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MEAN-PERIODIC SOLUTIONS OF RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we present a spectral criterion for existence of mean-periodic solutions of retarded functional differential equations with a time-independent main part.

Introduction. Consider the retarded functional differential equation

$$x'(t) = \int_{-r}^{0} dH(\theta)x(t+\theta) + f(t). \tag{1}$$

We will investigate the problem for existence and uniqueness of solutions which belong to certain class of functions when f belongs to the same one. Here x and f are l-vector valued functions and H is $l \times l$-matrix whose elements are real functions of bounded variation on $[-r, 0]$, $r > 0$.

Below we present samples of our problem. Let $P(\omega)$ be the space of the continuous ω-periodic functions and let $f \in P(\omega)$. Then Eq.(1) has a unique solution

$$x \in P(\omega) \cap C^1(\mathbf{R}, \mathbf{C}^l)$$

when the characteristic function of a complex variable λ

$$\chi(\lambda) \stackrel{\text{def}}{=} \det\left(E\lambda - \int_{-r}^{0} exp(\lambda\theta)dH(\theta)\right)$$

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has no zeros of the form
$$\frac{2k\pi\mathbf{i}}{\omega}, \qquad k \in \mathbf{Z}.$$
This result is a corollary of Theorem 9.1.2 of Hale [1]. Here E is the unit matrix of $\mathbf{C}^l$ and $\mathbf{i}$ is the imaginary unit. We denote by $C^q(\Delta, \mathbf{C}^l)$ the space of the functions $g : \Delta \to \mathbf{C}^l$ with continuous derivatives up to q-th order where $\Delta \subset \mathbf{R}$ is an interval.

Let AP be the space of the continuous almost-periodic functions, due to Bohr, and let $f \in AP$. Then from Theorem 9.1.1 [1] follows that Eq.(1) has a unique solution
$$x \in AP \cap C^1(\mathbf{R}, \mathbf{C}^l)$$
when the characteristic function χ has no zeros on the imaginary axis. In accordance with the same Theorem, this result remains valid also for bounded on the real axis functions f.

In both cases we have a spectral criterion for a solvability of our problem.

For any $a \in \mathbf{R}$ the ω-periodicity of $g \in C^0([a, \infty), \mathbf{C}^l)$ is presented by the property
$$\int_0^\omega g(t+s)d\eta(s) = 0, \qquad t \geq a,$$
where $\eta(0) = \eta(\omega) = 0$ and $\eta(s) = 1$ for $0 < s < \omega$. We are going to generalize this definition assuming that for a fixed integer p we have a distribution η with an action
$$\langle \eta(s), g(s) \rangle \stackrel{\text{def}}{=} \sum_{i=0}^{p} \int_0^\kappa g^{(i)}(s) d\eta_i(s)$$
where $\eta_i,\ 0 \leq i \leq p$, are real functions of bounded variation in $[0, \kappa],\ \kappa > 0$.

Definition 1. *Following* Schwartz [2], *for a fixed integer p, $\kappa \geq 0$ and for any real a, we call mean-periodic the functions which belong to the classes*
$$MP(\eta, a) \stackrel{\text{def}}{=} \left\{ g \in C^p([a, \infty), \mathbf{C}^l) \ : \ \langle \eta(s), g(t+s) \rangle = 0,\ t \geq a \right\},$$
$$MP(\eta, -\infty) \stackrel{\text{def}}{=} \left\{ g \in C^p(\mathbf{R}, \mathbf{C}^l) \ : \ \langle \eta(s), g(t+s) \rangle = 0,\ t \in \mathbf{R} \right\}.$$

The main purpose of this work is to find conditions under which Eq.(1) has a mean-periodic solution when f is mean-periodic of the same class. To the best of the author's knowledge this problem has not been discussed yet except for the periodic case. We will give a spectral criterion for its solvability. In this way we show that such a problem has an algebraic essence.

Certainly this paper is related to the problem of representation in projection series. More details on this topic the reader may find, for instance, in Banks and Manitius [3], Henry [4], as well in the fundamental works of Leont'ev [5, 6] and Sedleckii [7]. Also this paper is related to certain class of spectral-mapping problems; see, for instance, Hale [1:Ch.7,Ch.11], Henry [4] and the fundamental work of Hille and Philips [8:Ch.16].

The proofs that we present are straightforward but not so short. Perhaps one can make them more concise by means of constructive use of the spectral theory.

The main result. At first we have to define what we call a solution of Eq.(1). We use the conventional

Definition 2. *Let $f \in C^0([a,\infty),\mathbf{C}^l)$. A solution of* Eq.(1)*, which begins from the point a, is said to be a function*

$$x(\cdot;f,a) \in C^0([a-r,\infty),\mathbf{C}^l) \cap C^1([a,\infty),\mathbf{C}^l)$$

which satisfies Eq.(1) *for $t \geq a$.*

This definition is natural. We have a well-posed forward initial value problem /IVP/, i.e. for a given continuous x at $[a-r,a]$, Eq.(1) has a unique solution which begins from the point a (see [1]).

Let B be an operator in a Banach space. By $\sigma(B)$ we point out the spectrum of B. (For unbounded B we may have $\sigma(B) = \emptyset$; note that A is unbounded.)

Let A be the generator of the semigroup $\mathcal{T}(t)$, $t \geq 0$, associated with Eq.(1). It is known (see [1]) that $\sigma(A)$ contains only eigenvalues which are just the zeros of the characteristic function χ. Clearly $\sigma(A) \neq \emptyset$. Introduce the entire function

$$\Phi(\lambda) \stackrel{\text{def}}{=} \langle \eta(s), exp(\lambda s) \rangle.$$

Let γ be the jump of $\eta_p(s)$ at $s = 0$, i.e.

$$\int_0^\kappa g(s)d\eta_p(s) \stackrel{\text{def}}{=} \gamma g(0) + \int_0^\kappa g(s)d\eta_p^\circ(s)$$

where η_p° is continuous at zero.

Our main result is

Theorem 1. *Let $\gamma\Phi(\lambda) \neq 0$ for $\lambda \in \sigma(A)$ and $f \in MP(\eta,a)$. Then* Eq.(1) *has a unique solution $x(\cdot;f,a) \in MP(\eta,a-r)$.*

The condition $\gamma \neq 0$ allows us to propose

Lemma 1. *Let $\gamma \neq 0$ and $g \in MP(\eta,a)$. Then there is a unique $\hat{g} \in MP(\eta,-\infty)$ such that*

$$\hat{g}(t) = g(t), \qquad t \geq a.$$

Moreover, $g \in C^q([a,\infty), \mathbf{C}^l)$ *implies* $\hat{g} \in C^q(\mathbf{R}, \mathbf{C}^l)$. *Certainly the latter is essential only with* $q > p$.

Proof. Consider a backward IVP for the equation of a neutral type

$$\sum_{i=0}^{p} \int_0^{\kappa} y^{(i)}(t+\theta)d\eta_i(\theta) = 0, \qquad t \le a, \tag{2}$$

with an initial condition $y(\theta) = g(\theta),\ \theta \in [a, a+\kappa]$. We have

$$\sum_{i=0}^{p} \int_0^{\kappa} g^{(i)}(a+\theta)d\eta_i(\theta) = 0. \tag{3}$$

The condition $\gamma \neq 0$ makes the backward IVP for Eq.(2) to be well-posed (see [1]). Therefore Eq.(2), with the pointed out initial condition, has a solution which is the function $\hat{g}$ that we are looking for. The smoothness of $\hat{g}$ at a follows from (3).

The proof of the second part can be done in a conventional way taking into account the smoothness of the initial condition on $[a, a+\kappa]$. □

This result allows us to assume a function of $MP(\eta, \cdot)$, with $\gamma \neq 0$, as a function of $MP(\eta, -\infty)$. At this point we may improve Theorem 1 as follows.

Theorem 2. *Let* $\gamma\Phi(\lambda) \neq 0$ *for* $\lambda \in \sigma(A)$ *and* $f \in MP(\eta, -\infty)$. *Then* Eq.(1) *has a unique solution*

$$x^*(\cdot; f) \in MP(\eta, -\infty) \cap C^1(\mathbf{R}, \mathbf{C}^l)$$

which satisfies Eq.(1) *on the whole real axis. Furthermore,* $f \in C^q(\mathbf{R}, \mathbf{C}^l)$ *implies* $x^*(\cdot; f) \in C^{q+1}(\mathbf{R}, \mathbf{C}^l)$.

Proof. Let

$$x(\cdot; f, a) \in MP(\eta, a-r) \cap C^1([a,\infty), \mathbf{C}^l)$$

be the solution of Eq.(1), beginning from arbitrary chosen point a, whose existence is provided by Theorem 1. The uniqueness part of Lemma 1 gives

$$x(t; f, b) = x(t; f, a) \quad \text{for} \quad t \ge b \ge a.$$

Thus we are able to define

$$x^*(t; f) \stackrel{\text{def}}{=} x(t; f, a) \quad \text{for} \quad t \ge a.$$

The proof of the second part is evident. □

Remark. Suppose $\gamma \neq 0$. Then one can find that the condition "$\Phi(\lambda) \neq 0$ for $\lambda \in \sigma(A)$" fails only for a finite number elements of $\sigma(A)$ since, in this case, the zeros of Φ belong to certain half-plane $Re\lambda \geq \lambda_\Phi$ and, on the other hand, such a half-plane contains a finite number points of $\sigma(A)$.

Theorem 2 shows that our "mean-periodic" problem is a complete analog with respect to the "periodic" case and an analog with respect to the "almost-periodic" one. Note that the set of almost-periodic functions has no convenient description, in view of our aims, in "mean-periodic" terms.

The rest of the paper is dedicated to the proof of Theorem 1.

Proof of Theorem 1. For the sake of completeness, we will give some facts from [1] and [8]. Let X be the Banach space of the functions of $C^0([-r,0],\mathbf{C}^l)$ with the uniform norm. Assume $\psi \in X,\ f \in C^0([0,\infty),\mathbf{C}^l)$, and add to Eq.(1) the following initial condition

$$x(\theta) = \psi(\theta), \qquad \theta \in [-r,0]. \tag{4}$$

The IVP (1)-(4), i.e. Eq.(1) with the initial condition /IC/ (4), has a unique solution which begins from zero. The strongly continuous semigroup of linear operators $\mathcal{T}(t): X \to X,\ t \geq 0$, is produced by the solution of the homogeneous equation

$$y'(t) = \int_{-r}^{0} dH(\theta)y(t+\theta), \qquad t \geq 0, \tag{5}$$

with an IC $\psi \in X$ as follows

$$\Big(\mathcal{T}(t)\psi\Big)(\theta) = y(t+\theta), \qquad \theta \in [-r,0],$$

and the solution of IVP (1)-(4) turns into the form

$$x(t+\theta;f,0) = \Big(\mathcal{T}(t)\psi\Big)(\theta) + \int_0^t U(t+\theta-\tau)f(\tau)d\tau, \qquad t \geq 0,\ \theta \in [-r,0], \tag{6}$$

where $U: [-r,\infty) \to \mathbf{C}^{l\times l}$ is the fundamental matrix of Eq.(5). According to its definition, U is continuous on $[0,\infty)$, satisfies Eq.(5) almost for all $t \geq 0$ and

$$U(0) = E, \qquad U(\theta) = 0, \qquad \theta \in [-r,0). \tag{7}$$

The infinitesimal generator A is a differentiation in its domain which consists of the functions $\psi \in C^1([-r,0],\mathbf{C}^l)$ with

$$\psi'(0) = \int_{-r}^{0} dH(\theta)\psi(\theta). \tag{8}$$

The operator A is closed, with all its powers, and each half-plane $Re\lambda \geq \lambda_0$ contains a finite number of elements of $\sigma(A)$. Remember that $\mu \in \sigma(A)$ iff $\chi(\mu) = 0$. Every $\mu \in \sigma(A)$ is a pole of the resolvent of A. For a given $\psi \in X$, the map $\mathcal{T}(t)\psi : [0, \infty) \to X$ is continuous. Then, for a real β of bounded variation in $[0, \kappa]$, there exists

$$\int_0^\kappa \mathcal{T}(s)\psi d\beta(s)$$

which defines a bounded operator in X. We need a more general operator

$$\mathcal{P} \stackrel{\text{def}}{=} \sum_{i=0}^{p} \int_0^\kappa \mathcal{T}(s)A^i \cdot d\eta_i(s).$$

The domain of $\mathcal{P}$ coincides with the domain of A^p which is dense in X and belongs to $C^p([-r, 0], \mathbf{C}^l)$. We denote a domain of an operator by $D(\cdot)$.

Continue with

Lemma 2. *Let $\psi \in D(A^p)$, $f \in C^p([0, \infty), \mathbf{C}^l)$, and, in the case when $p \geq 1$, the following condition holds*

$$(*) \qquad f^{(i)}(0) = 0, \quad 0 \leq i \leq p - 1.$$

Then the solution y of Eq.(5)*, with an IC ψ, belongs to $C^p([-r, \infty), \mathbf{C}^l)$ and*

$$y^{(i)}(t + \theta) = \Big(\mathcal{T}(t)A^i\psi\Big)(\theta), \quad t \geq 0,\ \theta \in [-r, 0],\ 0 \leq i \leq p.$$

The solution $x(\cdot; f, 0)$ of Eq.(1)*, with an IC ψ, belongs to $C^p([-r, \infty), \mathbf{C}^l)$ and*

$$x^{(i)}(t + \theta; f, 0) = y^{(i)}(t + \theta) + \int_0^t U(t + \theta - \tau)f^{(i)}(\tau)d\tau, \quad t \geq 0,\ \theta \in [-r, 0],\ 0 \leq i \leq p.$$

P r o o f. Assume $p \geq 1$. Otherwise there is nothing to prove. In view of (6) we have

$$(9) \qquad x(t + \theta; f, 0) = y(t + \theta) + \int_0^t U(t + \theta - \tau)f(\tau)d\tau, \qquad t \geq 0,\ \theta \in [-r, 0].$$

Obviously y has a continuous derivative for $t + \theta \in [0, \infty)$. When $t + \theta \in [-r, 0]$, y has a continuous derivative since $\psi \in D(A) \subset C^1([-r, 0], \mathbf{C}^l)$. For a proof that $y \in C^1([-r, \infty), \mathbf{C}^l)$, it is enough to show that the left and the right derivatives of y at zero are equal which follows from (8). Therefore $y'(t)$, $t \geq -r$, is a solution of Eq.(5) with an IC $\psi' = A\psi$. Then

$$\Big(y(t + \theta)\Big)' = \Big(\mathcal{T}(t)A\psi\Big)(\theta), \qquad t \geq 0, \qquad \theta \in [-r, 0].$$

By (7) it is easy to verify that

$$\left(\int_0^t U(t+\theta-\tau)f(\tau)d\tau\right)' = \int_0^t U(t+\theta-\tau)f'(\tau)d\tau + U(t+\theta)f(0)$$

for $(t+\theta) \in [-r,0) \cup (0,\infty)$. The assumption $f(0)=0$ gives that the second addend in (9) also belongs to $C^1([-r,\infty),\mathbf{C}^l)$.

In this way we prove Lemma 2 for $i=1$. Repeating the construction above, we prove that Lemma 2 is valid for $i=0,1,\ldots,p$. $\square$

Lemma 3. *Let y be the solution of* Eq.(5) *with an IC $\psi \in X$. Then, for a real β of bounded variation on $[0,\kappa]$, it holds*

$$\int_0^\kappa y(s+\theta)d\beta(s) = \left(\int_0^\kappa \mathcal{T}(s)\psi d\beta(s)\right)(\theta), \qquad \theta \in [-r,0].$$

Here on the left-hand side of the equality stays the usual Stieltjes integral while on the right-hand side stays the abstract one.

Proof. Let $0 = s_0 < s_1 < \ldots < s_q = \kappa$ with $|s_i - s_{i-1}| \le \varepsilon$ and $\xi_i \in [s_i, s_{i-1}]$, $1 \le i \le q$. Then

$$\sum_{i=1}^q (\beta(s_i) - \beta(s_{i-1}))\, \mathcal{T}(\xi_i)\psi \overset{\varepsilon\to 0}{\rightarrow} \int_0^\kappa \mathcal{T}(s)\psi d\beta(s).$$

The convergence in X is the uniform convergence of functions on $[-r,0]$. Hence, for $\theta \in [-r,0]$, it holds

$$\left(\int_0^\kappa \mathcal{T}(s)\psi d\beta(s)\right)(\theta) \overset{\varepsilon\to 0}{\leftarrow} \sum_{i=1}^q (\beta(s_i) - \beta(s_{i-1}))\, \mathcal{T}(\xi_i)\psi(\theta) =$$

$$= \sum_{i=1}^q (\beta(s_i) - \beta(s_{i-1}))\, y(\xi_i+\theta) \overset{\varepsilon\to 0}{\rightarrow} \int_0^\kappa y(s+\theta)d\beta(s).$$

We are ready to offer a proof of Theorem 1 in which we assume that the initial point a is equal to zero. This assumption is not a restriction since the main part of Eq.(1) is autonomous.

Proof of Theorem 1. Existence. First, in the case when $p \ge 1$, we suppose that f satisfies $(*)$ of Lemma 2. When $p=0$ we do not need this regularity condition. Let $\psi \in D(A^p) \equiv D(\mathcal{P})$. Lemma 2 gives that the solution $x(\cdot; f, 0)$ of Eq.(1), with an IC ψ, belongs to $C^p([-r,\infty),\mathbf{C}^l)$. This allows us to set

$$z(t) \overset{\text{def}}{=} \langle \eta(s), x(t+s; f, 0)\rangle, \qquad t \ge -r.$$

It is easy to see that z is a solution of the homogeneous IVP

$$\begin{aligned} z'(t) &= \int_{-r}^{0} dH(\theta)z(t+\theta), \quad t \geq 0, \\ z(\theta) &= \langle \eta(s), x(s+\theta; f, 0)\rangle, \quad \theta \in [-r, 0]. \end{aligned}$$

By Lemmas 2 and 3 we find

$$z(\theta) = \Big(\mathcal{P}\psi\Big)(\theta) + F(\theta), \qquad \theta \in [-r, 0],$$

where

$$F(\theta) \stackrel{\text{def}}{=} \sum_{i=0}^{p} \int_{0}^{\kappa} \left(\int_{0}^{s} U(s+\theta-\tau) f^{(i)}(\tau) d\tau \right) d\eta_i(s) \ .$$

Hereafter, in Theorem 4, will be shown that the operator $\mathcal{P}$ is convertible under the conditions of Theorem 1. On the other hand, it is clear that $F \in X$. Choosing $\psi = -\mathcal{P}^{-1}F$ we obtain $z(\theta) = 0$, $\theta \in [-r, 0]$. The uniqueness of the solutions of IVP implies $z(t) = 0$ for $t \geq 0$. Therefore $x(\cdot; f, 0) \in MP(\eta, -r)$.

Further we leave the assumption that f satisfies $(*)$. Naturally, in what follows, we suppose $p \geq 1$.

Consider two cases.

I. *The function* Φ *has a finite number of zeros.*

Then in view of the fact that Φ is an entire function of an exponential type we get

$$\Phi(\lambda) = P(\lambda)exp(\nu\lambda)$$

where P is a polynomial and $\nu \in \mathbf{R}$. Furthermore, one can find that P is of p-th degree with a major coefficient γ and $\nu = 0$. Then $MP(\eta, \cdot)$ consists of the functions satisfying the homogeneous equation with scalar coefficients

$$\gamma g^{(p)}(t) + \sum_{i=0}^{p-1} e_i g^{(i)}(t) = 0$$

i.e. the set of the quasipolynomials

$$\sum_{j} \sum_{i} t^i exp(\mu_j t) c_{ij}, \qquad \forall c_{ij} \in \mathbf{C}^l,$$

where $\mu_1, \ldots$ are the zeros of P /which coincide with the zeros of Φ/ and the sums are taken in the well-known way. It is not difficult to show that the map

$$g(t) \to g'(t) - \int_{-r}^{0} dH(\theta)g(t+\theta),$$

with $\chi(\mu) \neq 0$, transforms one-to-one the quasipolynomial class

$$\sum_{i=0}^{q} t^i exp(\mu t) c_i, \qquad \forall c_i \in \mathbf{C}^l.$$

Then the solution that we are looking for can be found as a quasipolynomial since $\chi(\mu_j) \neq 0,\ j = 1, \dots,$ in accordance with the conditions of Theorem 1.

II. *The function Φ has an infinite number of zeros.*

Let $\lambda_1, \dots, \lambda_p$ are different zeros of Φ and

$$\overline{f}(t) \stackrel{\text{def}}{=} \sum_{i=1}^{p} exp(\lambda_i t) a_i, \qquad \forall a_i \in \mathbf{C}^l.$$

The functions of this type belong to $MP(\eta, -\infty)$. Let also $a_i,\ i = 1, \dots, p$, are chosen such that

$$f^*(t) \stackrel{\text{def}}{=} f(t) - \overline{f}(t)$$

satisfies $(*)$. It can be done since the determinant of the system

$$f^{(j)}(0) - \sum_{i=1}^{p} (\lambda_i)^j\, a_i = 0, \qquad 0 \leq j \leq p-1,$$

is not equal to zero. Let $x^*(\cdot; f^*, 0)$ be a $MP(\eta, -r)$-solution of Eq.(1) which corresponds to f^*. Then we can find a $MP(\eta, -r)$-solution for f in a form

$$x^*(t; f^*, 0) + \sum_{i=1}^{p} exp(\lambda_i t) b_i$$

where $b_i \in \mathbf{C}^l,\ 1 \leq i \leq p$, must be determined by the equalities

$$\left(E\lambda_i - \int_{-r}^{0} dH(s) exp(\lambda_i s) \right) b_i = a_i, \qquad 1 \leq i \leq p.$$

Suppose that some matrix in the latter degenerates. Then the corresponding $\lambda_.$ should be in $\sigma(A)$. This contradicts the condition $\Phi(\lambda_.) \neq 0$.

Uniqueness. Let $x_1(\cdot; f, 0), x_2(\cdot; f, 0) \in MP(\eta, -r)$ are solutions of Eq.(1) and denote with $\overline{x}(\cdot; f, 0)$ their difference. Let also

$$\psi_i(\theta) \stackrel{\text{def}}{=} x_i(\theta; f, 0), \qquad i = 1, 2, \qquad \theta \in [-r, 0].$$

Then $\overline{x}(\cdot; f, 0) \in C^p([-r, \infty), \mathbf{C}^l)$ is a solution of Eq.(1) with $f \equiv 0$. One can find immediately that $\psi_1 - \psi_2 \in D(A^p)$. Then as in the beginning of the proof we obtain

$\mathcal{P}(\psi_1 - \psi_2) = 0$. This fact, with the convertibility of $\mathcal{P}$, result in $\psi_1 = \psi_2$. Therefore $x_1(\cdot; f, 0) \equiv x_2(\cdot; f, 0)$. $\square$

Convertibility of $\mathcal{P}$. Rewrite Φ in the form

$$\Phi(\lambda) = \lambda^p \left(\gamma + \int_0^\kappa exp(\lambda s) d\eta_p^\circ(s) + \sum_{i=0}^{p-1} \lambda^{i-p} \int_0^\kappa exp(\lambda s) d\eta_i(s) \right) \stackrel{\text{def}}{=} \lambda^p \left(\gamma + \Psi(\lambda) \right).$$

There exist nondecreasing functions

$$\eta_i^+,\ \eta_i^-,\ \ 0 \le i \le p-1; \quad \eta_p^{\circ +},\ \eta_p^{\circ -};$$

for which

$$\eta_i(s) = \eta_i^+(s) - \eta_i^-(s), \qquad \eta_p^\circ(s) = \eta_p^{\circ +}(s) - \eta_p^{\circ -}(s), \qquad s \in [0, \kappa].$$

Put

$$\Psi^*(\lambda) \stackrel{\text{def}}{=} \int_0^\kappa exp(\lambda s) d\eta_p^{\circ +}(s) + \int_0^\kappa exp(\lambda s) d\eta_p^{\circ -}(s) +$$

$$+ \sum_{i=0}^{p-1} (-\lambda)^{i-p} \int_0^\kappa exp(\lambda s) d\eta_p^+(s) + \sum_{i=0}^{p-1} (-\lambda)^{i-p} \int_0^\kappa exp(\lambda s) d\eta_p^-(s).$$

Of course, when $p = 0$ there are no sums in the above expressions.

Lemma 4. *Let $\lambda \in (-\infty, 0)$. Suppose that m and k are whole nonnegative numbers. Then*

$$\left| \left(\lambda^{-p} \Psi^m(\lambda) \right)^{(k)} \right| \le \left((-\lambda)^{-p} \left(\Psi^*(\lambda) \right)^m \right)^{(k)}.$$

Moreover

$$\lim_{Re\mu \to -\infty} |\Psi^*(\mu)| = \lim_{Re\mu \to -\infty} |\Psi(\mu)| = 0$$

which in particular gives that, in the case when $\gamma \neq 0$, there is a constant $\omega > 0$ such that

$$|\gamma|^{-1} |\Psi(\mu)| \le \frac{1}{2}, \quad |\gamma|^{-1} |\Psi^*(\mu)| \le \frac{1}{2}, \qquad Re\mu \le -\omega.$$

Proof. Denote

$$S(0,0) = 1\ ;\ S(0,i) = 0\ ,\ i \ge 1\ ;\ S(j,0) = 1\ ,\ j \ge 1\ ;$$

$$S(j,i) = j. \dots .(j+i-1) \quad j, i \ge 1.$$

Let $\xi : [0,\kappa] \to \mathbf{R}$ be nondecreasing and $j \geq 0$. Then the right-hand side of the equality

$$\left((-\lambda)^{-j}\int_0^\kappa exp(\lambda s)d\xi(s)\right)^{(k)} = \sum_{i=0}^{k}\binom{k}{i}S(j,i)(-\lambda)^{-j-i}\int_0^\kappa s^{k-i}exp(\lambda s)d\xi(s)$$

is nonnegative for $\lambda < 0$. By the definitions of Ψ and Ψ^* we see that both sides of the inequality, in the first part of Lemma 4, contain the same addends which differ, eventually, by a sign. The previous conclusion implies that the addends in the right-hand side are certainly nonnegative.

The proof of the second part follows from the fact that η_p° is continuous at zero. $\square$

Following [8], we propose

Lemma 5. *Let $\mathcal{S}(t)$, $t \geq 0$, be a strongly continuous semigroup of linear operators in a Banach space and let B be its generator. Then the operator*

$$\mathcal{Q} \stackrel{\text{def}}{=} \sum_{i=0}^{p}\int_0^\kappa \mathcal{S}(s)B^i \cdot d\eta_i(s)$$

is closed.

Lemma 5 can be proved in a common way using an induction with respect to p. On the other hand, this assertion is clear. For these reasons we omit the proof.

Our preparation finishes with

Theorem 3. *Let $\gamma \neq 0$ and let ω be chosen as in Lemma 4. Let also $\mathcal{S}(t)$, $t \geq 0$, be a strongly continuous semigroup of linear operators in a Banach space Y with a generator B for which*

$$\|\mathcal{S}(t)\| \leq Mexp(-2\omega t), \qquad t \geq 0.$$

Then the operator $\mathcal{Q}$, defined in Lemma 5, is convertible.

Proof. The domain of $\mathcal{Q}$ coincides with $D(B^p)$ which is dense in Y. Under the conditions of Theorem 3, it follows that there are no points of $\sigma(B)$ in the half-plane $Re\lambda > -2\omega$. For a fixed $x \in Y$, it holds uniformly with respect to $t \in [0,\kappa]$ (see Yoshida [9])

(10) $$\lim_{n\to\infty} exp(B_n t)x = \mathcal{S}(t)x$$

where

$$B_n = nJ_n - nI, \quad J_n = n(nI - B)^{-1}, \qquad n \geq 0.$$

Here I is the identity. According to the general theory (see [8, 9]) we have

(11) $$\|(J_n)^k\| \leq Mn^k(n+2\omega)^{-k}, \qquad k,n \geq 0.$$

Note that M is a constant independent of n and k. Consider the operators

$$\mathcal{Q}_n \stackrel{\text{def}}{=} \sum_{i=0}^{p} \int_0^{\kappa} exp(B_n s)(B_n)^i d\eta_i(s), \qquad n \geq 0.$$

The boundedness of B_n implies $\mathcal{Q}_n = \Phi(B_n)$. Also $\mathcal{Q}_n$, $n \geq 0$, commute. Then

$$(B_n)^j x = (J_n)^j B^j x, \qquad x \in D(B^j), \qquad j, n \geq 0,$$

$$\lim_{n \to \infty} J_n x = x, \qquad x \in Y,$$

together with (10), yield

$$\lim_{n \to \infty} \mathcal{Q}_n x = \mathcal{Q} x, \qquad x \in D(B^p). \tag{12}$$

Let

$$\Delta(n) \stackrel{\text{def}}{=} \{\lambda \in \mathbf{C} : |\lambda + n| < n - \omega\}, \ \Gamma(n) \stackrel{\text{def}}{=} \{\lambda \in \mathbf{C} : |\lambda + n| = n - \omega\}, \quad n > \omega.$$

Below we assume $\boldsymbol{n}$ as an arbitrary whole number greater than $\mathbf{2}\boldsymbol{\omega}\mathbf{+1}$. Then $n - \omega > n^2(n + 2\omega)^{-1}$ and the inequalities (11) imply that, for $|n + \lambda| = |n - \omega|$ it holds

$$\|(n(n + \lambda)^{-1} J_n)^k\| < M(\mathbf{q}(n))^k, \qquad n, k \geq 0,$$

with a constant

$$\mathbf{q}(n) \stackrel{\text{def}}{=} \frac{n^2}{(n - \omega)(n + 2\omega)} < 1.$$

Therefore the equality

$$\lambda I - B_n = (n + \lambda)\left(I - n(n + \lambda)^{-1} J_n\right), \quad n + \lambda \neq 0,$$

provides that the representation

$$(\lambda I - B_n)^{-1} = \sum_{k=0}^{\infty} (n + \lambda)^{-k-1} n^k (J_n)^k \tag{13}$$

holds for $\lambda \notin \Delta(n) \cup \Gamma(n)$. Thus, for the spectrum of B_n, we obtain $\sigma(B_n) \subset \Delta(n)$. The disk $\Delta(n)$ belongs to the half-plane $Re\lambda \leq -\omega$. Moreover, according to Lemma 4, we have

$$\Phi(\lambda)^{-1} = \gamma^{-1} \lambda^{-p} \left(1 + (-1)^m \sum_{m=1}^{\infty} (\gamma^{-1} \Psi(\lambda))^m\right), \qquad Re\lambda \leq -\omega, \tag{14}$$

which implies that Φ^{-1} is analytic in a neighbourhood of $\Delta(n) \cup \Gamma(n)$. Then the operator $\mathcal{Q}_n^{-1}$ exists and

$$\mathcal{Q}_n^{-1} = \frac{1}{2\pi\mathbf{i}} \int_{\Gamma(n)} (\Phi(\lambda))^{-1}(\lambda I - B_n)^{-1} d\lambda.$$

The contour $\Gamma(n)$ is assumed to be counterclockwise. The uniform convergence in (14) allows us to interchange the sum and integral. Then

$$\mathcal{Q}_n^{-1} = \sum_{m=0}^{\infty} (-1)^m G_m$$

where

$$G_m \stackrel{\text{def}}{=} \frac{1}{\gamma 2\pi\mathbf{i}} \int_{\Gamma(n)} \lambda^{-p}(\gamma^{-1}\Psi(\lambda))^m (\lambda I - B_n)^{-1} d\lambda.$$

By (13) the latter becomes

$$G_m = \gamma^{-1-m} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\lambda^{-p}(\Psi(\lambda))^m\right)^{(k)} \Big|_{\lambda=-n} n^k (J_n)^k \tag{15}$$

The inequalities $\|(n-\omega)^{-k} n^k (J_n)^k\| \leq M,\ k \geq 0$, and Lemma 4 imply

$$\|G_m\| \leq M\gamma^{-1-m} \sum_{k=0}^{\infty} \frac{1}{k!} \left((-\lambda)^{-p}(\Psi^*(\lambda))^m\right)^{(k)} \Big|_{\lambda=-n} (n-\omega)^k$$

in which the sum is the Taylor's series of the function

$$(-\lambda)^{-p}(\Psi^*(\lambda))^m$$

centered at $-n$ and taken for $-\omega$. Again in accordance with Lemma 4

$$\|G_m\| \leq M|\gamma|^{-1}|\gamma|^{-m}\omega^{-p}(\Psi^*(-\omega))^m \leq M\omega^{-p}|\gamma|^{-1}2^{-m}, \qquad m \geq 0. \tag{16}$$

Adding these inequalities we get

$$\|\mathcal{Q}_n^{-1}\| \leq 2M\omega^{-p}|\gamma|^{-1},$$

i.e. the norm of $\mathcal{Q}_n^{-1}$ has an independent of n upper bound which is the central point of the proof. The inequalities

$$\|\mathcal{Q}_i^{-1}x - \mathcal{Q}_j^{-1}x\| \leq \|\mathcal{Q}_i^{-1}\mathcal{Q}_j^{-1}\| \|\mathcal{Q}_i x - \mathcal{Q}_j x\|, \qquad i,j \geq 2\omega+1, \qquad x \in D(\mathcal{Q}),$$

and (12) give that the sequence $\{\mathcal{Q}_n^{-1}\}$ converges on $D(\mathcal{Q})$ to a bounded operator $\mathcal{B} : Y \to Y$ for which we will prove that it is the converse one of $\mathcal{Q}$. By the equality

$$\mathcal{Q}_n^{-1}(\mathcal{Q}_n x - \mathcal{Q}x) + \mathcal{Q}_n^{-1}\mathcal{Q}x = x, \qquad x \in D(\mathcal{Q}),$$

after going over to a limit, we get

$$\mathcal{B}\mathcal{Q}x = x, \qquad x \in D(\mathcal{Q}).$$

Using induction one can show that, for $\mu \notin \sigma(B)$, it holds

$$B^p(\mu I - B)^{-k}x = (\mu I - B)^{-k}B^p x, \qquad x \in D(B^p), \qquad k \geq 0.$$

The operator B^p is closed. Then by (15) and (16) it is not difficult to see that the operator $\mathcal{Q}_n^{-1}$ transforms into itself $D(B^p)(\equiv D(\mathcal{Q}))$. This allows us to write

$$\mathcal{Q}_n^{-1}(\mathcal{Q}_n x - \mathcal{Q}x) + \mathcal{Q}\mathcal{Q}_n^{-1}x = x, \qquad x \in D(\mathcal{Q}).$$

Now going over to a limit and taking into account that $\mathcal{Q}$ is closed (Lemma 5) we obtain another equality

$$\mathcal{Q}\mathcal{B}x = x, \qquad x \in Y,$$

which completes the proof. $\square$

We are ready to prove

Theorem 4. *Let $\gamma\Phi(\lambda) \neq 0$ for $\lambda \in \sigma(A)$. Then the operator $\mathcal{P}$ is convertible.*

Proof. Let ω be chosen as in Lemma 4 and $\Lambda \stackrel{\text{def}}{=} \{\mu_1, \ldots, \mu_d\}$ be the set of the eigenvalues of A in the half-plane $Re\lambda \geq -2\omega$.

Without loss of generality, we suppose $\Lambda \neq \emptyset$ since we may choose ω à priori sufficiently large and $\sigma(A)$ is certainly nonempty.

The condition $\Phi(\mu_i) \neq 0$, $i = 1, \ldots, d$, gives the existence of a $\varepsilon > 0$ such that the circles $\Gamma_i \stackrel{\text{def}}{=} \{\lambda \in \mathbf{C} : |\lambda - \mu_i| = \varepsilon\}$ do not intersect and $\Phi(\lambda) \neq 0$ for $|\lambda - \mu_i| \leq \varepsilon$. Let

$$\Gamma_\Lambda \stackrel{\text{def}}{=} \bigcup_{i=1}^{d} \Gamma_i.$$

Then we are able to introduce the projector

$$P_\Lambda \stackrel{\text{def}}{=} \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_\Lambda} (\lambda I - A)^{-1} d\lambda.$$

The contour Γ_Λ is assumed to be counterclockwise. As well we can represent X in a direct sum

$$X = X_\Lambda \oplus X_0$$

of invariant with respect to A and $\mathcal{T}(t),\ t \geq 0$, subspaces with

$$X_\Lambda \stackrel{\text{def}}{=} \{\psi \in X : P_\Lambda \psi = \psi\}, \qquad X_0 \stackrel{\text{def}}{=} \{\psi \in X : P_\Lambda \psi = 0\}.$$

Also these subspaces are invariant with respect to $\mathcal{P}$. Further

$$\mathcal{T}(t) = \mathcal{T}_\Lambda(t) + \mathcal{T}_0(t), \qquad t \geq 0,$$

where

$$\mathcal{T}_\Lambda(t) \stackrel{\text{def}}{=} \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_\Lambda} exp(\lambda t)(\lambda I - A)^{-1} d\lambda$$

and $\mathcal{T}_0$ is a strongly continuous semigroup of linear operators in X_0. Let A_0 be the generator of $\mathcal{T}_0$ which, evidently, is the restriction of A in X_0. As in Lemma 7.2.1[1], one can get that $\sigma(A_0)$ contains, eventually, only eigenvalues among the zeros of the characteristic function χ.

In accordance with the same Lemma, each $\mu \in \sigma(A)$ is a pole of the resolvent with degree that is equal to the multiplicity of μ as a zero of χ. Therefore

$$X_\Lambda = \otimes_{i=1}^{d} Ker(\mu_i I - A)^{\nu(i)}$$

where $\nu(i)$ is the multiplicity of μ_i. It is clear that each μ_i is not an eigenvalue of A_0. The operator $\mathcal{T}_0(t)$ is compact for $t \geq r$ since $\mathcal{T}(t)$ has the same property [1]. The latter provides that the semigroup $\mathcal{T}_0(t)$ is uniformly continuous for $t \geq r$. For such a semigroup it holds a spectral-mapping Theorem 16.4.1 [8] which implies

$$\sigma(\mathcal{T}_0(r)) = \{exp(\mu r) : \mu \in \sigma(A_0)\} \cup \{0\}.$$

At this point we may have $\sigma(A_0) = \emptyset$. Then Lemma 7.4.2 [1] gives the existence of a constant M_0 with

$$\|\mathcal{T}_0(t)\| \leq M_0 exp(-2\omega t), \qquad t \geq 0. \tag{17}$$

The convertibility of $\mathcal{P}$ means the same for the operators

$$\mathcal{P}_\Lambda \stackrel{\text{def}}{=} \sum_{i=0}^{p} \int_0^{\kappa} \mathcal{T}_\Lambda(s) A^i \cdot d\eta_i(s) \quad \text{and} \quad \mathcal{P}_0 \stackrel{\text{def}}{=} \sum_{i=0}^{p} \int_0^{\kappa} \mathcal{T}_0(s) A^i \cdot d\eta_i(s)$$

in the corresponding subspaces X_Λ and X_0. One can find immediately that

$$\mathcal{P}_\Lambda^{-1} = \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_\Lambda} (\Phi(\lambda))^{-1} (\lambda I - A)^{-1} d\lambda.$$

The convertibility of $\mathcal{P}_0$ follows from (17) and Theorem 3. $\square$

Notes and examples. The condition $\Phi(\lambda) \neq 0$ for $\lambda \in \sigma(A)$ is necessary for the convertibility of $\mathcal{P}$, in view of the fact that the numbers $\{\Phi(\lambda) : \lambda \in \sigma(A)\}$ are in the spectrum of $\mathcal{P}$. For $p = 0$, Theorem 3 is a corollary of Theorem 16.4.1 [8], since $\mathcal{T}(t)$ is uniformly continuous for $t \geq r$.

When $r \downarrow 0$, Eq.(1) reduces to a system of ordinary differential equations

$$x'(t) = Ax(t) + f(t)$$

where A is a real $l \times l$-matrix. In this case the operator $\mathcal{P} = \Phi(A)$ is bounded and the classical spectral-mapping theorems give that $\sigma(\mathcal{P}) = \{\Phi(\lambda) : \lambda \in \sigma(A)\}$, i.e. $\mathcal{P}$ is convertible iff $\Phi(\lambda) \neq 0$ for $\lambda \in \sigma(A)$. Note that the condition $\gamma \neq 0$ has no role here.

As we note in the Remark after Theorem 2, the equality $\Phi(\lambda) = 0$ may hold only for a finite number elements of $\sigma(A)$ when $\gamma \neq 0$. This allows us to accept the conditions of Theorems 1 and 2 as effective.

Example 1. Let $p = 0$ and let $\eta(\equiv \eta_0)$ be the function defined in the first section. Then $MP(\eta, -\infty)$ consists of the continuous ω-periodic functions and $\Phi(\lambda) = 1 - exp(\lambda\omega)$. Theorem 2 gives the known result that if f is a ω-periodic continuous function and $2k\pi\mathbf{i}/\omega \notin \sigma(A),\ k \in \mathbf{Z}$, then Eq.(1) has a unique ω-periodic solution defined on the whole real axis.

Example 2. Let

$$\langle \eta, g\rangle = P\Big(\frac{d}{ds}\Big)g(s)\Big|_{s=0}$$

where $P(\equiv \Phi)$ is a polynomial. Then $MP(\eta, -\infty)$ consists of the quasipolynomials

$$\sum_j \sum_i t^i exp(\mu_j t) b_{ji}, \qquad \forall b_{ji} \in \mathbf{C}^l,$$

where $\mu_1, \dots$ are the roots of P. The conditions of Theorem 2 reduces to $\mu_j \notin \sigma(A),\ j = 1, \dots$.

The following example is not illustrative as the previous ones.

Example 3. Assume $\gamma \neq 0$ and that η_p has also a jump at κ. Then the forward and the backward IVP for Eq.(2) are well-posed. Thus for a given $\phi \in C^p([0,\kappa], \mathbf{C}^l)$ with

$$\sum_{i=0}^{p} \int_0^{\kappa} \phi^{(i)}(\theta) d\eta_i(\theta) = 0 \tag{18}$$

there is a unique $f(\cdot;\phi) \in MP(\eta, -\infty)$ which coincides with ϕ at $[0,\kappa]$. Theorem 2 says that Eq.(1), with $f(\cdot;\phi)$, has a unique $MP(\eta, -\infty)$-solution, when $\Phi(\lambda) \neq 0$ for λ with $\chi(\lambda) = 0$. We may consider this solution as generated by ϕ.

Note that the convertibility of the operator $\mathcal{Q}$, defined in Theorem 3, is independent of the value of the constant M.

In the general case, the set $MP(\eta, a)$ has a complicated structure. In particular, $MP(\eta, a)$ contains the linear closure, with respect to the natural topology of $C^p([a,\infty), \mathbf{C}^l)$, of the quasiexponential functions $t^j exp(\mu t)v$. Here μ is a zero of Φ, j is less than the multiplicity of μ and $v \in \mathbf{C}^l$. In the regular case, i.e. when η_p have jumps at the both ends of $[0,\kappa]$, $MP(\eta, \cdot)$ admits an internal description as a functions class produced by the initial conditions ϕ satisfying (18).

At the beginning of this section we commented the necessity of the condition $\Phi(\lambda) \neq 0$, $\lambda \in \sigma(A)$, with respect to the solvability of the mean-periodic problem. Actually we do not give a proof of this fact. Note only that the situation is similar to the case of ordinary differential systems.

Besides in Lemma 1, which helps our main result to appear in a natural form (Theorem 2), we use essentially another condition $\gamma \neq 0$ in the proof of Theorem 3. These facts justify $\gamma \neq 0$ as a reasonable assumption in the presented construction. It will be of interest to investigate the rate of its necessity.

It should be interesting also to prove Theorems 1 and 2 for neutral functional differential equations. This problem can be treated in a similar way if we are able to describe a convenient analog of the projector P_Λ.

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