SOME RESULTS ON THE COMMUTATIVE NEUTRIX PRODUCT OF DISTRIBUTIONS

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Abstract. Let \( f, g \) be distributions in \( \mathcal{D}' \) and let \( f_n = f * \delta_n, g_n = g * \delta_n \), where \( \{\delta_n\} \) is a certain sequence converging to the Dirac delta-function. The neutrix product \( f \square g \) is said to exist and be equal to \( h \) if

\[
\lim_{n \to \infty} \langle f_n g_n, \phi \rangle = \langle h, \phi \rangle
\]

for all \( \phi \) in \( \mathcal{D} \). Neutrix products of the form \( \ln x \square \delta(x) \) and \( x^{-s} \square \delta(x) \) are evaluated from which further neutrix products are obtained.

The following definition of a neutrix was given by van der Corput [1]:

**Definition 1.** Let \( N \) be an additive group of functions defined on a set \( N' \) with values in an additive group \( N'' \) with the property that the only constant function in \( N \) is the zero function. Then \( N \) is said to be a neutrix and the functions in \( N \) are said to be negligible.

**Example 1.** Let \( N' = N'' = R \), the real numbers and let \( N \) be the set of real-valued functions of the form

\[ N = \{ a \sin x + b \cos x : a, b \in R \} \]

Then \( N \) is a neutrix.

Now suppose \( N' \) is a subspace of a topological space \( X \) having an accumulation point \( y \) which is not in \( N' \). Let \( N'' = R \) (or \( C \) the complex numbers). Let \( N \) be an additive group of real (or complex) valued functions defined on \( N' \), with the property that if \( N \) contains a function \( \nu(x) \) which converges to a finite limit \( c \) as \( x \) tends to \( y \), then \( c = 0 \). Then \( N \) is a neutrix, since if \( f \) is in \( N \) and \( f(x) = c \) for all \( x \) in \( N' \), then \( \lim_{x \to y} f(x) = c \) implies \( c = 0 \).

This leads us to the following definition:
Definition 2. Let $f$ be a real (or complex) valued function on $\mathbb{N}'$ and suppose there exists $c$ in $\mathbb{R}$ (or $\mathbb{C}$) such that $f(x) - c$ is in $\mathbb{N}$. Then $c$ is called the neutrix limit of $f(x)$ as $x$ tends to $y$ and we write

$$\operatorname{N-lim}_{x \to y} f(x) = c.$$ 

Notice that if a neutrix limit $c$ exists then it is unique, since if $f(x) - c$ and $f(x) - c'$ are in $\mathbb{N}$, then

$$c - c' \in \mathbb{N} \Rightarrow c = c'.$$

Also notice that if $\mathbb{N}$ is a neutrix containing the set of all functions which converge to zero in the normal sense as $x$ tends to $y$, then

$$\lim_{x \to y} f(x) = c \Rightarrow \operatorname{N-lim}_{x \to y} f(x) = c.$$

From now on, the neutrix $\mathbb{N}$ we will use will have domain the positive integers, range the real numbers with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \quad \lambda > 0, \quad r = 1, 2, \ldots$$

and all functions which converge to zero in the normal sense as $n$ tends to infinity.

Example 2. The Gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt,$$

but more generally we have

$$\Gamma^{(r)}(x) = \operatorname{N-lim}_{n \to \infty} \int_{1/n}^{1/n} t^{x-1} \ln^r t e^{-t} \, dt$$

for $x \neq 0, -1, -2, \ldots$ and $r = 0, 1, 2, \ldots$, see [7].

Example 3. The Beta function $B(x, y)$ is defined for $x, y > 0$ by

$$B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} \, dt,$$

but more generally, if

$$B_{r,s}(x, y) = \frac{\partial^{r+s}}{\partial x^r \partial y^s} B(x, y),$$

we have

$$B_{r,s}(x, y) = \operatorname{N-lim}_{n \to \infty} \int_{1/n}^{1-1/n} t^{x-1} \ln^r t (1 - t)^{y-1} \ln^s (1 - t) \, dt.$$
for \( x, y \neq 0, -1, -2, \ldots \) and \( r, s = 0, 1, 2, \ldots \), see [8].

**Example 4.** The distribution \( x^\lambda \) is defined

\[
\langle x^\lambda, \phi(x) \rangle = \int_0^\infty x^\lambda \phi(x) \, dx
\]

for \( x > -1 \) and by

\[
\langle x^\lambda, \phi(x) \rangle = \int_0^\infty x^\lambda \left[ \phi(x) - \sum_{i=0}^{m-1} \frac{x^i}{i!} \phi^{(i)}(0) \right] \, dx
\]

for \( -m - 1 < \lambda < -m \) and arbitrary \( \phi \) in \( D \), but more generally,

\[
\langle x^\lambda \ln^r x, \phi(x) \rangle = N - \lim_{n \to \infty} \int_{1/n}^\infty x^\lambda \ln^r x \phi(x) \, dx
\]

for \( \lambda \neq -1, -2, \ldots \) and \( r = 0, 1, 2, \ldots \), see [6].

We now let \( \rho(x) \) be any infinitely differentiable function having the following properties:

1. \( \rho(x) = 0 \) for \( |x| \geq 1 \),
2. \( \rho(x) \geq 0 \),
3. \( \rho(x) = \rho(-x) \),
4. \( \int_{-1}^1 \rho(x) \, dx = 1 \).

Putting \( \delta_n(x) = n \rho(nx) \) for \( n = 1, 2, \ldots \), it follows that \( \{\delta_n(x)\} \) is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function \( \delta(x) \).

Now let \( D \) be the space of infinitely differentiable functions with compact support and let \( D' \) be the space of distributions defined on \( D \). Then if \( f \) is an arbitrary distribution in \( D' \), we define

\[
f_n(x) = (f \ast \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle
\]

for \( n = 1, 2, \ldots \). It follows that \( \{f_n(x)\} \) is a regular sequence of infinitely differentiable functions converging to the distribution \( f \).

The following definition for the product of two distributions was given in [3].

**Definition 3.** Let \( f \) and \( g \) be distributions in \( D' \) and let \( f_n = f \ast \delta_n \) and \( g_n = g \ast \delta_n \). We say that the neutrix product \( f \Box g \) of \( f \) and \( g \) exists and is equal to the distribution \( h \) on the interval \( (a, b) \) if

\[
N - \lim_{n \to \infty} \langle f_n g_n, \phi \rangle = \langle h, \phi \rangle
\]
for all functions \( \phi \) in \( \mathcal{D} \) with support contained in the interval \((a, b)\). If
\[
\lim_{n \to \infty} \langle f_n g_n, \phi \rangle = \langle h, \phi \rangle,
\]
we simply say that the product \( f \cdot g \) exists and equals \( h \), see [2].

This definition of the neutrix product is clearly commutative. A non-commutative neutrix product, denoted by \( f \circ g \), was considered in [5].

We now prove the following theorem.

**Theorem 1.** Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \) and suppose that the neutrix products \( f \square g^{(i)} \) exist on the interval \((a, b)\) for \( i = 0, 1, \ldots, r \). Then the neutrix products \( f^{(k)} \square g \) exist on the interval \((a, b)\) and

\[
(1) \quad f^{(k)} \square g = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [f \square g^{(i)}]^{(k-i)}
\]

for \( k = 1, 2, \ldots, r \).

**Proof.** Let \( \phi \) be an arbitrary function in \( \mathcal{D} \) with support contained in the interval \((a, b)\) and suppose that the neutrix products \( f \square g^{(i)} \) exist on the interval \((a, b)\) for \( i = 0, 1, \ldots, r \). Put
\[
f_n = f \ast \delta_n, \quad g_n = g \ast \delta_n.
\]

Then
\[
\langle f \square g, \phi \rangle = N \lim_{n \to \infty} \langle f_n, g_n \phi \rangle,
\]
\[
\langle f \square g', \phi \rangle = N \lim_{n \to \infty} \langle f, g'_n \phi \rangle.
\]

Further
\[
\langle (f \square g)', \phi \rangle = -\langle f \square g, \phi' \rangle = -N \lim_{n \to \infty} \langle f_n, g_n \phi' \rangle
\]
\[
= -N \lim_{n \to \infty} \langle f_n, (g_n \phi)' - g'_n \phi \rangle
\]
\[
= N \lim_{n \to \infty} \langle f'_n, g_n \phi \rangle + N \lim_{n \to \infty} \langle f_n, g'_n \phi \rangle
\]

and so
\[
N \lim_{n \to \infty} \langle f'_n, g_n \phi \rangle = \langle (f \square g)', \phi \rangle - \langle f \square g', \phi \rangle.
\]

This proves that the neutrix product \( f' \square g \) exists and satisfies equation (1) for the case \( k = 1 \). Thus

\[
(2) \quad (f \square g)' = f' \square g + f \square g'.
\]
Now suppose that equation (1) holds for some $k < r$. Then by our assumption, the neutrix product $f^{(k)} \Box g$ exists and using equation (2) we have
\[
[f^{(k)} \Box g]' = f^{(k+1)} \Box g + f^{(k)} \Box g' = f^{(k+1)} \Box g + \sum_{i=0}^{k} \binom{k}{i} (-1)^i [f \Box g^{(i)}]^{(k-i)} \]
\[
= \sum_{i=0}^{k+1} \binom{k+1}{i} (-1)^i [f \Box g^{(i)}]^{(k-i+1)}.
\]
Thus
\[
f^{(k+1)} \Box g = \sum_{i=0}^{k+1} \binom{k+1}{i} (-1)^i [f \Box g^{(i)}]^{(k-i+1)}.
\]
Equation (1) now follows by induction.

The following two theorems hold, see [4] and [12] respectively.

**Theorem 2.** The neutrix product $x^r_+ \Box \delta^{(s)}(x)$ exists and
\[
x^r_+ \Box \delta^{(s)}(x) = \frac{(-1)^r s!}{2(s-r)!} \delta^{(s-r)}(x),
\]
for $r = 0, 1, 2, \ldots$ and $s = r + 1, r + 2, \ldots$.

**Theorem 3.** The neutrix product $x^{-r} \Box \delta^{(s)}(x)$ exists and
\[
x^{-r} \Box \delta^{(s)}(x) = c_{rs} \delta^{(r+s)}(x),
\]
where
\[
c_{rs} = \frac{(-1)^{s-1}}{(r-1)!(r+s)!} \int_{-1}^{1} v^{r+s} \rho^{(s)}(v) \int_{-1}^{1} \ln |v-u| \rho^{(r)}(u) \, du \, dv,
\]
for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$ In particular
\[
x^{-r} \delta^{(r-1)}(x) = \frac{(-1)^r r!}{(2r)!} \delta^{(2r-1)}(x),
\]
for $r = 1, 2, \ldots$. Further,
\[
\frac{(-1)^s}{(s-1)!} x^{-r} \Box \delta^{(s-1)}(x) + \frac{(-1)^r}{(r-1)!} x^{-s} \Box \delta^{(r-1)}(x) = \frac{(-1)^{r+s}}{(r+s-1)!} \delta^{(r+s-1)}(x),
\]
for \( r, s = 1, 2, \ldots \).

Note that in the following, the distributions \( x_+^{-r} \) and \( x_-^{-r} \) are defined by

\[
x_+^{-r} = \frac{(-1)^{r-1}}{(r-1)!} (\ln x_+)^{(r)}, \quad x_-^{-r} = -\frac{1}{(r-1)!} (\ln x_-)^{(r)}
\]

for \( r = 1, 2, \ldots \) and not as in Gel’fand and Shilov [9].

The neutrix product \( x_+^{-r} \delta^{(r-1)}(x) \) was considered in [3] where it was proved that

\[
x_+^{-r} \delta^{(r-1)}(x) = \frac{(-1)^r r!}{2(2r)!} \delta^{(2r-1)}(x)
\]

for \( r = 1, 2, \ldots \).

We now prove the following generalization of this result.

**Theorem 4.** The neutrix products \( \ln x_+ \delta^{(s)}(x) \), \( \ln x_- \delta^{(s)}(x) \), \( \ln |x| \delta^{(s)}(x) \), \( x_+^{-r} \delta^{(s)}(x) \) and \( x_-^{-r} \delta^{(s)}(x) \) exist and

\[
\ln x_+ \delta^{(s)}(x) = b_s \delta^{(s)}(x) = (-1)^s \ln x_- \delta^{(s)}(x) = \frac{1}{2} \ln |x| \delta^{(s)}(x),
\]

where

\[
b_s = \frac{1}{s!} \int_{-1}^{1} v^s \rho^{(s)}(v) \int_{-1}^{v} \ln(v-u) \rho(u) \, du \, dv
\]

for \( s = 0, 1, 2, \ldots \) and

\[
x_+^{-r} \delta^{(s)}(x) = \frac{1}{2} c_{rs} \delta^{(r+s)}(x) = (-1)^r x_-^{-r} \delta^{(s)}(x)
\]

for \( r = 1, 2, \ldots \) and \( s = 0, 1, 2, \ldots \). In particular

\[
x_+^{-r} \delta^{(r-1)}(x) = \frac{(-1)^r r!}{2(2r)!} \delta^{(2r-1)}(x),
\]

for \( r = 1, 2, \ldots \). Further,

\[
\frac{(-1)^s}{(s-1)!} x_+^{-r} \delta^{(s-1)}(x) + \frac{(-1)^r}{(r-1)!} x_-^{-s} \delta^{(r-1)}(x) = \frac{(-1)^{r+s}}{(r + s - 1)!} \delta^{(r+s-1)}(x),
\]

for \( r, s = 1, 2, \ldots \).
Proof. We put

\[(\ln x_+)_n = \ln x_+ \ast \delta_n(x) = \int_{-1/n}^x \ln(x-t)\delta_n(t) \, dt\]

on the interval \((-1/n, 1/n)\). Then

\[
\int_{-1/n}^{1/n} (\ln x_+)_n \delta_n^{(s)}(x) x^i \, dx = \int_{-1/n}^{1/n} x^i \delta(x) \int_{-1/n}^x \ln(x-t)\delta_n(t) \, dt \, dx
\]

\[
= n^{s-i} \int_{-1}^1 v^i \rho(s)(v) \int_{-1}^v \ln(v-u)\rho(u) \, du \, dv - n^{s-i} \ln n \int_{-1}^1 v^i \rho(s)(v) \int_{-1}^v \rho(u) \, du \, dv,
\]

on making the substitutions \(nt = u\) and \(nx = v\), for \(i = 0, 1, 2, \ldots\)

It follows that

\[
(14) \quad \lim_{n \to \infty} \int_{-1/n}^{1/n} (\ln x_+)_n \delta_n^{(s)}(x) x^i \, dx = 0,
\]

for \(i = 0, 1, 2, \ldots, s-1\) and

\[
(15) \quad \lim_{n \to \infty} \int_{-1/n}^{1/n} (\ln x_+)_n \delta_n^{(s)}(x) x^s \, dx = \int_{-1}^1 v^s \rho(s)(v) \int_{-1}^v \ln(v-u)\rho(u) \, du \, dv = s! b_s,
\]

\[
(16) \quad \lim_{n \to \infty} \int_{-1/n}^{1/n} (\ln x_+)_n \delta_n^{(s)}(x) x^{s+1} \, dx = 0.
\]

Now let \(\phi\) be an arbitrary function in \(D\). Then

\[
\phi(x) = \sum_{i=0}^s \frac{\phi^{(i)}(0)}{i!} x^i + \frac{\phi^{(s+1)}(\xi x)}{(s+1)!} x^{s+1},
\]

where \(0 < \xi < 1\). Using equations (14), (15) and (16), it follows that

\[
\lim_{n \to \infty} (\ln x_+)_n \delta_n^{(s)}(x) \phi(x) = b_s \phi^{(s)}(0) = b_s \delta^{(s)}(x),
\]

proving equation (7) for \(s = 0, 1, 2, \ldots\).

Equation (8) follows on replacing \(x\) by \(-x\) in equation (7) and equation (9) then follows on noting that \(\ln |x| = \ln x_+ + \ln x_-\).

Theorem 1 now shows us that the neutrix product \(x_+^{r} \square \delta^{(s)}(x)\) exists and

\[
x_+^{r} \square \delta^{(s)}(x) = \sum_{i=0}^r \binom{r}{i} \frac{(-1)^{r+i-1}}{(r-1)!} b_{s+i} \delta^{(r+s)}(x)
\]

\[
= (-1)^r x_+^{-r} \square \delta^{(s)}(x)
\]
on replacing $x$ by $-x$. From equation (4) we have

$$x^{-r} \square \delta^{(s)}(x) = x_{-}^{-r} \square \delta^{(s)}(x) + (-1)^r x_{-}^{-r} \square \delta^{(s)}(x) = c_{rs} \delta^{(r+s)}(x).$$

Equations (10), (11), (12) and (13) now follow and further we have

$$c_{rs} = 2 \sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r+i-1}}{(r-1)!} b_{s+i}$$

for $r = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$ In particular

$$\sum_{i=0}^{r} \binom{r}{i} \frac{(-1)^{r+i-1}}{(r-1)!} b_{r+i-1} = \frac{(-1)^{r} r!}{2(2r)!},$$

for $r = 1, 2, \ldots$, since

$$c_{r,r-1} = \frac{(-1)^{r} r!}{2(2r)!}.$$

Thus each $b_{2s+1}$ can be solved as a linear sum of $b_0, b_2, \ldots, b_{2s}$ and so each $c_{rs}$ is a linear sum of $b_0, b_2, \ldots, b_{2s}, \ldots$

**Theorem 5.** The neutrix products $x_{-}^{-r} \square x_{-}^{s}$ and $x_{-}^{-r} \square x_{+}^{s}$ exist and

$$x_{-}^{-r} \square x_{-}^{s} = \sum_{i=s+1}^{r} \frac{(-1)^{r-s+i} s!}{(r-1)!} b_{i-s-1} \delta^{(r-s-1)}(x)$$

$$= (-1)^{r-s-1} x_{-}^{-r} \square x_{+}^{s},$$

for $r = 1, 2, \ldots$ and $s = 0, 1, \ldots, r-1$.

**Proof.** The product of $\ln x_{+}$ and $x_{-}^{s}$ is a straightforward product of locally summable functions, see [2], and

$$\ln x_{+} \cdot x_{-}^{s} = 0$$

for $s = 0, 1, 2, \ldots$ Putting $g(x) = x_{-}^{s}$, we have

$$g^{(i)}(x) = \begin{cases} 
    \frac{(-1)^{s} s!}{(s-i)!} x_{-}^{s-i}, & 0 \leq i \leq s, \\
    (-1)^{s+1} s! \delta^{(i-s-1)}(x), & i > s.
\end{cases}$$

Thus, by equation (19) we have

$$\ln x_{+} \cdot g^{(i)}(x) = 0.$$
for $i = 0, 1, \ldots, s$ and by equation (7) we have
\[
\ln x^+ \square g^{(i)}(x) = (-1)^{s+1}s!b_{i-s-1}\delta^{(i-s-1)}(x)
\]
for $i = s + 1, s + 2, \ldots$. It now follows from equation (1) that
\[
(ln x^+)^{(r)} \square g(x) = (-1)^{r-1}(r-1)!x^+ \square x^s
\]
\[
= \sum_{i=0}^{r} \binom{r}{i} (-1)^i [\ln x^+ \square g^{(i)}(x)]^{(r-i)}
\]
\[
= \sum_{i=s+1}^{r} \binom{r}{i} (-1)^{s+i-1}s!b_{i-s-1}\delta^{(r-s-1)}(x).
\]
Equation (17) follows immediately and equation (18) follows on replacing $x$ by $-x$.

**Theorem 6.** The neutrix products $x^+ \square x^+, x^- \square x^-, x^r \square x^s$ and $x^{-r} \square x^s$ exist and
\[
x^+ \square x^+ = x^+ - (-1)^{r+s}\frac{\psi(r-s-1) + \psi(r-1)}{(r-s-1)!}\delta^{(r-s-1)}(x) + \sum_{i=s+1}^{r} \frac{(-1)^{r+i}s!b_{i-s-1}\delta^{(r-s-1)}(x)}{(r-1)!}
\]
\[
x^- \square x^- = x^- + \frac{\psi(r-s-1) + \psi(r-1)}{(r-s-1)!}\delta^{(r-s-1)}(x) + \sum_{i=s+1}^{r} \frac{(-1)^{s+i}s!b_{i-s-1}\delta^{(r-s-1)}(x)}{(r-1)!}
\]
\[
x^r \square x^s = x^r - (-1)^{r+s}\frac{\psi(r-s-1) + \psi(r-1)}{(r-s-1)!}\delta^{(r-s-1)}(x) + \sum_{i=s+1}^{r} \frac{2(-1)^{r+i}s!b_{i-s-1}\delta^{(r-s-1)}(x)}{(r-1)!}
\]
\[
x^{-r} \square x^s = (-1)^r x^- + (-1)^r\frac{\psi(r-s-1) + \psi(r-1)}{(r-s-1)!}\delta^{(r-s-1)}(x) + \sum_{i=s+1}^{r} \frac{2(-1)^{r+s+i}s!b_{i-s-1}\delta^{(r-s-1)}(x)}{(r-1)!}
\]
for $r = 1, 2, \ldots$ and $s = 0, 1, \ldots, r - 1$.

**Proof.** The product of $x^+_r$ and the infinitely differentiable function $x^s$ is given by
\[
x^+_r x^s = x^+ - (-1)^{r+s}\frac{\psi(r-s-1) + \psi(r-1)}{(r-s-1)!}\delta^{(r-s-1)}(x),
\]
for \( r = 1, 2, \ldots \) and \( s = 0, 1, \ldots, r - 1 \), see [9]. It follows that
\[
x_+^{-r} \, \square x_+^s = x_+^{-r} \cdot x^s - (-1)^s x_+^{-r} \, \square x_-
\]
\[
= x_+^{-r+s} - (-1)^{r+s} \frac{\psi(r - s - 1) + \psi(r - 1)}{(r - s - 1)!} \delta(r - s - 1)(x) +
\]
\[
- \sum_{i=s+1}^{r} \frac{(-1)^{r+i}s!}{(r - 1)!} b_{i-s-1} \delta(r - s - 1)(x)
\]
proving equation (20).

Equation (21) now follows from equation (20) on replacing \( x \) by \( -x \). Equation (22) follows from equations (18) and (20) and then equation (23) follows from equation (22) on replacing \( x \) by \( -x \).

**Theorem 7.** The neutrix products \((x_+^r \ln x_+ \, \square \delta(s)(x))\), \((x_-^r \ln x_- \, \square \delta(s)(x))\) and \((x^r \ln |x|) \, \square \delta(s)(x)\) exist and

\[
(x_+^r \ln x_+) \, \square \delta(s)(x) = \binom{s}{r} (-1)^r r! b_0 \delta(s-r)(x) + \sum_{i=r+1}^{s} \binom{s}{i} \frac{1}{2} (-1)^r (i - r - 1)! c_{i-r,0} \delta(s-r)(x),
\]

\[
(x_-^r \ln x_-) \, \square \delta(s)(x) = \binom{s}{r} r! b_0 \delta(s-r)(x) + \sum_{i=r+1}^{s} \frac{1}{2} r! c_{i-r,0} \delta(s-r)(x),
\]

\[
(x^r \ln |x|) \, \square \delta(s-r)(x) = \binom{s}{r} (-1)^r r! b_0 \delta(s-r)(x) + \sum_{i=r+1}^{s} (-1)^r (i - r - 1)! c_{i-r,0} \delta(s-r)(x),
\]

for \( r = 0, 1, 2, \ldots \) and \( s = r, r + 1, \ldots \).

**Proof.** We define the function \( f(x_+, r) \) by
\[
f(x_+, r) = \frac{x_+^r \ln x_+ - \psi(r) x_+^r}{r!}
\]
and it follows easily by induction that
\[
f^{(i)}(x_+, r) = f(x_+, r - i),
\]
for \( i = 0, 1, \ldots, r \). In particular,
\[
f^{(r)}(x_+, r) = \ln x_+,
\]
so that
\[ f^{(i)}(x_+, r) = (-1)^{i-r-1}(i-r-1)!x_+^{-i+r}, \]
for \( i = r + 1, r + 2, \ldots \). Now \( f^{(i)}(x_+, r) \) is a continuous function which is zero at the origin for \( i = 0, 1, \ldots, r - 1 \) and so
\[ f^{(i)}(x_+, r).\delta(x) = 0, \]
for \( r = 0, 1, \ldots, r - 1 \). Using equation (7) we have
\[ f^{(r)}(x_+, r)\square\delta(x) = b_0\delta(x) \]
and using equation (10) we have
\[ f^{(i)}(x_+, r)\square\delta(x) = -\frac{1}{2}(-1)^{i-r-1}(i-r-1)!c_{i-r,0}\delta^{(i-r)}(x) \]
for \( i = r + 1, r + 2, \ldots \)

Using equations (1), (27), (28) and (29) we have
\[
\begin{align*}
f((x_+, r)\square\delta(x)) &= \sum_{i=0}^{s} \binom{s}{i} (-1)^i [f^{(i)}(x_+, r)\square\delta(x)]^{(s-i)} \\
&= \sum_{i=r}^{s} \binom{s}{i} (-1)^i [f^{(i)}(x_+, r)\square\delta(x)]^{(s-i)} \\
&= \binom{s}{r} (-1)^r b_0\delta^{(s-r)}(x) + \\
&\quad + \sum_{i=r+1}^{s} \binom{s}{i} \frac{1}{2}(-1)^r (i-r-1)!c_{i-r,0}\delta^{(r-s)}(x).
\end{align*}
\]
Thus
\[
(x_+^r \ln x_+)}\square\delta^{(s)}(x) = r!f(x_+, r)\square\delta^{(s)}(x) + \psi(r)x_+^{r}\square\delta^{(s)}(x)
\]
and equation (24) follows on using equation (3).

Equation (25) now follows from equation (24) on replacing \( x \) by \(-x\) and equation (26) follows on noting that
\[
x^r \ln |x| = x_+^r \ln x_+ + (-1)^r x_-^r \ln x_-.
\]

For further related results, see Gramchev [10], and for a survey of recent results and theories in the product of distributions, see Oberguggenberger [11].
REFERENCES


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