \\ &\le 2(1+T)|Q(t)-Q(s)|\end{aligned}$$

for all $(t,x)\in D_{\bar{T}}$ and all integers $n\ge 1$, where $Q(t)=\int_0^t K(\tau,z(\tau))\,d\tau$, and $z(t)$ is the local solution of (8). Thus we get for some positive D

$$|v_{mn}(t,x)-v_{mn}(s,x)|\le D|Q(t)-Q(s)|^{1/2} \tag{13}$$

for all $m\ge n\ge 0$ and $(t,x),(s,x)\in D_{\bar{T}}$. From (12) and (13) we have

$$0\le v_k(t)<P$$

and

$$|v_k(t)-v_k(s)|\le D|Q(t)-Q(s)|^{1/2},$$

for all integers $k\ge 0$ and $t,s\in[0,\bar{T}]$, which implies by Ascoli-Arzela's theorem that there is a subsequence $\{v_{k(l)}(t)\}$ which converges uniformly to some continuous function $v(t)$ defined on $[0,\bar{T}]$.

Now, since $m-1\ge n-1\ge k(l)$, by condition 1^0 of H_3 we obtain for $m\ge n\ge k(l+1)$,

$$\begin{aligned}v_{mn}(t,x) &= \mathbf{E}|u_m(t,x)-u_n(t,x)|^2 \\ &\le 2\mathbf{E}(|\int_0^t [f[u_{m-1}](s;t,x)-f[u_{n-1}](s;t,x)]\,ds|^2 \\ &+ |\int_0^t \mathbf{E}\{(g[u_{m-1}](s;t,x)-g[u_{n-1}](s;t,x))\,|\,\mathcal{F}_s\}\,dw(s)|^2) \\ &\le A\int_0^t M(s,\mathbf{E}|u_{m-1}(s,y(s;t,x))-u_{n-1}(s,y(s;t,x))|^2)\,ds \\ &\le A\int_0^t M(s,v_{k(l)}(s))\,ds,\end{aligned}$$

for all $(t,x)\in D_{\bar{T}}$, where $A=2(1+T)$. Thus

$$v_{k(l+1)}(t)\le A\int_0^t M(s,v_{k(l)}(s))\,ds,\ t\in[0,\bar{T}].$$

Thus, by the Lebesgue dominated convergence theorem and the continuity of $M(t,u)$ in u for each fixed $t \in [0,\bar{T}]$ we have

$$v(t) \le A \int_0^t M(s, v(s))\, ds, \ t \in [0,\bar{T}].$$

Hence, by condition 2^0 of H_3, we obtain that $v(t) \equiv 0$ on $[0,\bar{T}]$.

Now, for $m \ge n \ge k(l+1)$, applying Doob's martingale inequality and Schwarz's inequality we obtain

$$\begin{aligned}
&\mathbf{E}\big(\sup_{(t,x)\in D_{\bar{T}}} |u_m(t,x) - u_n(t,x)|^2\big) \\
&\le 2\mathbf{E}\Big(\sup_{t,x} \Big| \int_0^t (f[u_{m-1}](s;t,x) - f[u_{n-1}](s;t,x))\, ds\Big|^2 \\
&+ \sup_{t,x} \Big| \int_0^t \mathbf{E}\{[g[u_{m-1}](s;t,x) - g[u_{n-1}](s;t,x)] \mid \mathcal{F}_s\}\, dw(s)\Big|^2\Big) \\
&\le 2(4+T) \int_0^{\bar{T}} M(s, \sup_x \mathbf{E}|u_{m-1}(s,x) - u_{n-1}(s,x)|^2)\, ds \\
&\le 2(4+T) \int_0^{\bar{T}} M(s, v_{k(l)}(s))\, ds \to 0
\end{aligned}$$

as $l \to \infty$, which imples that the sequence $\{u_n(t,x)\}$ is a Cauchy sequence in the Banach space B. Therefore, there exists a stochastic process $u(t,x)$ such that

$$\mathbf{E}(\sup_{t,x} |u_n(t,x) - u(t,x)|^2) \to 0$$

as $n \to \infty$. As usual we can prove that $u(t,x)$ is a local solution of (3).

It remains to prove the uniqueness. Suppose $u(t,x)$ and $z(t,x)$ are two solutions of (3) on $D_{\bar{T}}$. Then, we get

$$\sup_x \mathbf{E}|u(t,x) - z(t,x)|^2 \le 2(1+T) \int_0^t M(s, \sup_x \mathbf{E}|u(s,x) - z(s,x)|^2)\, ds$$

for all $t \in [0,\bar{T}]$. This and condition 2^0 of H_3 implies that $\mathbf{E}|u(t,x) - z(t,x)|^2 = 0$ for all $(t,x) \in D_{\bar{T}}$. This shows the uniqueness of the local solution, and the theorem is proved. □

4. The global existence of solutions. Now, we shall present the existence and uniqueness of a global solution of (3).

Theorem 2. *Let:*

1^0 *Conditions* $1^0 - 3^0$ *of* H_1 *and* $1^0 - 3^0$ *of* H_2 *be satisfied with* $r = \infty$,

$\varrho = \infty$ and $T = \infty$, respectively;

2^0 *For any fixed $T > 0$, the differential equation* (8) *has a global solution for any initial value $z_0 \geq 0$;*

3^0 *For any fixed $T > 0$, conditions 4^0 of H_1 and 2^0 of H_3 hold;*

4^0 *$\varphi(y(0;t,x))$ be independent of the Brownian motion $w(s), s \geq 0$, and* $\sup_{t,x} \mathbf{E}|\varphi(y(0;t,x))|^2 < \infty$.

Then, the sequence $\{u_n(t,x)\}$ defined by (7) *converges on any subset D_T of $\mathbf{R}_+ \times \mathbf{R}^m$, to the unique solution of* (3).

Proof. Let S be the set of times τ such that the sequence $\{u_n(t,x)\}$ converges on the interval $[0,\tau]$. Let $\tau_1 = \sup\{\tau \in S\}$. By Theorem 1 we have that $\tau_1 > 0$. Suppose now that $\tau_1 < \infty$. Then we can take a time T_0 such that $\tau_1 < T_0 < \infty$. Thus, by Assumptions 1^0 and 2^0, we have a solution $z(t)$ of (8) with $T = T_0$ which exists on $[0, T_0]$, and the estimate (9) holds on D_{T_0}. The remainder of proof follows as in Theorem 1, replacing $\bar{T}$ by T_o, which completes the proof of Theorem 2.

Corollary. *For the stochastic equations* (3) *and* (4), *suppose that:*

1^0 *There are continuous, non-decreasing and concave functions $\alpha_j : \mathbf{R}_+ \to \mathbf{R}_+$ with $\alpha_j(0) = 0, j = 1,2$, and $\lambda_j \in L^1_{loc}(\mathbf{R}_+, \mathbf{R}_+)$ such that*

$$|a(t,z) - a(t,\bar{z})|^2 + |b(t,z) - b(t,\bar{z})|^2 \leq \lambda_1(t)\alpha_1(|z-\bar{z}|^2),$$
$$|f(t,x,z) - f(t,x,\bar{z})|^2 + |g(t,x,z) - g(t,x,\bar{z})| \leq \lambda_2(t)\alpha_2(|z-\bar{z}|^2),$$

for all $(t,x,z), (t,x,\bar{z}) \in \mathbf{R}_+ \times \mathbf{R}^m \times \mathbf{R}^n$;

2^0 $|a(\cdot,0)|, |b(\cdot,0)|, |f(\cdot,x,0)|, |g(\cdot,x,0)| \in L^2_{loc}(\mathbf{R}_+,\mathbf{R}_+)$, $x \in \mathbf{R}^m$;

3^0 $\displaystyle\int_{0+} (1/\alpha_j(u))\,du = \infty$, $j = 1,2$;

4^0 *Condition 3^0 of H_2 is satisfied and* $\sup_{t,x} \mathbf{E}|\varphi(y(0;t,x))|^2 < \infty$.

Then, on any finite interval $[0,T]$, there are unique solutions of (3) *and* (4).

Proof. Since $\alpha_j(u)$, $j = 1,2$, are concave on $\mathbf{R}_+$, there are positive real numbers $c_j > 0, d_j > 0$, $j = 1,2$, such that $\alpha_j(u) \leq c_j u + d_j$, $j = 1,2$.

From condition 1^0 we have

$$|a(t,z) - a(t,0)|^2 + |b(t,z) - b(t,0)|^2 \leq \lambda_1(t)\alpha_1(|z|^2),$$

and

$$|f(t,x,z) - f(t,x,0)|^2 + |g(t,x,z) - g(t,x,0)|^2 \leq \lambda_2(t)\alpha_2(|z|^2).$$

Thus we obtain that

$$|a(t,z)|^2 + |b(t,z)|^2 \leq 2(|a(t,z) - a(t,0)|^2 + |b(t,z) - b(t,0)|^2 + |a(t,0)|^2 + |b(t,0)|^2)$$

$$
\begin{aligned}
&\leq 2\lambda_1(t)\alpha_1(|z|^2) + 2(|a(t,0)|^2 + |b(t,0)|^2) \\
&\leq 2c_1\lambda_1(t)|z|^2 + 2c_1\lambda_1(t) + 2(|a(t,0)|^2 + |b(t,0)|^2) \\
&= \beta_1(t)|z|^2 + \gamma_1(t),
\end{aligned}
$$

and

$$|f(t,x,z)|^2 + |g(t,x,z)|^2 \leq \beta_2(t)|z|^2 + \gamma_2(t),$$

where $\beta_j(t) = 2c_j\lambda_j(t)$, $\gamma_1(t) = 2c_1\lambda_1(t)+2(|a(t,0)|^2+|b(t,0)|^2)$, and $\gamma_2(t) = 2c_2\lambda_2(t)+ 2\sup_x(|f(t,x,0)|^2 + |g(t,x,0)|^2)$ are locally integrable with respect to $t \in \mathbf{R}_+$.

Next, set $H(t,u) = \beta_1(t)u + \gamma_1(t)$,and $K(t,u) = \beta_2(t)u + \gamma_2(t)$, $u \geq 0$, for all $t \in \mathbf{R}_+$. Then, since H and K are linear in u, condition 2^o of Theorem 2 is satisfied with functions H and K, respectively. Obviously the rest assumptions of Lemma 1 and Theorem 2 are satisfied. Therefore, by Lemma 1 and Theorem 2, we get the desirable conclusion which completes the proof of the corollary.

Remark 3. If $\alpha_j(u) = u\,(u \geq 0)$ and $\lambda_j(t) = L_j\,(L_j > 0)$, then condition 1^o implies a Lipschitz condition [5].

REFERENCES

[1] L.ARNOLD. Stochastic Differential Equations: Theory and Applications. John Wiley, New York, 1974.

[2] E.A.CODDINGTON, N.LEVINSON. Theory of Ordinary Differential Equations. McGraw-Hill, New York, 1955.

[3] T.FUNAKI. Construction of a solution of random transport equation with boundary condition. *J. Math. Soc. Japan*, **31** (1979), 719-744.

[4] U.FRISCH. Wave Propagation in Random Media. In: Probabilistic Methods in Applied Mathematics (ed. A. Bharucha-Ried), vol. 1. Academic Press, New York, 1968, 75-198.

[5] I.I.GIKHMAN. The Cauchy problem for second-order nonlinear parabolic stochastic systems. Statistics and control of random processes (Preila,1987),"Nauka", Moscow, 1989, 31-36 (in Russian).

[6] I.I.GIKHMAN, T.M.MIESTECHKINA. Cauchy problem for first order stochastic partial differential equations. *Teor. Sluchainykh Protsessov*, **11** (1983), 25-28 (in Russian).

[7] J.B.Keller. Wave Propagation in Random Media. *Proc. Sympos. Appl. Math.*, **13** (1962), 227-246.

[8] S.Ogawa. A partial differential equation with the white noise as a coefficient. *Z. Wahr. verw. Geb.*, **28** (1973), 53-71.

[9] A.E.Rodkina. On existence and uniqueness of solution of stochastic differential equation with hereditary. *Stochastics*, **12** (1984), 187-200.

[10] T.Taniguchi. Successive approximations to solutions of stochastic differential equations. *J. Differential Equations*, **96** (1992), 152-169.

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