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ON AN EXTREMAL PROBLEM CONCERNING BERNSTEIN OPERATORS

Heinz H. Gonska and Ding-Xuan Zhou*

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ABSTRACT. The best constant problem for Bernstein operators with respect to the second modulus of smoothness is considered. We show that for any $\frac{1}{2} \le a < 1$, there is an $N(a) \in \mathbf{N}$ such that for $n \ge N(a)$,

$$\sup_{1-a\le \frac{k}{n}\le a} \left| B_n\left(f, \frac{k}{n}\right) - f\left(\frac{k}{n}\right)\right| \le c\omega_2\left(f, \frac{1}{\sqrt{n}}\right),$$

where c is a constant, $0 < c < 1$.

1. Introduction and Main Result. The Bernstein operators are given by

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} \equiv \sum_{k=0}^{n} f\left(\frac{k}{n}\right)P_{n,k}(x). \tag{1.1}$$

These operators have many applications and nice properties. It is well-known that the rate of convergence of Bernstein operators can be estimated by means of first and

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second moduli of smoothness as follows:

$$\|B_n(f) - f\|_{C[0,1]} \le C_1\omega_1\left(f, \frac{1}{\sqrt{n}}\right), \tag{1.2}$$

$$\|B_n(f) - f\|_{C[0,1]} \le C_2\omega_2\left(f, \frac{1}{\sqrt{n}}\right). \tag{1.3}$$

Here C_1, C_2 are constants independent of $f \in C[0,1]$ and $n \in \mathbf{N}$, the moduli of smoothness are defined by

$$\omega_1(f,t) := \sup_{0<h\le t} \sup_{0\le x<x+h\le 1} |f(x+h) - f(x)|, \tag{1.4}$$

$$\omega_2(f,t) := \sup_{0<h\le t} \sup_{x\pm h\in[0,1]} |f(x+h) - 2f(x) + f(x-h)|. \tag{1.5}$$

An interesting extremal problem concerning Bernstein operators is the so-called best constant problem, i.e., to determine the best constants C_1 and C_2 in (1.2) and (1.3), respectively. The first best constant was given early in 1961 by Sikkema [8] who showed that the best constant C_1 in (1.2) is $\dfrac{4306 + 837\sqrt{6}}{5832} \le 1.09$ (see E. Blaswich's thesis [1] for a detailed verification of Sikkema's results). There has also been an extensive study of the second constant. It is shown in [2] and [3, 4] that $C_2 \ge 1$ and C_2 can be 3.25. The latter was improved by Păltănea [7] who showed that C_2 can be chosen as 1.115. A conjecture [5] is that the best constant C_2 is 1.

The purpose of this paper is to show that at the "interpolatory points" of any closed interval apart from the endpoints, the constant C_2 in (1.3) can be chosen less than one for sufficiently large n.

Theorem. *Let* $r := \dfrac{\sum_{j=2}^{\infty} j^2 e^{-2(j-1)^2} + 1}{2 + 2\sum_{j=2}^{\infty} e^{-2(j-1)^2}} < 1$. *Then for any* $\frac{1}{2} \le a < 1$, $\epsilon > 0$, *there is an* $N(a,\epsilon) \in \mathbf{N}$ *such that for* $n \ge N(a,\epsilon)$, $f \in C[0,1]$,

$$\sup_{1-a\le \frac{k}{n} \le a} \left| B_n\left(f, \frac{k}{n}\right) - f\left(\frac{k}{n}\right)\right| \le (r+\epsilon)\omega_2\left(f, \frac{1}{\sqrt{n}}\right). \tag{1.6}$$

We prove the theorem by splitting the Bernstein operator into two parts called main part and tail part, respectively.

Let $0<\beta<\frac{1}{10}$, $k_0\in\mathbf{N}$ be such that $\frac{1}{2}\le\frac{k_0}{n}\le a$. Then for $n\ge(1-a)^{\frac{2}{2\beta-1}}$,

$$\begin{aligned}
B_n\left(f,\frac{k_0}{n}\right)-f\left(\frac{k_0}{n}\right) &= \sum_{k=0}^{n}\left(f\left(\frac{k}{n}\right)-f\left(\frac{k_0}{n}\right)\right)P_{n,k}\left(\frac{k_0}{n}\right)\\
&= \sum_{1\le l\le n^{\frac{1}{2}+\beta}}\left\{f\left(\frac{k_0+l}{n}\right)-2f\left(\frac{k_0}{n}\right)+f\left(\frac{k_0-l}{n}\right)\right\}P_{n,k_0-l}\left(\frac{k_0}{n}\right)\\
&\quad+\left\{\sum_{1\le l\le n^{\frac{1}{2}+\beta}}\left(f\left(\frac{k_0+l}{n}\right)-f\left(\frac{k_0}{n}\right)\right)\right.\\
&\quad\times\left(P_{n,k_0+l}\left(\frac{k_0}{n}\right)-P_{n,k_0-l}\left(\frac{k_0}{n}\right)\right)\\
&\quad\left.+\sum_{|k-k_0|>n^{\frac{1}{2}+\beta}}\left(f\left(\frac{k}{n}\right)-f\left(\frac{k_0}{n}\right)\right)P_{n,k}\left(\frac{k_0}{n}\right)\right\}\\
&:= L_n(f)+T_n(f). \qquad (1.7)
\end{aligned}$$

Here the linear functionals $L_n(f)$ and $T_n(f)$ are called the main part and tail part, respectively. In the following sections we estimate these two parts separately.

2. Preliminary Results. We need some preliminary results.

For the estimation of the tail functional $T_n(f)$, we need Peetre's K-functional

$$K_2(f,t):=\sup_{g\in C^2[0,1]}\{\|f-g\|_\infty+t\|g''\|_\infty\}. \qquad (2.1)$$

We have the following well-known equivalence:

Lemma 2.1. *For $f\in C[0,1]$, $0<t\le\frac{1}{2}$, we have*

$$M_0^{-1}\omega_2(f,t)\le K_2(f,t^2)\le M_0\omega_2(f,t), \qquad (2.2)$$

where the constant M_0 does not depend on f and t.

Depending on Stirling's formula, the following asymptotic expression for the Bernstein basis plays an essential role in our estimations.

Theorem 2.2. *Let $\frac{1}{2}\le a<1$, $0<\beta<\frac{1}{6}$, $x_0\in[\frac{1}{2},a]$, $n\in\mathbf{N}$, $k_0\in\mathbf{N}$ be such that $\left|\frac{k_0}{n}-x_0\right|\le\frac{1}{n}$, $\frac{1}{2}\le\frac{k_0}{n}\le a$. Then there is a constant M depending on a and β such that for any $1\le l\le n^{\frac{1}{2}+\beta}$,*

$$1-Mn^{3\beta-\frac{1}{2}}\le\frac{P_{n,k_0+l}(x_0)}{P_{n,k_0-l}(x_0)}\le1+Mn^{3\beta-\frac{1}{2}}. \qquad (2.3)$$

In what follows we denote by M a positive constant which can be different at each occurrence.

Proof. We prove only the right inequality of (2.3), since the proof of the left inequality is simpler by the same method.

We recall Stirling's formula

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left(1+O\left(\frac{1}{n}\right)\right),$$

say, for $n \geq N_0 > M$,

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left(1-\frac{M}{n}\right) \leq n! \leq \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left(1+\frac{M}{n}\right). \tag{2.4}$$

Let $n \geq \max\left\{4^{\frac{2}{1-2\beta}}, \left(\frac{1-a}{2}\right)^{\frac{2}{2\beta-1}}, \frac{2N_0}{1-a}\right\}$. Then, for $1 \leq l \leq n^{\frac{1}{2}+\beta}$,

$$k_0-l,\ k_0+l,\ n-k_0-l,\ n-k_0+l \geq \frac{1+a}{2}n \geq N_0.$$

Hence,

$$\begin{aligned}
\frac{P_{n,k_0+l}(x_0)}{P_{n,k_0-l}(x_0)} &= \frac{(k_0-l)!(n-k_0+l)!x_0^{2l}}{(k_0+l)!(n-k_0-l)!(1-x_0)^{2l}} \\
&\leq \frac{(k_0-l)^{k_0-l}(n-k_0+l)^{n-k_0+l}\sqrt{(k_0-l)(n-k_0+l)}}{(k_0+l)^{k_0+l}(n-k_0-l)^{n-k_0-l}\sqrt{(k_0+l)(n-k_0-l)}} \\
&\quad \frac{x_0^{2l}(1+\frac{M}{k_0-l})(1+\frac{M}{n-k_0+l})}{(1-x_0)^{2l}(1-\frac{M}{k_0+l})(1-\frac{M}{n-k_0-l})} \\
&\leq \left\{\sqrt{\left(1-\frac{2l}{k_0+l}\right)\left(1+\frac{2l}{n-k_0-l}\right)}\frac{(1+\frac{M}{k_0-l})(1+\frac{M}{n-k_0+l})}{(1-\frac{M}{k_0+l})(1-\frac{M}{n-k_0-l})}\right\} \\
&\quad \times \left\{\left(\frac{k_0-l}{k_0+l}\left(\frac{n-k_0+l}{n-k_0-l}\right)^{\frac{n-k_0}{k_0}}\right)^{k_0}\right\} \\
&\quad \times \left\{\left(\left(\frac{x_0(1-\frac{k_0}{n}+\frac{l}{n})}{(\frac{k_0}{n}+\frac{l}{n})(1-x_0)}\right)^2 \frac{(k_0+l)(n-k_0-l)}{(k_0-l)(n-k_0+l)}\right)^l\right\} \\
&:= I_1+I_2+I_3. \qquad (2.5)
\end{aligned}$$

We turn to estimate the three terms in (2.5).

For the first term, by Taylor expansion, we have

$$\begin{aligned} I_1 &\le \left(1+\frac{l}{n-k_0-l}\right)\frac{\left(1+\frac{2M}{(1-a)n}\right)^2}{\left(1-\frac{2M}{(1-a)n}\right)^2} \\ &\le (1+Mn^{\beta-\frac{1}{2}}). \end{aligned} \tag{2.6}$$

For the second term, we note that $0<\frac{n-k_0}{k_0}\le\frac{1}{2}$. Therefore, for $n\ge\left(\frac{4}{1-a}\right)^{\frac{2}{1-2\beta}}$,

$$\begin{aligned} &\frac{k_0-l}{k_0+l}\left(\frac{n-k_0+l}{n-k_0-l}\right)^{\frac{n-k_0}{k_0}} \\ \le\ &\frac{k_0-l}{k_0+l}\left\{1+\frac{2l(n-k_0)}{(n-k_0-l)k_0}+\frac{1}{2}\frac{n-k_0}{k_0}\left(\frac{n-k_0}{k_0}-1\right)\left(\frac{2l}{n-k_0-l}\right)^2+M\left(\frac{2l}{n-k_0-l}\right)^3\right\} \\ =\ &1+\frac{2l^2(2k_0-n)}{k_0(k_0+l)(n-k_0-l)}-\frac{2(k_0-l)(n-k_0)(2k_0-n)l^2}{k_0^2(k_0+l)(n-k_0-l)^2}+M\left(\frac{2l}{n-k_0-l}\right)^3 \\ =\ &1+\frac{2l^3(2k_0-n)}{k_0^2(k_0+l)(n-k_0-l)^2}(n-2k_0)+M\left(\frac{2l}{n-k_0-l}\right)^3 \\ \le\ &1+M\left(\frac{l}{n}\right)^3. \end{aligned}$$

Hence,

$$\begin{aligned} I_2 &\le \left(1+M\left(\frac{l}{n}\right)^3\right)^{k_0} \\ &\le e^{M\frac{l^3}{n^3}k_0} \\ &\le e^{Mn^{3\beta-\frac{1}{2}}} \\ &\le 1+Mn^{3\beta-\frac{1}{2}}. \end{aligned} \tag{2.7}$$

Finally, for the third term, we have

$$\left(\frac{x_0(1-\frac{k_0}{n}+\frac{l}{n})}{(\frac{k_0}{n}+\frac{l}{n})(1-x_0)}\right)^2\frac{(k_0+l)(n-k_0-l)}{(k_0-l)(n-k_0+l)}$$

$$
\begin{aligned}
&\le \left(1+2\frac{x_0-\frac{k_0+l}{n}}{\frac{k_0+l}{n}}\right)\left(1+2\frac{x_0-\frac{k_0}{n}+\frac{l}{n}}{1-x_0}\right)\left(1+\frac{2l}{k_0-l}\right)\left(1-\frac{2l}{n-k_0+l}\right)+M\left(\frac{l}{n}\right)^2\\
&\le 1+2\left\{\frac{nx_0-k_0-l}{k_0+l}+\frac{l}{k_0-l}+\frac{x_0-\frac{k_0}{n}+\frac{l}{n}}{1-x_0}-\frac{l}{n-k_0+l}\right\}+M\left(\frac{l}{n}\right)^2\\
&\le 1+2\left\{\frac{2l^2}{k_0^2-l^2}+\frac{l(l+1)}{n(1-x_0)(n-k_0+l)}\right\}+\frac{M}{n}+M\left(\frac{l}{n}\right)^2\\
&\le 1+M\left(\left(\frac{l}{n}\right)^2+\frac{1}{n}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
I_3 &\le \left(1+M\left(\left(\frac{l}{n}\right)^2+\frac{1}{n}\right)\right)^l\\
&\le e^{M(\frac{l^3}{n^2}+\frac{l}{n})}\\
&\le 1+Mn^{3\beta-\frac{1}{2}}. \qquad (2.8)
\end{aligned}
$$

Combining (2.6), (2.7), and (2.8), we derive from (2.5) that for $n \ge N(a,\beta,N_0) \in \mathbf{N}$,

$$
\frac{P_{n,k_0+l}(x_0)}{P_{n,k_0-l}(x_0)} \le 1+Mn^{3\beta-\frac{1}{2}}.
$$

Therefore, (2.3) holds for any $n \in \mathbf{N}$ and $1 \le l \le n^{\frac{1}{2}+\beta}$.

The proof of Theorem 2.2 is complete. □

With all the above preparations, we can give our estimations.

3. The Tail Part. In this section we estimate the error of the tail part of (1.7). Here the tail functional $T_n : C[0,1] \to \mathbf{R}$ is defined by

$$
\begin{aligned}
T_n(f) :&= \sum_{1\le l\le n^{\frac{1}{2}+\beta}} \left(f\left(\frac{k_0+l}{n}\right)-f\left(\frac{k_0}{n}\right)\right)\left(P_{n,k_0+l}\left(\frac{k_0}{n}\right)-P_{n,k_0-l}\left(\frac{k_0}{n}\right)\right)\\
&+\sum_{|k-k_0|>n^{\frac{1}{2}+\beta}} \left(f\left(\frac{k}{n}\right)-f\left(\frac{k_0}{n}\right)\right)P_{n,k}\left(\frac{k_0}{n}\right). \qquad (3.1)
\end{aligned}
$$

Our estimate of the error can be stated as follows.

Theorem 3.1. *Let* $0 < \beta < \frac{1}{10}$, $\frac{1}{2} \le a < 1$, $k_0 \in \mathbf{N}$ *be such that* $\frac{1}{2} \le \frac{k_0}{n} \le a$. *Then, for* $f \in C[0,1]$, *we have*

$$|T_n(f)| \le M(n^{5\beta-\frac{1}{2}} + n^{-2\beta})\omega_2\left(f, \frac{1}{\sqrt{n}}\right). \tag{3.2}$$

To prove Theorem 3.1, we use Peetre's K-functional and the following two lemmas.

Lemma 3.2. *Under the assumptions of Theorem* 3.1, *we have*

$$|T_n(f)| \le Mn^{-2\beta}\|f\|_\infty. \tag{3.3}$$

P r o o f. Using Theorem 2.2 for $x_0 = \frac{k_0}{n}$ in (3.1), we have

$$\begin{aligned}
|T_n(f)| &\le 2\|f\|_\infty \sum_{1\le l\le n^{\frac{1}{2}+\beta}} P_{n,k_0-l}\left(\frac{k_0}{n}\right)\left|\frac{P_{n,k_0+l}(\frac{k_0}{n})}{P_{n,k_0-l}(\frac{k_0}{n})} - 1\right| \\
&\quad +2\|f\|_\infty \sum_{|k-k_0|>n^{\frac{1}{2}+\beta}} \left|\frac{k-k_0}{n^{\frac{1}{2}+\beta}}\right|^2 P_{n,k}\left(\frac{k_0}{n}\right) \\
&\le 2\|f\|_\infty Mn^{3\beta-\frac{1}{2}} + 2\|f\|_\infty n^{-1-2\beta}\left\{\sum_{k=0}^{n}\left(k - n\frac{k_0}{n}\right)^2 P_{n,k}\left(\frac{k_0}{n}\right)\right\} \\
&\le M(n^{3\beta-\frac{1}{2}} + n^{-2\beta})\|f\|_\infty \\
&\le 2Mn^{-2\beta}\|f\|_\infty.
\end{aligned}$$

Here we used the second moment of the Bernstein operator:

$$B_n((\cdot - x)^2, x) = \frac{x(1-x)}{n}, \quad x \in [0,1]. \tag{3.4}$$

The proof of Lemma 3.2 is complete. □

Lemma 3.3. *Under the assumptions of Theorem* 3.1, *we have for* $f \in C^2[0,1]$,

$$|T_n(f)| \le M(n^{5\beta-\frac{1}{2}} + n^{-2\beta})\frac{\|f''\|_\infty}{n}. \tag{3.5}$$

P r o o f. We observe first that for any linear function l,

$$T_n(l) = 0.$$

Let $f \in C^2[0,1]$. We use Taylor expansion

$$f(x) = f\left(\frac{k_0}{n}\right) + f'\left(\frac{k_0}{n}\right)\left(x - \frac{k_0}{n}\right) + \int_{\frac{k_0}{n}}^{x} (x-u)f''(u)du. \tag{3.6}$$

Then we have

$$\begin{aligned}
|T_n(f)| &= \Bigg| \sum_{1\le l\le n^{\frac12+\beta}} \int_{\frac{k_0}{n}}^{\frac{k_0+l}{n}} \left(\frac{k_0+l}{n} - u\right) f''(u)duP_{n,k_0-l}\left(\frac{k_0}{n}\right)\left(\frac{P_{n,k_0+l}(\frac{k_0}{n})}{P_{n,k_0-l}(\frac{k_0}{n})} - 1\right) \\
&\quad + \sum_{|k-k_0|>n^{\frac12+\beta}} \int_{\frac{k_0}{n}}^{\frac{k}{n}} \left(\frac{k}{n} - u\right) f''(u)duP_{n,k}\left(\frac{k_0}{n}\right)\Bigg| \\
&\le M \sum_{1\le l\le n^{\frac12+\beta}} \left(\frac{l}{n}\right)^2 P_{n,k_0-l}\left(\frac{k_0}{n}\right) n^{3\beta-\frac12}\|f''\|_\infty \\
&\quad + \sum_{|k-k_0|>n^{\frac12+\beta}} \left(\frac{k-k_0}{n}\right)^2 \left(\frac{k-k_0}{n^{\frac12+\beta}}\right)^2 P_{n,k}\left(\frac{k_0}{n}\right)\|f''\|_\infty \\
&\le Mn^{5\beta-\frac32}\|f''\|_\infty + \sum_{|k-k_0|>n^{\frac12+\beta}} \left(\frac{k}{n} - \frac{k_0}{n}\right)^4 P_{n,k}\left(\frac{k_0}{n}\right) n^{1-2\beta}\|f''\|_\infty \\
&\le M(n^{5\beta-\frac12} + n^{-2\beta})\frac{\|f''\|_\infty}{n}.
\end{aligned}$$

Here we used the forth moment of the Bernstein operator (see [6]):

$$B_n((\cdot - x)^4, x) = \frac{3(x(1-x))^2}{n^2} - \frac{2(x(1-x))^2}{n^3} + \frac{x(1-x)(1-2x)^2}{n^3}. \tag{3.7}$$

The proof of Lemma 3.3 is complete. □

Now we can give the proof of Theorem 3.1.

P r o o f o f T h e o r e m 3.1. Let $f \in C[0,1]$. By Lemma 3.2, 3.3 and 2.1, taking the infimum over $g \in C^2[0,1]$, we have

$$\begin{aligned}
|T_n(f)| &\le \inf_{g\in C^2[0,1]} \{|T_n(f-g)| + |T_n(g)|\} \\
&\le \inf_{g\in C^2[0,1]} \left\{Mn^{-2\beta}\|f-g\|_\infty + M\left(n^{5\beta-\frac12} + n^{-2\beta}\right)\frac{\|g''\|_\infty}{n}\right\} \\
&\le M\left(n^{5\beta-\frac12} + n^{-2\beta}\right) K_2\left(f, \frac1n\right) \\
&\le M\left(n^{5\beta-\frac12} + n^{-2\beta}\right) \omega_2\left(f, \frac{1}{\sqrt{n}}\right).
\end{aligned}$$

This completes the proof of Theorem 3.1. □

4. Estimate for the Main Part. In this section, we estimate the main part of (1.7).

From the splitting formula (1.7), we have

$$
\begin{aligned}
|L_n(f)| &= \left| \sum_{1\le l\le n^{\frac{1}{2}+\beta}} \left\{ f\left(\frac{k_0+l}{n}\right) - 2f\left(\frac{k_0}{n}\right) + f\left(\frac{k_0-l}{n}\right)\right\} P_{n,k_0-l}\left(\frac{k_0}{n}\right)\right| \\
&\le \sum_{1\le j\le n^\beta} \sum_{(j-1)\sqrt{n}<l\le j\sqrt{n}} \omega_2\left(f,\frac{l}{n}\right) P_{n,k_0-l}\left(\frac{k_0}{n}\right) \\
&\le \sum_{1\le j\le n^\beta} j^2 K_j \omega_2\left(f,\frac{1}{\sqrt{n}}\right). \qquad (4.1)
\end{aligned}
$$

Here,

$$K_j := \sum_{(j-1)\sqrt{n}<l\le j\sqrt{n}} P_{n,k_0-l}\left(\frac{k_0}{n}\right). \qquad (4.2)$$

For these terms, we have

Lemma 4.1. *Let* $\frac{1}{2}\le a<1$, $0<\beta<\frac{1}{6}$, $k_0\in\mathbf{N}$ *be such that* $\frac{1}{2}\le\frac{k_0}{n}\le a$. *Then for any* $\delta>0$, *there is an* $N\in\mathbf{N}$ *such that for any* $n>N, 1\le j\le n^\beta$, *there holds*

$$\frac{K_1}{K_j} \ge (1-\delta)(e-\delta)^{2(j-1)^2}. \qquad (4.3)$$

Proof. Let $n\in\mathbf{N}$, $1\le j\le n^\beta$. We set $L=[(j-1)\sqrt{n}\,]$. Then we have

$$
\begin{aligned}
K_j &\le \sum_{L+1\le l\le L+[\sqrt{n}]+1} P_{n,k_0-l}\left(\frac{k_0}{n}\right) \\
&\le \sum_{l=1}^{[\sqrt{n}]} P_{n,k_0-L-l}\left(\frac{k_0}{n}\right)\frac{[\sqrt{n}\,]+1}{[\sqrt{n}\,]}. \qquad (4.4)
\end{aligned}
$$

Let $1\le l\le[\sqrt{n}\,]$. By Stirling's formula, we have

$$
\begin{aligned}
\frac{P_{n,k_0-l}(\frac{k_0}{n})}{P_{n,k_0-L-l}(\frac{k_0}{n})} &= \frac{(k_0-L-l)^{k_0-L-l}(n-k_0+L+l)^{n-k_0+L+l}}{(k_0-l)^{k_0-l}(n-k_0+l)^{n-k_0+l}} \\
&\quad \left(\frac{\frac{k_0}{n}}{1-\frac{k_0}{n}}\right)^L \sqrt{\frac{(k_0-L-l)(n-k_0+L+l)}{(k_0-l)(n-k_0+l)}}\left(1+O\left(\frac{1}{n}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{k_0(n-k_0+L+l)}{(k_0-L-l)(n-k_0)}\right)^L \\
&\quad \left\{\frac{k_0-L-l}{k_0-l}\left(\frac{n-k_0+L+l}{n-k_0+l}\right)^{\frac{n-k_0+l}{k_0-l}}\right\}^{k_0-l} \\
&\quad \left\{\sqrt{\frac{(k_0-L-l)(n-k_0+L+l)}{(k_0-l)(n-k_0+l)}}\left(1+O\left(\frac{1}{n}\right)\right)\right\} \\
&:= J_1J_2J_3. \qquad (4.5)
\end{aligned}
$$

We turn to estimate the three terms separately.

For the first term J_1, we have

$$
\begin{aligned}
J_1 &= \left\{1+\frac{\frac{L+l}{n}}{(1-\frac{k_0}{n})(\frac{k_0-L-l}{n})}\right\}^L \\
&\geq \left\{1+\frac{\frac{L+l}{n}}{(1-\frac{k_0}{n})\frac{k_0}{n}}\right\}^L \\
&= \left\{1+\frac{\frac{L+l}{n}}{(1-\frac{k_0}{n})\frac{k_0}{n}}\right\}^{\frac{\left(1-\frac{k_0}{n}\right)\frac{k_0}{n}}{\frac{L+l}{n}}\frac{\frac{L+l}{n}L}{\left(1-\frac{k_0}{n}\right)\frac{k_0}{n}}}.
\end{aligned}
$$

Therefore, for any $\eta_1>0$, there is an $N_1\in\mathbf{N}$ such that for $n\geq N_1$, there holds

$$
\begin{aligned}
J_1 &\geq (e-\eta_1)^{\frac{L^2+Ll}{n\left(1-\frac{k_0}{n}\right)\frac{k_0}{n}}} \\
&\geq (1-\eta_1)(e-\eta_1)^{\frac{(j-1)^2}{\left(1-\frac{k_0}{n}\right)\frac{k_0}{n}}}. \qquad (4.6)
\end{aligned}
$$

For the second term J_2, since $\dfrac{1-a}{a}\leq\dfrac{n-k_0+l}{k_0-l}\leq 4$ for $n\geq 16$, we have

$$
\begin{aligned}
J_2 &= \left\{\frac{k_0-L-l}{k_0-l}\left(1+\frac{L}{k_0-l}\right.\right. \\
&\quad \left.\left.+\frac{1}{2}\frac{(n-k_0+l)(n-2k_0+2l)}{(k_0-l)^2}\left(\frac{L}{n-k_0+l}\right)^2+O\left(\left(\frac{L}{n-k_0+l}\right)^3\right)\right)\right\}^{k_0-l} \\
&= \left\{1-\frac{L^2}{(k_0-l)^2}+\frac{1}{2}\frac{L^2}{(k_0-l)^2}\frac{n-2k_0+2l}{n-k_0+l}+O\left(\left(\frac{L}{n-k_0+l}\right)^3\right)\right\}^{k_0-l}
\end{aligned}
$$

$$= \left\{\left(1-\frac{L^2}{(k_0-l)^2}\frac{n}{2(n-k_0+l)}\right)\left(1+O\left(\left(\frac{L}{n-k_0+l}\right)^3\right)\right)\right\}^{k_0-l}$$

$$= \left(1-\frac{L^2 n}{2(k_0-l)^2(n-k_0+l)}\right)^{-\frac{2(k_0-l)^2(n-k_0+l)}{L^2 n}\left(-\frac{L^2 n}{2(k_0-l)(n-k_0+l)}\right)}$$

$$\left(1+O\left(\left(\frac{L}{n-k_0+l}\right)^3\right)\right)^{k_0-l}.$$

We observe that

$$\left(\frac{L}{n-k_0+l}\right)^3 \le \left(\frac{(j-1)\sqrt{n}}{(1-a)n}\right)^3 \le \frac{1}{(1-a)^3}n^{3\beta-\frac{3}{2}},$$

while for $n \ge 16$,

$$k_0 - l \ge \frac{n}{2} - \sqrt{n} \ge \frac{n}{4}.$$

Also, for $n \ge 16$,

$$\frac{L^2 n}{2(k_0-l)^2(n-k_0+l)} \le \frac{(j-1)^2 n^2}{2n^3(\frac{1}{2}-\frac{1}{\sqrt{n}})(1-a)} \le \frac{2}{1-a}n^{2\beta-1}.$$

Therefore, for any $\eta_2 > 0$, there is an $N_2 \in \mathbf{N}$ such that for $n \ge N_2$, there holds

$$J_2 \ge (1-\eta)(e+\eta_2)^{-\frac{(j-1)^2}{2\left(1-\frac{k_0}{n}\right)\frac{k_0}{n}}}. \tag{4.7}$$

The third term J_3 is easier to estimate. In fact, for $n \ge 16$,

$$\frac{k_0-L-l}{k_0-l} \ge 1-\frac{(j-1)\sqrt{n}}{k_0-l} \ge 1-\frac{4(j-1)}{\sqrt{n}} \ge 1-4n^{\beta-\frac{1}{2}}.$$

and

$$\frac{n-k_0+L+l}{n-k_0+l} \le 1+\frac{(j-1)\sqrt{n}}{n-k_0+l} \le 1+\frac{j-1}{(1-a)\sqrt{n}} \le 1+\frac{1}{1-a}n^{\beta-\frac{1}{2}}.$$

Therefore, for any $\eta_3 > 0$, there is an $N_3 \in \mathbf{N}$ such that for $n \ge N_3$,

$$\frac{[\sqrt{n}\,]}{[\sqrt{n}\,]+1}J_3 \ge 1-\eta_3. \tag{4.8}$$

Combining (4.5) with (4.6), (4.7) and (4.8), we can derive our estimate as follows.

For any $\delta > 0$, we choose $\eta_1, \eta_2, \eta_3 > 0$ be such that

$$\frac{(e-\eta_1)^2}{e+\eta_2} \geq e - \delta$$

and

$$(1-\eta_1)(1-\eta_2)(1-\eta_3) \geq 1-\delta.$$

Then, letting $N = \max\{N_1, N_2, N_3\}$, we know that for $n \geq N$, $1 \leq l \leq [\sqrt{n}]$, $1 \leq j \leq n^\beta$, $\frac{1}{2} \leq \frac{k_0}{n} \leq a$,

$$\begin{aligned} \frac{P_{n,k_0-l}(\frac{k_0}{n})}{P_{n,k_0-[(j-1)\sqrt{n}]-l}(\frac{k_0}{n})} &\geq \frac{[\sqrt{n}]+1}{[\sqrt{n}]}(1-\delta)(e-\delta)^{\frac{(j-1)^2}{2\left(1-\frac{k_0}{n}\right)\frac{k_0}{n}}} \\ &\geq \frac{[\sqrt{n}]+1}{[\sqrt{n}]}(1-\delta)(e-\delta)^{2(j-1)^2}. \end{aligned}$$

Hence,

$$\frac{K_1}{K_j} \geq (1-\delta)(e-\delta)^{2(j-1)^2}.$$

The proof of Lemma 4.1 is complete. □

With the above preparations, we can now prove our main result.

P r o o f o f T h e o r e m. Let $r = \dfrac{\sum_{j=2}^{\infty} j^2 e^{-2(j-1)^2} + 1}{2 + 2\sum_{j=2}^{\infty} e^{-2(j-1)^2}}$, $\frac{1}{2} \leq a < 1$.

We note that the series $\sum_{j=2}^{\infty} j^2 x^{-2(j-1)^2}$ and $\sum_{j=2}^{\infty} x^{-2(j-1)^2}$ are uniformly convergent for $2 \leq |x| \leq 4$. Therefore, for any $\epsilon > 0$, we can find $\delta > 0$ and $M_1 \in \mathbf{N}$ such that for $n \geq M_1$,

$$\frac{\sum_{2\leq j\leq n^\beta} j^2 \frac{1}{1-\delta}(e-\delta)^{-2(j-1)^2} + 1}{2 + 2\sum_{2\leq j\leq n^\beta} \frac{1}{1-\delta}(e-\delta)^{-2(j-1)^2}} \leq r + \frac{\epsilon}{2}. \tag{4.9}$$

Under this choice, by Lemma 4.1, we can find $M_2 \in \mathbf{N}$ such that (4.3) holds. Then, for $n \geq \max\{M_1, M_2\}$,

$$\sum_{1\leq j\leq n^\beta} \left(j^2 - 2\left(r + \frac{\epsilon}{2}\right)\right) K_j$$

$$\begin{aligned}
&\le \sum_{2\le j\le n^\beta}\left(j^2-2\left(r+\frac{\epsilon}{2}\right)\right)\frac{1}{1-\delta}(e-\delta)^{-2(j-1)^2}K_1+\left(1-2\left(r+\frac{\epsilon}{2}\right)\right)K_1\\
&= \left\{\sum_{2\le j\le n^\beta} j^2\frac{1}{1-\delta}(e-\delta)^{-2(j-1)^2}+1-\right.\\
&\quad\left.\left(r+\frac{\epsilon}{2}\right)\left(2+2\sum_{2\le j\le n^\beta}\frac{1}{1-\delta}(e-\delta)^{-2(j-1)^2}\right)\right\}K_1\\
&\le 0.
\end{aligned}$$

This implies

$$\begin{aligned}
|L_n(f)| &\le \left\{2\left(r+\frac{\epsilon}{2}\right)\sum_{1\le j\le n^\beta}K_j+\sum_{1\le j\le n^\beta}\left(j^2-2\left(r+\frac{\epsilon}{2}\right)\right)K_j\right\}\omega_2\left(f,\frac{1}{\sqrt{n}}\right)\\
&\le \left(r+\frac{\epsilon}{2}\right)2\left(\sum_{1\le j\le n^\beta}K_j\right)\omega_2\left(f,\frac{1}{\sqrt{n}}\right)\\
&\le \left(r+\frac{\epsilon}{2}\right)\sum_{1\le j\le n^\beta}\ \sum_{(j-1)\sqrt{n}<l\le j\sqrt{n}}\left\{P_{n,k_0-l}\left(\frac{k_0}{n}\right)+P_{n,k_0+l}\left(\frac{k_0}{n}\right)\right\}\omega_2\left(f,\frac{1}{\sqrt{n}}\right)\\
&\le \left(r+\frac{\epsilon}{2}\right)\omega_2\left(f,\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

By Theorem 3.1, for $\epsilon>0$, we can find $M_3\in\mathbf{N}$ such that for $n\ge M_3$,

$$|T_n(f)|\le\frac{\epsilon}{2}\omega_2\left(f,\frac{1}{\sqrt{n}}\right).$$

Therefore, for $n\ge\max\{M_1,M_2,M_3\}$, $1-a\le\dfrac{k_0}{n}\le a$, we have

$$\left|B_n\left(f,\frac{k_0}{n}\right)-f\left(\frac{k_0}{n}\right)\right|\le(r+\epsilon)\omega_2\left(f,\frac{1}{\sqrt{n}}\right),$$

and the proof of our theorem is complete.

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Heinz H. Gonska, Ding-Xuan Zhou
Department of Mathematics
University of Duisburg
D-47057 Duisburg, Germany

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