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SOME REDUCTION FORMULAS FOR THE H-FUNCTION OF TWO VARIABLES

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ABSTRACT. Six reduction formulas for the H-function of two variables are derived in this article. The results obtained are believed to be new.

1. Introduction and Preliminaries. The H-function of two variables introduced by P. Mittal and K. Gupta [8, p. 117] is defined in terms of the concise notation of H. Srivastava and R. Panda [12, p. 266] in the following form:

$$
\begin{aligned}
H[x,y] &= H^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{array}{c|c} x & (a_j;\alpha_j,A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2};(e_j,E_j)_{1,p_3} \\ y & (b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2};(f_j,F_j)_{1,q_3}\end{array}\right] \\
&= (2\pi i)^{-2}\int_{L_1}\int_{L_2}\varphi(s,t)\theta_1(s)\theta_2(t)x^s y^t\,ds\,dt,
\end{aligned}
\tag{1.1}
$$

where, for convenience, $(a_j;\alpha_j,A_j)_{1,p_1}$ and $(c_j,\gamma_j)_{1,p_2}$ abbreviate the parameter sequences $(a_1;\alpha_1,A_1),\ \dots,\ (a_{p_1};\alpha_{p_1},A_{p_1})$ and $(c_1;\gamma_1),\ \dots,\ (c_{p_2};\gamma_{p_2})$ respectively, and similar representations hold for $(b_j;\beta_j,B_j)_{1,q_1}$; $(d_j,\delta_j)_{1,q_2}$ and so on.

We have used the following notations:

$$
\varphi(s,t) = \frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j s+A_j t)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j s-A_j t)\prod_{j=1}^{q_1}(1-b_j+\beta_j s+B_j t)},
\tag{1.2}
$$

$$
\theta_1(s) = \frac{\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j s)\prod_{j=1}^{n_2}\Gamma(1-c_j+\gamma_j s)}{\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j s)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\gamma_j s)},
\tag{1.3}
$$

and

$$\theta_2(s) = \frac{\prod_{j=1}^{m_3} \Gamma(f_j - F_j t) \prod_{j=1}^{n_3} \Gamma(1 - e_j + E_j t)}{\prod_{j=m_3+1}^{q_3} \Gamma(1 - f_j + F_j t) \prod_{j=n_3+1}^{p_3} \Gamma(e_j - E_j t)}. \tag{1.4}$$

Also x and y in (1.1) are not equal to zero and an empty product is always interpreted as unity. L_1 and L_2 are suitable contours of Barnes type.

Further we suppose that all the parameters a_j, b_j, c_j, d_j, e_j, f_j are complex numbers and all the associated coefficients α_j, A_j, β_j, B_j, γ_j, δ_j, E_j, F_j are real and positive such that

$$\xi_1 = \sum_{j=1}^{p_1} \alpha_j + \sum_{j=1}^{p_2} \gamma_j - \sum_{j=1}^{q_1} \beta_j - \sum_{j=1}^{q_2} \delta_j \leq 0, \tag{1.5}$$

$$\xi_2 = \sum_{j=1}^{p_1} A_j + \sum_{j=1}^{p_2} E_j - \sum_{j=1}^{q_1} B_j - \sum_{j=1}^{q_3} F_j \leq 0 \tag{1.6}$$

and

$$w_1 = -\sum_{j=n_1+1}^{p_1} \alpha_j - \sum_{j=1}^{q_1} \beta_j + \sum_{j=1}^{m_2} \delta_j - \sum_{j=m_2+1}^{q_2} \delta_j + \sum_{j=1}^{n_2} \gamma_j - \sum_{j=n_2+1}^{p_2} \gamma_j > 0, \tag{1.7}$$

$$w_2 = -\sum_{j=n_1+1}^{p_1} A_j - \sum_{j=1}^{q_1} B_j + \sum_{j=1}^{m_3} F_j - \sum_{j=m_3+1}^{q_3} F_j + \sum_{j=1}^{n_3} E_j - \sum_{j=n_3+1}^{p_3} E_j > 0. \tag{1.8}$$

The double Mellin-Barnes contour integral (1.1) converges absolutely [8, p. 119] under comditions (1.7) and (1.8) and defines an analytic function of two complex variables in the region defined by $|\arg x| < \frac{1}{2}\pi w_1$, $|\arg y| < \frac{1}{2}\pi w_2$.

The well-known Fox's H-function [5] of one variable is defined and represented by means of the following Mellin-Barnes type integral:

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_j, e_j)_{2,p} \\ (b_j, f_j)_{1,q} \end{matrix}\right] = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - f_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - f_j s)} \times \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + e_j s)}{\prod_{j=n+1}^{p} \Gamma(a_j - e_j s)} z^s ds \tag{1.9}$$

where L is a suitable contour separating the poles of gamma functions $\prod_{j=1}^{m}\Gamma(b_j - B_j s)$, for $j = 1, \dots m$ from those of the gammas functions $\prod_{j=1}^{n}\Gamma(1 - a_j + A_j s)$, for $j = 1, \dots n$. Asymptotic expansions and analytic continuations of the H-function are given by B. Braaksma [3]. A detailed account of the H-function is available from the monograph of A. Mathai and R. Saxena [7].

Reduction formulas for the H-function of two variables are obtained by R. Buschman [4] and N. Rathie [10, 11] and T. Hai and B. Yakubovich [6, p. 86-88].

In this article, we derive six reduction formulas for the H-function of two variables with the help of certain known summation formulas for the generalized hypergeometric series.

2. Main Results. The following reduction formulas for the H-function of two variables will be established here:

$$(i) \qquad H^{0,1;1,1;m,n}_{1,1;1,1;p,q}\left[\begin{array}{c|c} 1 & (1-\gamma;1,k):(\beta,1);(e_j,E_j)_{1,p} \\ y & (1-\gamma-\beta;1,k):(0,1);(f_j,F_j)_{1,q}\end{array}\right]$$

$$(2.1) \qquad = \Gamma(1-\beta)H^{m,n+2}_{p+2,q+2}\left[y\left|\begin{array}{l}(1-\gamma,k),(-\frac{\gamma}{2},\frac{k}{2}),(e_j,E_j)_{1,p} \\ \\ (f_j,F_j)_{1,q},(-\gamma,k),(1-\beta-\frac{\gamma}{2},\frac{k}{2})\end{array}\right.\right],$$

where $Re(1-\beta) > 0$.

$$(ii) \qquad H^{0,0;m,n;1,0}_{1,0;p,q;0,1}\left[\begin{array}{c|c} x & (\alpha;\rho,1):(c_j,\gamma_j)_{1,p};\dots \\ y & \dots:(d_j,\delta_j)_{1,q};(\beta,1)\end{array}\right]$$

$$(2.2) \qquad = y^{\beta}(1-y)^{\alpha-\beta-1}H^{m,n}_{p+1,q}\left[x(1-y)^{-\rho}\left|\begin{array}{l}(c_j,\gamma_j)_{1,p},(\alpha-\beta,\rho) \\ (d_j,\delta_j)_{1,q}\end{array}\right.\right],$$

where $|y| < 1$.

$$(iii) \qquad H^{0,2;m,n;1,0}_{2,0;p,q;1,3}\left[\begin{array}{c|c} x & (\alpha;\rho,1),(\beta;\sigma,1):(c_j,\gamma_j)_{1,p};(h,1) \\ 1 & \dots:(d_j,\delta_j)_{1,q};(0,1),(h,1),(f,1)\end{array}\right]$$

$$(2.3) \qquad = \frac{1}{\Gamma(h)\Gamma(1-h)}H^{m+1,n+2}_{p+4,q+1}\left[x\left|\begin{array}{l}(\alpha,\rho),(\beta,\sigma),(c_j,\gamma_j)_{1,p},(\alpha-f,\rho),(\beta-f,\sigma) \\ (\alpha+\rho-f-1,\rho+\sigma),(d_j,\delta_j)_{1,q}\end{array}\right.\right],$$

where $Re(1-h) > 0$.

(iv)
$$H^{0,3;1,0;m,n}_{3,0;0,3;p,q}\left[\begin{array}{c|l} 1 & (1-\alpha;1,\rho),(1-\beta;1,\sigma),(1-\gamma;1,\xi):\ldots;(e_j,E_j)_{1,p} \\ y & \ldots:(0,1),(1-b,1),(1-c,1);(f_j,F_j)_{1,q} \end{array}\right]$$

$$= 2^{-2\alpha}\left\{ H^{m+1,n+4}_{p+7,q+2}\left[\frac{y}{(4)^\rho}\left| \begin{array}{l} (1-\alpha,\rho),(1-\beta,\sigma),(2-\gamma,\xi), \\ (\frac{c}{2}-\frac{\alpha}{2},\frac{\rho}{2}), \end{array}\right.\right.\right.$$

$$\left.\begin{array}{l} (1-\frac{b}{2}-\frac{\beta}{2},\frac{\sigma}{2}),(e_j,E_j)_{1,p},(b-\alpha,\rho),(c-\alpha,\rho),(\frac{b}{2}-\frac{\beta}{2},\frac{\sigma}{2}) \\ (f_j,F_j)_{1,q},(1-\frac{c}{2}-\frac{\alpha}{2},\frac{\rho}{2}) \end{array}\right]$$

$$+H^{m+1,n+4}_{p+7,q+2}\left[\frac{y}{(4)^\rho}\left| \begin{array}{l} (1-\alpha,\rho),(1-\beta,\sigma),(2-\gamma,\xi),(1-\frac{c}{2}-\frac{\beta}{2},\frac{\sigma}{2}), \\ (\frac{b}{2}-\frac{\alpha}{2},\frac{\rho}{2}),(f_j,F_j)_{1,q}; \end{array}\right.\right.$$

$$\left.\left.\begin{array}{l} (e_j,E_j)_{1,p},(b-\alpha,\rho),(c-\alpha,\rho),(\frac{c}{2}-\frac{\beta}{2},\frac{\sigma}{2}) \\ (1-\frac{b}{2}-\frac{\alpha}{2},\frac{\rho}{2}) \end{array}\right]\right\} \tag{2.4}$$

(v) L.H.S. of (iv)

$$= 2^{-2\alpha-2}\left\{ C H^{m+2,n+3}_{p+5,q+4}\left[\frac{y}{(4)^\rho}\left| \begin{array}{l} (1-\alpha,\rho),(1-\beta,\sigma),(1-\gamma,\xi),(e_j,E_j)_{1,p'} \\ (\frac{b}{2}-\frac{\alpha}{2}+\frac{1}{2},\frac{\rho}{2}),(\frac{c}{2}-\frac{\alpha}{2},\frac{\rho}{2}), \end{array}\right.\right.\right.$$

$$\left.\begin{array}{l} (b-\alpha,\rho),(c-\alpha,\rho) \\ (f_j,F_j)_{1,q},(-\frac{c}{2}-\frac{\alpha}{2},\frac{\rho}{2}),(\frac{1}{2}-\frac{b}{2}-\frac{\alpha}{2},\frac{\rho}{2}) \end{array}\right] - bH^{m+2,n+3}_{p+5,q+4}\left[\frac{y}{(4)^\rho}\right.$$

$$\left.\left.\left|\begin{array}{l} (1-\alpha,\rho),(1-\beta,\sigma),(1-\gamma,\xi),(e_j,E_j)_{1,p},(b-\alpha,\rho),(c-\alpha,\rho) \\ (\frac{b}{2}-\frac{\alpha}{2},\frac{\rho}{2}),(\frac{c}{2}+\frac{1}{2}-\frac{\alpha}{2},\frac{\rho}{2}),(f_j,F_j)_{1,q},(-\frac{b}{2}-\frac{\alpha}{2},\frac{\rho}{2}),(\frac{1}{2}-\frac{c}{2}-\frac{\alpha}{2},\frac{\rho}{2}) \end{array}\right.\right]\right\}; \tag{2.5}$$

(vi) L.H.S. of (iv)

$$= 2^{-2\alpha-1}\left\{ H^{m,n+5}_{p+7,q+2}\left[\frac{y}{(4)^{\rho}}\left| \begin{matrix} (1-\alpha,\rho),(1-\beta,\sigma),(1-\gamma,\xi), \\ (f_j,F_j)_{1,q},(1-\frac{c}{2}-\frac{\alpha}{2},\frac{\rho}{2}), \end{matrix}\right.\right.\right.$$

$$\left.\begin{matrix} (\frac{1}{2}-\frac{b}{2}-\frac{\beta}{2},\frac{\sigma}{2}),(1-\frac{c}{2}-\frac{\beta}{2},\frac{\sigma}{2}),(e_j,E_j)_{1,p},(b-\alpha,\rho),(c-\alpha,\rho) \\ (\frac{1}{2}-\frac{b}{2}-\frac{\alpha}{2},\frac{\rho}{2}) \end{matrix}\right]$$

$$-bH^{m,n+5}_{p+7,q+2}\left[\frac{y}{(4)^{\rho}}\left| \begin{matrix} (1-\alpha,\rho),(1-\beta,\sigma),(1-\gamma,\xi),(1-\frac{b}{2}-\frac{\beta}{2},\frac{\sigma}{2}), \\ (f_j,F_j)_{1,q},(1-\frac{b}{2}-\frac{\alpha}{2},\frac{\rho}{2}), \end{matrix}\right.\right.$$

$$\left.\left.\begin{matrix} (\frac{1}{2}-\frac{c}{2}-\frac{\beta}{2},\frac{\sigma}{2}),(e_j,E_j)_{1,p},(b-\alpha,\rho),(c-\alpha,\rho) \\ (\frac{1}{2}-\frac{c}{2}-\frac{\alpha}{2},\frac{\rho}{2}) \end{matrix}\right]\right\}. \tag{2.6}$$

Proof. To prove (2.1), we denote the L.H.S. of (2.1) by I, and express the H-function of two variables in terms of the double contour integral (1.1) to obtain

$$I = (2\pi i)^{-2}\int_{L_1}\int_{L_2}\theta_2(t)\frac{\Gamma(\gamma+s+kt)\Gamma(1-\beta+s)\Gamma(-s)}{\Gamma(\gamma+\beta+s+kt)}y^t ds dt \tag{2.7}$$

where

$$\theta_2(t) = \frac{\prod_{j=1}^{m}\Gamma(f_j-F_jt)\prod_{j=1}^{n}\Gamma(1-e_j+E_jt)}{\prod_{j=m+1}^{q}\Gamma(1-f_j+F_jt)\prod_{j=n+1}^{p}\Gamma(e_j-E_jt)}$$

Evaluating the s-integral in (2.7) with the help of the well-known result

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\Gamma(-s)\Gamma(\lambda+s)\Gamma(\mu+s)}{\Gamma(\nu+s)}z^s ds = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\nu)}{}_2F_1(\lambda,\mu;\nu;-z)$$

and using Gauss's theorem [9, p.68] and the interpreting the result with the help of (1.9), we finally arrive at the result (2.1). The proof of (2.2), (2.3), (2.4), (2.5) and (2.6) can be obtained similarly by using the results [7, p. 145], [7, p. 151] and [1, p. 81 (Eq. 3.1 to 3.3)], respectively.

3. Special Case. If we set $\sigma = \xi = 0$ in (2.4), we obtain the following reduction formula

$$H^{0,1;1,2;m,n}_{1,0;2,3;p,q}\left[\begin{array}{c|c} 1 & (1-\alpha,1;\rho);(1-\beta,1),(1-\gamma,1),(e_j,E_j)_{1,p} \\ y & \ldots;(0,1),(1-b,1),(1-c,1),(f_j,F_j)_{1,q} \end{array}\right]$$

$$= 2^{-2\alpha}\Gamma(\beta)\Gamma(\gamma-1)\left\{\frac{\Gamma(\frac{b}{2}+\frac{\beta}{2})}{\Gamma(\frac{b}{2}-\frac{\beta}{2})}H^{m+1,n+1}_{p+3,q+2}\left[\frac{y}{(4)^\rho}\left|\begin{array}{l}(1-\alpha,\rho),\\ (\frac{c}{2}-\frac{\alpha}{2},\frac{\rho}{2}),\end{array}\right.\right.\right.$$

$$\left.\begin{array}{l}(e_j,E_j)_{1,p},(b-\alpha,\rho),(c-\alpha,\rho)\\ (f_j,F_j)_{1,q},(1-\frac{c}{2}-\frac{\alpha}{2},\frac{\rho}{2})\end{array}\right] + \frac{\Gamma(\frac{c}{2}+\frac{\beta}{2})}{\Gamma(\frac{c}{2}-\frac{\beta}{2})}H^{m+1,n+1}_{p+3,q+2}\left[\frac{y}{(4)^\rho}\right.$$

$$\left.\left.\left|\begin{array}{l}(1-\alpha,\rho),(e_j,E_j)_{1,p},(b-\alpha,\rho),(c-\alpha,\rho)\\ (\frac{b}{2}-\frac{\alpha}{2},\frac{\rho}{2}),(f_j,F_j)_{1,q},(1-\frac{b}{2}-\frac{\alpha}{2},\frac{\rho}{2})\end{array}\right.\right]\right\}. \tag{3.1}$$

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