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INTEGRAL MANIFOLDS AND PERTURBATIONS OF THE NONLINEAR PART OF SYSTEMS OF AUTONOMOUS DIFFERENTIAL EQUATIONS WITH IMPULSES AT FIXED MOMENTS*

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Abstract. Sufficient conditions are obtained for the existence of local integral manifolds of autonomous systems of differential equations with impulses at fixed moments. In case of perturbations of the nonlinear part an estimate of the difference between the manifolds is obtained.

1. Introduction. The impulsive differential equations are adequate mathematical models of evolutionary processes which are subjected to short-time effects during their evolution. They are successfully used in science and technology [1], [4].

In spite of the great possibilities for applications, the theory of these equations is developed rather slowly. This is due to their specific properties such as "beating", merging of the solutions, loss of the property of autonomy, etc. We shall note that the theory of the impulsive differential equations is considerably richer than the theory of the ordinary differential equations.

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The intensive investigations in the field of the integral manifolds are brought forth by the necessity of studying these processes in their entirety. In many cases the standard conditions for existence of integral manifolds are satisfied only locally, i.e. only for on bounded domains.

In the present paper sufficient conditions are obtained for existence of local integral manifolds of autonomous systems of differential equations with impulses at fixed moments, and in case of perturbations of the nonlinear part an estimate of the difference between the respective manifolds is obtained.

2. Preliminary notes. Statement of the problem. Let us assume that the sequence of numbers $\{\tau_k\}$, $\tau_k \in \mathbb{R}$, $\tau_k < \tau_{k+1}$ $(k \in \mathbb{Z})$ has no finite accumulation point.

Consider the autonomous system of differential equations with impulses at fixed moments

$$
\begin{aligned}
\frac{dz}{dt} &= \mathbf{A}z + F(z), \quad t \neq \tau_k \\
\Delta z &= \mathbf{B}z + \Phi_k(z), \quad t = \tau_k, \ k \in \mathbb{Z}
\end{aligned}
\tag{1}
$$

where $z \in \mathbb{R}^{m+n}$, $F : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$, $\Phi_k : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$, where $\mathbb{R}^{m+n}$ is an $m+n$ – dimensional Euclidean space with norm $\|\cdot\|$, $\mathbf{A}$ and $\mathbf{B}$ are constant matrices, $\mathbf{A} = \mathrm{diag}\,[A_1, A_2]$, $\mathbf{B} = \mathrm{diag}\,[B_1, B_2]$, where A_1, B_1 are $m \times m$ – dimensional, and A_2, B_2 are $n \times n$ – dimensional, $\Delta z = z(\tau_k + 0) - z(\tau_k - 0)$.

Definition 1. *The function $z(t) : \mathbb{R} \to \mathbb{R}^{m+n}$ is said to be a solution of* (1) *if it satisfies the following conditions:*

1) *$z(t)$ is a piecewise continuous function with points of discontinuity of the first kind τ_k $(k \in \mathbb{Z})$ at which it is continuous on the left, i.e.*

$$z(\tau_k - 0) = z(\tau_k), \ z(\tau_k + 0) = z(\tau_k) + \Phi_k(z(\tau_k)) + \mathbf{B}z(\tau_k);$$

2) *$z(t)$ is differentiable for $t \neq \tau_k$ and*

$$\frac{dz}{dt} = \mathbf{A}z(t) + F(z(t)).$$

Denote by $\mathbf{U}_r$ the closed ball in $\mathbb{R}^m$ with a center at the origin and radius r, i.e.

$$\mathbf{U}_r = \{x : x \in \mathbb{R}^m, \|x\| \leq r, \ r > 0\}.$$

Denote by $\mathbf{V}_\rho$ the closed ball in $\mathbb{R}^n$ with a center at the origin and radius ρ, i.e.

$$\mathbf{V}_\rho = \{x : x \in \mathbb{R}^n, \|x\| \le \rho,\ \rho > 0\}.$$

Consider the Banach space $\mathbf{E}$ of all functions $\widehat{\varphi}$ mapping the set $\mathbb{R}^m$ into $\mathbb{R}^n$, which are continuous with respect to x and are bounded, with norm

$$|\widehat{\varphi}| = \sup\{\|\varphi(x)\| : x \in \mathbb{R}^m\}.$$

By $\mathbf{L}(\rho, \eta)$ we denote the subset of $\mathbf{E}$ consisting of all functions for which

$$\|\widehat{\varphi}(x)\| \le \rho, \quad \|\widehat{\varphi}(x) - \widehat{\varphi}(\overline{x})\| \le \eta\|x - \overline{x}\|; \quad x, \overline{x} \in \mathbb{R}^m,$$

where $\eta =$ const > 0, $\rho =$ const > 0.

It is easily checked that $\mathbf{L}(\rho, \eta)$ is closed in $\mathbf{E}$.

We write down system (1) in the form

$$\begin{aligned} &\frac{dx}{dt} = A_1 x + f(x, y) && \frac{dy}{dt} = A_2 y + g(x, y), \quad t \ne \tau_k \\ &\Delta x = B_1 x + I_k(x, y) && \Delta y = B_2 y + J_k(x, y), \quad t \ne \tau_k,\ k \in \mathbb{Z} \end{aligned} \tag{2}$$

where $z = (x, y)$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$. The functions f and I_k map $\mathbb{R}^{m+n}$ into $\mathbb{R}^m$ and g and J_k map $\mathbb{R}^{m+n}$ into $\mathbb{R}^n$.

Definition 2. *An arbitrary manifold J in the extended phase space of system (1) is said to be an* **integral manifold** *if for an arbitrary solution $z = (x(t), y(t))$ of (1) for which $(t_0, z(t_0)) \in J$ it follows that $(t, z(t)) \in J$ for $t \ge t_0$.*

Definition 3. *The integral manifold J,*

$$J = \{(x, y) : y = \widehat{\varphi}(x), \quad x \in \mathbf{U}_r, \quad \widehat{\varphi} \in \mathbf{L}(\rho, \eta)\}$$

is said to be a **local integral manifold of class $\mathbf{L}(\rho, \eta)$** *or a local (ρ, η)-manifold.*

In the present paper sufficient conditions are obtained for existence of integral manifolds of system (1) for any $\rho > 0$ and $\eta > 0$.

We shall say that conditions (H) are satisfied if the following conditions hold:

H1. There exist constants p and ε such that uniformly on t and $s \in \mathbb{R}$ the following inequality is valid

$$i(s,t) \leq p(t-s) + \varepsilon,$$

where by $i(s,t)$ we have denoted the number of the points τ_k in the interval $[s,t]$.

H2. $\det(E_m + B_1) \neq 0$, the matrices A_1 and B_1 commute and there exists a constant $\alpha > 0$ such that the eigenvalues λ_i $(i = 1,\dots,m)$ of the matrix $\Lambda_1 = A_1 + \mathrm{pln}\,(E_m + B_1)$ satisfy the inequalities

$$|\mathrm{Re}\lambda_i| < \alpha, \quad (i = 1,\dots,m),$$

where E_m is a unit $m \times m$ matrix.

H3. $\det(E_n + B_2) \neq 0$, the matrices A_2 and B_2 commute and there exists a constant $\gamma > 0$ such that the eigenvalues λ_j $(j = 1,\dots,n)$ of the matrix $\Lambda_2 = A_2 + \mathrm{pln}\,(E_n + B_2)$ satisfy the inequalities

$$|\mathrm{Re}\lambda_j| > \gamma, \quad (j = 1,\dots,n),$$

where E_n is a unit $n \times n$ matrix.

H4. The functions f and I_k are continuous, bounded and Lipschitz continuous, i.e. there exist constants Q, l_1 and l_2 such that

$$\sup_{(x,y)\in \mathbf{U}_r \times \mathbf{V}_\rho} \|f(x,y)\| + \sup_{(x,y)\in \mathbf{U}_r \times \mathbf{V}_\rho,\ k\in\mathbb{Z}} \|I_k(x,y)\| \leq Q,$$

$$\|f(\overline{x},\overline{y}) - f(x,y)\| + \|I_k(\overline{x},\overline{y}) - I_k(x,y)\| \leq l_1\|\overline{x} - x\| + l_2\|\overline{y} - y\|,$$

where $\overline{x}, x \in \mathbf{U}_r$, $\overline{y}, y \in \mathbf{V}_\rho$.

H5. The functions g and J_k are continuous, bounded and Lipschitz continuous, i.e. there exist constants Q, δ_1 and δ_2 such that

$$\sup_{(x,y)\in \mathbf{U}_r \times \mathbf{V}_\rho} \|g(x,y)\| + \sup_{(x,y)\in \mathbf{U}_r \times \mathbf{V}_\rho,\ k\in\mathbb{Z}} \|J_k(x,y)\| \leq Q,$$

$$\|g(\overline{x},\overline{y}) - g(x,y)\| + \|J_k(\overline{x},\overline{y}) - J_k(x,y)\| \leq \delta_1\|\overline{x} - x\| + \delta_2\|\overline{y} - y\|$$

where $\overline{x}, x \in \mathbf{U}_r$, $\overline{y}, y \in V_\rho$.

Henceforth we shall use the following lemmas.

Lemma 1 [2]. *Let conditions* H1 *and* H2 *be satisfied. Then the Cauchy matrix*

$$W(t,s) = e^{\Lambda_1(t-s)}(E_m + B_1)^{-p(t-s)+i(s,t)} \quad (t \geq s)$$

of the linear system

$$\frac{dx}{dt} = A_1 x, \quad t \neq \tau_k,$$

$$\Delta x = B_1 x, \quad t = \tau_k, \ k \in \mathbb{Z}$$

satisfies the inequality

$$\|W(t,s)\| \leq N e^{\alpha_1(t-s)}, \qquad t, s \in \mathbb{R},$$

where $\alpha_1 = \alpha_1(\varepsilon) > 0$, $N = N(\varepsilon) > 0$.

For the system

$$\frac{dy}{dt} = A_2 y, \quad t \neq \tau_k,$$

$$\Delta y = B_2 y, \quad t = \tau_k, \ k \in \mathbb{Z}$$

we construct Green's function

$$G(t,s) = \begin{cases} -(E_n + B_2)^{-p(t-s)+i(t,s)}\mathbf{S}^{-1}\mathrm{diag}\,(e^{\Lambda_2^+(t-s)}, 0)\mathbf{S}, & t < s, \\ \\ -(E_n + B_2)^{-p(t-s)+i(t,s)}\mathbf{S}^{-1}\mathrm{diag}\,(0, e^{\Lambda_2^-(t-s)}, 0)\mathbf{S}, & t > s, \end{cases}$$

where $\mathbf{S}$ is a nonsingular matrix such that

$$\Lambda_2 = \mathbf{S}^{-1}\mathrm{diag}\,(\Lambda_2^+, \Lambda_2^-)\mathbf{S}.$$

Λ_2^+ is a square matrix with positive real parts of its eigenvalues and Λ_2^- is a square matrix with negative real parts of its eigenvalues.

Lemma 2 [5]. *Let conditions* H1 *and* H3 *be satisfied. Then for Green's function* $G(t,s)$ *there exist constants* $K > 0$ *and* $\Delta_1 = \Delta_1(\gamma) > 0$ *such that*

$$\|G(t,s)\| \leq K e^{-\Delta_1 |t-s|}, \qquad t, s \in \mathbb{R}.$$

Lemma 3 [2]. *Let the inequality*

$$\mathbf{U}(t) \le \int_{t_0}^{t} \mathbf{U}(s)\nu(s)ds + F(t) + \sum_{t_0<\tau_k<t} \beta_k \mathbf{U}(\tau_k) + \sum_{t_0<\tau_k<t} \gamma_k(t)$$

be satisfied, where the function $\mathbf{U}(t)$ *is piecewise continuous with discontinuities of the first kind at the points* τ_k, $\nu(t) \ge 0$ *is a locally integrable function,* $F(t)$ *and* $\gamma_k(t)$ *are non decreasing functions for* $t \in [t_0, \infty]$ *and* $\gamma_k(t) \ge 0$, $\beta_k \ge 0$.

Then

$$\mathbf{U}(t) \le \left(F(t) + \sum_{t_0<\tau_k<t} \gamma_k(t) \right) \prod_{t_0<\tau_k<t} (1+\beta_k) exp \left(\int_{t_0}^{t} \nu(s)ds \right).$$

3. Main Results. Consider the system

$$\begin{aligned} &\frac{dx}{dt} = A_1 x + \widehat{f}(x,y) \qquad && \frac{dy}{dt} = A_2 y + \widehat{g}(x,y), \quad t \ne \tau_k \\ &\Delta x = B_1 x + \widehat{I}_k(x,y) && \Delta y = B_2 y + \widehat{J}_k(x,y), \quad t = \tau_k,\ k \in \mathbb{Z} \end{aligned} \tag{3}$$

where

$$\widehat{f}(x,y) = \begin{cases} f(x,y), & x \in \mathbf{U}_r,\ y \in \mathbf{V}_\rho, \\ f\left(\frac{x}{\|x\|} r, y \right), & x \in \mathbb{R}^m \setminus \mathbf{U}_r,\ y \in \mathbf{V}_\rho \end{cases}$$

and the functions $\widehat{g}(x,y)$ and $\widehat{I}_k(x,y)$, $\widehat{J}_k(x,y)$, $k \in \mathbb{Z}$ are defined analogously.

Let $\widehat{\varphi}_j \in \mathbf{L}(\rho,\eta)$, $j = 1,2$. We denote by $\psi_j(t) = \psi_j(t,s,\omega_j|\widehat{\varphi}_j)$, $\omega_j \in \mathbb{R}^m$, $j = 1,2$ the solution of the Cauchy problem for the system

$$\begin{aligned} \frac{d\psi}{dt} &= A_1 \psi + \widehat{f}(\psi, \varphi(\psi)), \quad t \ne \tau_k \\ \Delta \psi &= B_1 \psi + \widehat{I}_k(\psi, \varphi(\psi)), \quad t = \tau_k,\ \psi_j(s) = \omega_j,\ k \in \mathbb{Z}. \end{aligned} \tag{4}$$

Lemma 4. *Let conditions* H1, H2 *and* H4 *be satisfied. Then the following inequality holds*

$$\begin{aligned}\|\psi_1-\psi_2\| \leq & \left\{ Nl_2 \left[\frac{1}{\alpha}\left(e^{\alpha_1|t-s|}-1\right) + \sum_{s\leq\tau_k<t} e^{\alpha_1|t-\tau_k|} \right] \; |\widehat{\varphi}_1-\widehat{\varphi}_2| + \right. \\ & \left. + Ne^{\alpha_1|t-s|}\; \|\omega_1-\omega_2\| \right\} [1+N(l_1+l_2\eta)]^{i(s,t)}\, e^{N(l_1+l_2\eta)|t-s|}.\end{aligned} \tag{5}$$

P r o o f. From [3] it follows that $\widehat{f}$ and $\widehat{I}_k$ satisfy conditions H4 with the same constants. Then, from [1] and (4) it follows that

$$\begin{aligned}\psi_j = \;& W(t,s)\omega_j + \int_s^t W(t,r)\widehat{f}\left(\psi_j(r), \widehat{\varphi}_j\left(\psi_j(r)\right)\right) dr + \\ & \sum_{s\leq\tau_k<t} W(t,\tau_k)\widehat{I}_k\left(\psi_j(\tau_k), \widehat{\varphi}_j\left(\psi_j(\tau_k)\right)\right).\end{aligned}$$

Hence, for $t \geq s$

$$\begin{aligned}\|\psi_1-\psi_2\| \;\leq\;& \|W(t,s)\|\,\|\omega_1-\omega_2\| + \\ & + \int_s^t W(t,r)\left\|\widehat{f}\left(\psi_1(r), \widehat{\varphi}_1\left(\psi_2(r)\right)\right) - \widehat{f}\left(\psi_2(r), \widehat{\varphi}_2\left(\psi_2(r)\right)\right)\right\| ds + \\ & + \sum_{s\leq\tau_k<t} W(t,\tau_k)\left\|\widehat{I}_k\left(\psi_1(\tau_k), \widehat{\varphi}_1\left(\psi_2(\tau_k)\right)\right) - \widehat{I}_k\left(\psi_2(\tau_k), \widehat{\varphi}_2\left(\psi_2(\tau_k)\right)\right)\right\| \leq \\ \leq\;& Ne^{\alpha_1(t-s)}\|\omega_1-\omega_2\| + \int_s^t Ne^{\alpha_1(t-r)}(l_1+l_2\eta)\|\psi_1-\psi_2\|dr + \\ & + \int_s^t Ne^{\alpha_1(t-r)} l_2|\varphi_1-\varphi_2|dr + \\ & + \sum_{s\leq\tau_k<t} Ne^{\alpha_1(t-\tau_k)}(l_1+l_2\eta)\|\psi_1-\psi_2\| + \\ & + \sum_{s\leq\tau_k<t} Ne^{\alpha_1(t-\tau_k)} l_2|\varphi_1-\varphi_2|.\end{aligned}$$

Set

$$\begin{aligned}
\mathbf{U}(t) &= e^{-\alpha_1 t}\|\psi_1-\psi_2\|,\\
F(t) &= Nl_2\int_s^t e^{-\alpha_1 r}|\varphi_1-\varphi_2|dr + Ne^{-\alpha_1 t}\|\omega_1-\omega_2\|,\\
\gamma_k(t) &= Nl_2 e^{-\alpha_1\tau_k}|\varphi_1-\varphi_2|,\\
\beta_k &= N(l_1+l_2\eta),\\
\mathbf{V}(t) &= N(l_1+l_2\eta).
\end{aligned}$$

Then, from Lemma 3, it follows that

$$\begin{aligned}
\|\psi_1-\psi_2\| \le{} & \Bigg\{Nl_2\left[\frac{1}{\alpha_1}\left(e^{\alpha_1(t-s)}-1\right)+\sum_{s\le\tau_k<t} e^{\alpha_1(t-\tau_k)}\right]|\widehat{\varphi}_1-\widehat{\varphi}_2| + \\
& +Ne^{\alpha_1(t-s)}\|\omega_1-\omega_2\|\Bigg\}\left[1+N(l_1+l_2\eta)\right]^{i(s,t)}e^{N(l_1+l_2\eta)(t-s)}.
\end{aligned}$$

The proof for $t<s$ is analogous. $\square$

Theorem 1. *Let the following conditions be satisfied:*

1. *Conditions* (H) *hold.*

2. *The constants defined in conditions* (H), *Lemma* 1 *and Lemma* 2 *are such that the following inequalities are valid:*

$$\beta = \Delta_1-\alpha_1-N(l_1+l_2\eta)-p\ln[1+N(l_1+l_2\eta)]>0, \tag{6}$$

$$2KN(\delta_1+\delta_2\eta)e^{\varepsilon\ln[1+N(l_1+l_2\eta)]}\left(\frac{1}{\beta}-\frac{p+\varepsilon}{1-e^{-\beta}}\right)\le\eta, \tag{7}$$

$$\begin{aligned}
& 2KNl_1(\delta_1+\delta_2\eta)e^{\varepsilon\ln[1+N(l_1+l_2\eta)]}\left[\frac{1}{\alpha_1}\left(\frac{1}{\beta}-\frac{1}{\beta+\alpha_1}\right)+\frac{\varepsilon}{\beta}+\frac{p}{\beta^2}+\right.\\
& \left.+\frac{p+\varepsilon}{\alpha_1}\left(\frac{1}{1-e^{-\beta}}-\frac{1}{1-e^{-(\beta-\alpha_1)}}\right)+\frac{\varepsilon(p+\varepsilon)}{1-e^{-\beta}}+\frac{e^{-\beta}}{(1-e^{-\beta})^2}\right]+
\end{aligned} \tag{8}$$

$$+K\delta_2\left(\frac{1}{\Delta_1}+\frac{p+\varepsilon}{1-e^{-\Delta_1}}\right)<1,$$

$$2KQ\left(\frac{1}{\Delta_1}+\frac{p+\varepsilon}{1-e^{-\Delta_1}}\right)\leq\rho. \tag{9}$$

Then, system (2) *has a local* (ρ,η)*-manifold.*

P r o o f. From [3] it follows that $\widehat{g}(\cdot)$ and $\widehat{J}_k(\cdot)$ satisfy H5 with the same constants. In $L(\rho,\eta)$ we define an operator $\mathbf{T}$ by the formula

$$\begin{aligned}\mathbf{T}\varphi &= \int_{-\infty}^{\infty}G(t,s)\widehat{g}\left(\psi(s),\varphi\left(\psi(s)\right)\right)ds+\\ &+\sum_{-\infty<\tau_k<\infty}G(t,\tau_k)\widehat{J}_k\left(\psi(\tau_k),\varphi\left(\psi(\tau_k)\right)\right).\end{aligned}$$

Then, from Lemma 2 and (9) we have

$$\begin{aligned}\|\mathbf{T}\varphi\| &\leq \int_{-\infty}^{\infty}Ke^{-\Delta_1|t-s|}Qds+\sum_{-\infty<\tau_k<\infty}Ke^{-\Delta_1|t-\tau_k|}Q\leq\\ &\leq 2KQ\left(\frac{1}{\Delta_1}+\frac{p+\varepsilon}{1-e^{-\Delta_1}}\right)\leq\rho.\end{aligned} \tag{10}$$

From (5) and (6) it follows that

$$\|\mathbf{T}\varphi_1(\psi_1)-\mathbf{T}\varphi_2(\psi_2)\|\leq\int_{-\infty}^{\infty}\|G(t,s)\|\,\|\widehat{g}(\psi_1,\varphi_1(\psi_1))-\widehat{g}(\psi_2,\varphi_2(\psi_2))\|ds+$$

$$+\sum_{-\infty<\tau_k<\infty}\|G(t,\tau_k)\|\,\|\widehat{J}_k(\psi_1(\tau_k),\varphi_1(\psi_1(\tau_k)))-\widehat{J}_k(\psi_2(\tau_k),\varphi_2(\psi_2(\tau_k)))\|\leq$$

$$\leq\int_{-\infty}^{\infty}Ke^{-\Delta_1|t-s|}\{\delta_1\|\psi_1(s)-\psi_2(s)\|+\delta_2\|\varphi_1(\psi_1(s))-\varphi_2(\psi_2(s))\|\}ds+$$

$$+\sum_{-\infty<\tau_k<\infty}Ke^{-\Delta_1|t-\tau_k|}\{\delta_1\|\psi_1(\tau_k)-\psi_2(\tau_k)\|+\delta_2\|\varphi_1[\psi_1(\tau_k)]-\varphi_2[\psi_2(\tau_k)]\|\}\leq$$

$$\leq\int_{-\infty}^{\infty}Ke^{-\Delta_1|t-s|}\left[(\delta_1+\delta_2\eta)\|\psi_1(s)-\psi_2(s)\|+\delta_2|\varphi_1-\varphi_2|\right]ds+$$

$$+\sum_{-\infty<\tau_k<\infty}Ke^{-\Delta_1|t-\tau_k|}\left[(\delta_1+\delta_2\eta)\|\psi_1(\tau_k)-\psi_2(\tau_k)\|+\delta_2|\varphi_1-\varphi_2|\right]\leq$$

$$\leq \int_{-\infty}^{t} Ke^{-\Delta_1(t-s)}\Bigg\{(\delta_1+\delta_2\eta)\Bigg[Nl_1\Big(\frac{1}{\alpha_1}(e^{\alpha_1(t-s)}-1)+$$

$$+\sum_{s\leq\tau_k<t} e^{\alpha_1(t-\tau_k)}\Big)\,|\varphi_1-\varphi_2|+Ne^{\alpha_1(t-s)}\|\omega_1-\omega_2\|\Bigg]\times$$

$$\times e^{\varepsilon\ln[1+N(l_1+l_2\eta)]}e^{\{N(l_1+l_2\eta)+p\ln[1+N(l_1+l_2\eta)]\}(t-s)}+$$

$$+\delta_2|\varphi_1-\varphi_2|\Bigg\}ds+\sum_{\tau_k<t} Ke^{-\Delta_1(t-\tau_k)}\Bigg\{(\delta_1+\delta_2\eta)\Bigg[Nl_1\Big(\frac{1}{\alpha_1}(e^{\alpha_1(t-s)}-1)+$$

$$+\sum_{\tau_k\leq\tau_j<t} e^{\alpha_1(t-\tau_j)}\Big)|\varphi_1-\varphi_2|+Ne^{\alpha_1(t-\tau_k)}\|u_1-u_2\|\Bigg]\times$$

$$\times e^{\varepsilon\ln[1+N(l_1+l_2\eta)]}e^{\{N(l_1+l_2\eta)+p\ln[1+N(l_1+l_2\eta)]\}(t-\tau_k)}+\delta_2|\varphi_1-\varphi_2|\Bigg\}+$$

$$+\int_{t}^{\infty} Ke^{\Delta_1(t-s)}\Bigg\{(\delta_1+\delta_2\eta)\Bigg[Nl_1\Big(\frac{1}{\alpha_1}(e^{-\alpha_1(t-s)}-1)+\sum_{t\leq\tau_k<s} e^{-\alpha_1(t-\tau_k)}\Big)\times$$

$$\times|\varphi_1-\varphi_2|+Ne^{-\alpha_1(t-s)}\|\omega_1-\omega_2\|\Bigg]\times \tag{11}$$

$$\times\, e^{\varepsilon\ln[1+N(l_1+l_2\eta)]}e^{\{N(l_1+l_2\eta)+p\ln[1+N(l_1+l_2\eta)]\}(t-s)}+\delta_2|\varphi_1-\varphi_2|\Bigg\}ds+$$

$$\sum_{t<\tau_k} Ke^{\Delta_1(t-\tau_k)}\Bigg\{(\delta_1+\delta_2\eta)\Bigg[Nl_1\Big(\frac{1}{\alpha_1}(e^{-\alpha_1(t-\tau_k)}-1)+$$

$$+\sum_{t\leq\tau_j<\tau_k} e^{-\alpha_1(t-\tau_j)}\Big)\,|\varphi_1-\varphi_2|+Ne^{-\alpha_1(t-\tau_k)}\|\omega_1-\omega_2\|\Bigg]\times$$

$$\times e^{-\varepsilon\ln[1+N(l_1+l_2\eta)]}e^{\{-N(l_1+l_2\eta)+p\ln[1+N(l_1+l_2\eta)]\}(t-s)}+$$

$$+\delta_2|\varphi_1-\varphi_2|\Bigg\}=N^*\|\omega_1-\omega_2\|+K^*|\varphi_1-\varphi_2|,$$

where we denote:

$$N^* = 2KN(\delta_1 + \delta_2\eta)e^{\varepsilon \ln[1+N(l_1+l_2\eta)]}\left(\frac{1}{\beta} + \frac{p+\varepsilon}{1-e^{-\beta}}\right),$$

$$\begin{aligned} K^* &= 2KNl_1(\delta_1 + \delta_2\eta)e^{\varepsilon \ln[1+N(l_1+l_2\eta)]}\left[\frac{1}{\alpha_1}\left(\frac{1}{\beta} - \frac{1}{\beta+\alpha_1}\right) + \frac{\varepsilon}{\beta} + \frac{p}{\beta^2} + \right. \\ &+ \left. \frac{p+\varepsilon}{\alpha_1}\left(\frac{1}{1-e^{-\beta}} - \frac{1}{1-e^{-(\beta+\alpha_1)}}\right) + \frac{\varepsilon(p+\varepsilon)}{1-e^{-\beta}} + \frac{e^{-\beta}}{(1-e^{-\beta})^2}\right] + \\ &+ 2K\delta_2\left(\frac{1}{\Delta_1} + \frac{p+\varepsilon}{1-e^{-\Delta_1}}\right). \end{aligned}$$

From (11) for $\psi_1 = \psi_2$ we obtain that

$$\|\mathbf{T}\varphi_1 - \mathbf{T}\varphi_2\| \le N^*\|\omega_1 - \omega_2\|. \tag{12}$$

Then, from (7), (10) and (12), it follows that $\mathbf{T}$ is a mapping from $\mathbf{L}(\rho,\eta)$ into $\mathbf{L}(\rho,\eta)$.

For $u_1 = u_2$ from (11) we obtain that

$$\|\mathbf{T}\varphi_1 - \mathbf{T}\varphi_2\| \le K^*\|\varphi_1 - \varphi_2\|. \tag{13}$$

From (13) and (8) it follows that $\mathbf{T}$ is a contracting operator. Then the fixed point $\widehat{\varphi}$ of the operator $\mathbf{T}$ provides an integral manifold of (3). Hence the restriction of the function $\widehat{\varphi}$ to $\mathbf{U}_r$ defines a local integral manifold of system (2). □

Further on consider the system

$$\begin{aligned} &\frac{dx}{dt} = A_1x + f(x,y) \qquad &&\frac{dy}{dt} = A_2y + g(x,y) + h(x,y), \qquad &&t \ne \tau_k \\ &\Delta x = B_1x + I_k(x,y) \qquad &&\Delta y = B_2y + J_k(x,y) + h_k(x,y), \quad &&t = \tau_k,\ k \in \mathbb{Z} \end{aligned} \tag{14}$$

where $h : \mathbf{U}_r \times \mathbf{V}_\rho \to \mathbb{R}^n$, $h_k : \mathbf{U}_r \times \mathbf{V}_\rho \to \mathbb{R}^n$, $k \in \mathbb{Z}$.

Definition 4. *We called system* (14) **perturbed** *with respect to system* (2).

Let $\widehat{\Phi}$ be a function defining an integral manifold of the system

$$(15)\qquad \begin{aligned} &\frac{dx}{dt} = A_1x + \widehat{f}(x,y) && \frac{dy}{dt} = A_2y + \widehat{g}(x,y) + \widehat{h}(x,y), && t \neq \tau_k \\ &\Delta x = B_1x + \widehat{I}_k(x,y) && \Delta y = B_2y + \widehat{J}_k(x,y) + \widehat{h}_k(x,y), && t = \tau_k,\ k \in \mathbb{Z} \end{aligned}$$

where

$$\widehat{h}(x,y) = \begin{cases} h(x,y), & x \in \mathbf{U}_r, \quad y \in \mathbf{V}_\rho, \\ h\left(\dfrac{x}{\|x\|}r, y\right), & x \in \mathbb{R}^m \setminus \mathbf{U}_r, \quad y \in \mathbf{V}_\rho. \end{cases}$$

$$\widehat{h}_k(x,y) = \begin{cases} h_k(x,y), & x \in \mathbf{U}_r, \quad y \in \mathbf{V}_\rho, \\ h_k\left(\dfrac{x}{\|x\|}r, y\right), & x \in \mathbb{R}^m \setminus \mathbf{U}_r, \quad y \in \mathbf{V}_\rho. \end{cases}$$

By $\psi(t,s,\omega|\widehat{\Phi})$ we denote the solution of the Cauchy problem for the system

$$\begin{aligned} \frac{d\psi}{dt} &= A_1\psi + \widehat{f}(\psi, \widehat{\Phi}(\psi)),\ t \neq \tau_k, \\ \Delta\psi &= B_1\psi + \widehat{I}_k(\psi, \widehat{\Phi}(\psi)),\ t = \tau_k,\ k \in \mathbb{Z}. \end{aligned}$$

Lemma 5. *Let the following conditions be satisfied:*

1. *Conditions* H1, H2 *and* H4 *are valid.*
2. *System* (15) *has a* (ρ,η)*-manifold with function* $\widehat{\Phi}(x)$.

Then the following inequality holds

$$\begin{aligned} &\|\psi(t,s,\omega|\widehat{\Phi}) - \psi(t,s,\omega|\widehat{\varphi})\| \leq \\ &\leq \left[\frac{Nl_2}{\alpha_1}\left(e^{\alpha_1|t-s|} - 1\right)|\widehat{\Phi} - \widehat{\varphi}| + \sum_{s \leq \tau_k < t} Ne^{\alpha_1|t-\tau_k|} l_2 |\widehat{\Phi} - \widehat{\varphi}|\right] \times \\ &\times [1 + N(l_1 + l_2\eta)]^{i(s,t)}\, e^{N(l_1+l_2\eta)|t-s|} \end{aligned}$$

P r o o f. It is carried out analogously to the proof of Lemma 4. □

Theorem 2. *Let the following conditions be given:*
1. *The conditions of Theorem* 1 *are satisfied.*
2. *The functions h and h_k are bounded with constant Q_1, $k \in \mathbb{Z}$.*
3. *The relation $l_2 \le l_1$ holds.*
4. *System* (14) *has a local (ρ, η)-manifold with function $\widehat{\Phi}(x)$.*

Then the following inequality is valid

$$\|\widehat{\Phi} - \widehat{\varphi}\| \le (1-q)^{-1} 2K \left(\frac{1}{\Delta_1} + \frac{p+\varepsilon}{1-\varepsilon^{-\Delta_1}} \right) \times$$
$$\times \max \left(\sup_{(x,y)\in \mathbf{U}_r \times \mathbf{V}_\rho} \|h(x,y)\|, \sup_{(x,y)\in \mathbf{U}_r \times \mathbf{V}_\rho,\ k\in\mathbb{Z}} \|h_k(x,y)\| \right),$$

where

$$\begin{aligned} q \;=\; & 2K(\delta_1+\delta_2\eta) N l_2 e^{\varepsilon \ln[1+N(l_1+l_2\eta)]} \left[\frac{1}{\alpha_1}\Big(\frac{1}{\beta} - \frac{1}{\beta+\alpha_1}\Big) + \right. \\ & +(p+\varepsilon)\Big(\frac{1}{1-e^{-\beta}} - \frac{1}{1-e^{-(\beta+\alpha_1)}}\Big) + \varepsilon\Big(\frac{1}{\beta} - \frac{p+\varepsilon}{1-e^{-\beta}}\Big) + \\ & \left. +p\Big(\frac{1}{\beta^2} + \frac{e^{-\beta}}{(1-e^{-\beta})^2}\Big)\right] + 2K\delta_2\Big(\frac{1}{\Delta_1} + \frac{p+\varepsilon}{1-e^{-\Delta_1}}\Big). \end{aligned}$$

P r o o f. From [3] it follows that $\widehat{h}$ and $\widehat{h}_k$ satisfy Condition 2 or Theorem 2 with the same constants.

Then

$$\begin{aligned} \|\widehat{\Phi} - \widehat{\varphi}\| \;\le\; & \int_{-\infty}^{\infty} \|G(t,s)\| \left\{ \left\| \widehat{g}\left(\psi(\widehat{\varphi}), \widehat{\varphi}\right) - \widehat{g}\left(\psi(\widehat{\Phi}), \widehat{\Phi}\right) \right\| + \left\| \widehat{h}\left(\psi(\widehat{\Phi}), \widehat{\Phi}\right) \right\| \right\} ds + \\ & + \sum_{-\infty<\tau_k<\infty} \|G(t,\tau_k)\| \left\{ \left\| \widehat{I}_k\left(\psi(\widehat{\varphi}), \widehat{\varphi}\right) - \widehat{I}_k\left(\psi(\widehat{\Phi}), \widehat{\Phi}\right) \right\| + \left\| \widehat{h}\left(\psi(\widehat{\Phi}), \widehat{\Phi}\right) \right\| \right\} \le \\ \le\; & \int_{-\infty}^{\infty} K e^{-\Delta_1|t-s|} \left[(\delta_1+\delta_2\eta) \left\| \psi(\widehat{\Phi}) - \psi(\widehat{\varphi}) \right\| + \delta_2 \left\| \widehat{\Phi} - \widehat{\varphi} \right\| + \left\| \widehat{h} \right\| \right] ds + \end{aligned}$$

$$+\sum_{-\infty<\tau_k<\infty} Ke^{-\Delta_1|t-\tau_k|}\left[(\delta_1+\delta_2\eta)\left\|\psi(\widehat{\Phi})-\psi(\widehat{\varphi})\right\|+\delta_2\left\|\widehat{\Phi}-\widehat{\varphi}\right\|+\left\|\widehat{h}\right\|\right].$$

From Conditions 1, 3 of Theorem 2 it follows that $0 < q < 1$. Then, from Lemma 5, we obtain that

$$\begin{aligned}\|\widehat{\Phi}-\widehat{\varphi}\| &\leq (1-q)^{-1}2K\left(\frac{1}{\Delta_1}+\frac{p+\varepsilon}{1-\varepsilon^{-\Delta_1}}\right)\times \\ &\times \max\left(\sup_{(x,y)\in\mathbb{R}^m\times\mathbf{V}_\rho}\|\widehat{h}(x,y)\|, \sup_{(x,y)\in\mathbb{R}^m\times\mathbf{V}_\rho,\ k\in\mathbb{Z}}\|\widehat{h}_k(x,y)\|\right).\end{aligned} \tag{16}$$

Estimate (16) holds for all $x \in \mathbb{R}^m$, $y \in \mathbf{V}_\rho$. Then it is valid for any $x \in \mathbf{U}_r$, $y \in \mathbf{V}_\rho$.

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