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DEFORMATION LEMMA, LJUSTERNIK-SCHNIRELLMANN THEORY AND MOUNTAIN PASS THEOREM ON C^1–FINSLER MANIFOLDS

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Abstract. Let M be a complete C^1–Finsler manifold without boundary and $f : M \to \mathbb{R}$ be a locally Lipschitz function. The classical proof of the well known deformation lemma can not be extended in this case because integral lines may not exist. In this paper we establish existence of deformations generalizing the classical result. This allows us to prove some known results in a more general setting (minimax theorem, a theorem of Ljusternik-Schnirelmann type, mountain pass theorem). This approach enables us to drop the compactness assumptions characteristic for recent papers in the field using the Ekeland's variational principle as the main tool.

1. Introduction. In [9] R. S. Palais proved the following

Theorem 1.1. *Let M be a complete C^2–Finsler manifold (without boundary) of category k and $f : M \to \mathbb{R}$ be a C^1–function which is bounded below. If f satisfies the Palais-Smale condition then f has at least k distinct critical points.*

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As A. Szulkin points out in [13], Ivar Ekeland posed the question whether this theorem is still valid for a C^1-Finsler manifold M of category k. The classical proof can not be extended to this case bacause integral lines (along which important deformations are constructed) may not exist. A. Szulkin in [13] proved the result of Palais when M is a C^1-Finsler manifold containing a nonempty compact subset of category k, using the Ekeland's variational principle instead of the classical deformation lemma (see [1], [14]). In this paper we answer the Ekeland's question positively. We do it by proving and applying a suitable deformation lemma.

Another motivation for establishing existence of deformations of a C^1-Finsler manifold M is the series of results known as "mountain pass theorem". As far as we know one of the most general results in this series (concerning C^1-functions $f : M \to \mathbb{R}$) is the "min-max principle" of N. Ghoussoub (Theorem 1 and its quantitative version Theorem 1.ter in [5]). It is proved by using the Ekeland's variational principle as the main tool. Our deformation lemma allows us to prove this theorem dropping the compactness assumption on the elements of the deformation stable family $\mathcal{F}$ appearing in its formulation (see Theorem 1 in [5], theorem 1 in [7] and section 5 below). Moreover, we relax the smoothness condition on the function $f : M \to \mathbb{R}$, assuming it locally Lipschitz.

The first to consider locally Lipschitz functions (instead of C^1 ones) in the mountain pass setting was K. C. Chang (see [2]). The fact that "the separating mountain range has positive altitude" is crucial for the proof of his result (as well as for the proof of the classical mountain pass theorem, see [1]). In [7] N. Ghoussoub and D. Preiss established a general mountain pass principle for smooth functions (Gâteaux-differentiable with strong to weak* continuous derivative) in the case of "zero altitude mountain range", replacing the deformations by the Ekeland's variational principle and initiating new perturbation methods. In [11] we generalize the both above mentioned results using deformations. Independently M. Choulli, R. Deville and A. Rhandi applying again the Ekeland's variational principle, obtained in [3] the main result of [11] proving that it includes the general mountain pass principle of [7]. Now it can be considered as a corollary of Theorem 5.2 below.

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The paper is organized as follows. In Section 2 we introduce some basic notions of the theory of the Clarke subdifferential for the case of locally Lipschitz functions on

C^1-manifolds. In Section 3 we prove the main tool for our subsequent investigation – the deformation lemma. Section 4 is devoted to the Ljusternik-Schnirelmann theory for locally Lipschitz functions on C^1-Finsler manifolds and Section 5 – to the "min-max principle" of N. Ghoussoub and the mountain pass theorem in the same setting.

2. Preliminaries. In this section we introduce the necessary notions and the basic relations between them.

Let M be a C^1-Banach manifold modelled on a Banach space E. For $x_0 \in M$, we denote by $T_{x_0}(M)$ the tangent space to M at x_0.

Definition 2.1. *The function $f : M \to \mathbb{R}$ is called locally Lipschitz iff $f \circ \varphi^{-1} : \varphi(O_\varphi) \to \mathbb{R}$ is locally Lipschitz for every chart (O_φ, φ).*

Definition 2.2. *Let $x_0 \in M$, the chart (O_φ, φ) be such that $x_0 \in O_\varphi$, $F_\varphi(y) := f(\varphi^{-1}(y))$ and $\partial F_\varphi(\varphi(x_0))$ be the Clarke subdifferential (cf. [4], Section 2.1.1) of F_φ at $\varphi(x_0) \in E$. The set*

$$\{x^* \in T_{x_0}(M)^* : x^* = \varphi^*(x_0)(y^*) \quad \text{for some} \quad y^* \in \partial F_\varphi(\varphi(x_0))\}$$

where $\varphi^*(x_0) \in \mathcal{L}(E^*, T_{x_0}(M)^*)$ is the mapping adjoint to the differential $\varphi_*(x_0) \in \mathcal{L}(T_{x_0}(M), E)$ of φ at x_0, is called Clarke subdifferential of f at x_0 and is denoted by $\partial f(x_0)$.

Lemma 2.1. *The set $\partial f(x_0)$ does not depend on the chart (O_φ, φ).*

Proof. Let (O_φ, φ) and (O_ψ, ψ) be two charts with $x_0 \in O_\varphi \cap O_\psi$. Let $z^* = (\psi^{-1})^*(\psi(x_0))(x^*)$ where $x^* = \varphi^*(x_0)(y^*)$ for some $y^* \in \partial F_\varphi(\varphi(x_0))$ and $(\psi^{-1})^*(\psi(x_0)) \in \mathcal{L}(T_{x_0}(M)^*, E^*)$ is the mapping adjoint to the differential $(\psi^{-1})_*(\psi(x_0)) \in \mathcal{L}(E, T_{x_0}(M))$ of ψ^{-1} at $\psi(x_0)$. Then

$$z^* = (\psi^{-1})^*(\psi(x_0))[\varphi^*(x_0)(y^*)] = [(\psi^{-1})_*(\psi(x_0))]^* \circ [\varphi_*(x_0)]^*(y^*) =$$
$$= [\varphi_*(x_0) \circ (\psi^{-1})_*(\psi(x_0))]^*(y^*) = [(\varphi \circ \psi^{-1})_*(\psi(x_0))]^*(y^*).$$

Let us define $g : \psi(O_\varphi \cap O_\psi) \to \varphi(O_\varphi \cap O_\psi)$ by $g = \varphi \circ \psi^{-1}$. Then $z^* = [g_*(\psi(x_0))]^*(y^*)$, i.e. $z^* = y^* \circ g_*(\varphi(x_0))$. For $z \in \psi(O_\varphi \cap O_\psi)$ we have

$$F_\psi(z) = f(\psi^{-1}(z)) = f(\varphi^{-1}(\varphi(\psi^{-1}(z)))) = F_\varphi(g(z)).$$

Since $g \in C^1[\psi(O_\varphi \cap O_\psi), \varphi(O_\varphi \cap O_\psi)]$, it is strongly differentiable. Moreover $g_*(\psi(x_0)) = g'(\psi(x_0)) \in \mathcal{L}(E, E)$ is surjective. Therefore according to theorem 2.3.10 in [4] we have

$$\partial F_\psi(\psi(x_0)) = \partial(F_\varphi \circ g)(g^{-1}(\varphi(x_0))) = \partial F_\varphi(g(g^{-1}(\varphi(x_0)))) \circ g'(g^{-1}(\varphi(x_0))) =$$
$$= \partial F_\varphi(\varphi(x_0)) \circ g'(\psi(x_0)) = \partial F_\varphi(\varphi(x_0)) \circ g_*(\psi(x_0)).$$

Hence

$$z^* = y^*(g_*(\psi(x_0)) \in \partial F_\varphi(\varphi(x_0)) \circ g_*(\psi(x_0)) = \partial F_\psi(\psi(x_0)).$$

Since $x^* = \psi^*(x_0)(z^*)$, the lemma is proved. □

Definition 2.3. *Let $x_0 \in M, f : M \to \mathbb{R}$ be locally Lipschitz and $h_0 \in T_{x_0}(M)$. The number*

$$sup\{< x^*, h_0 >: x^* \in \partial f(x_0)\}$$

is called Clarke derivative of f at x_0 in direction h_0 and is denoted by $f^0(x_0, h_0)$.

Remark 2.1. Since $\partial F_\varphi(\varphi(x_0))$ is weak* compact in E^* and $\varphi^*(x_0) : E^* \to T_{x_0}(M)^*$ is weak* to weak* continuous, $\partial f(x_0)$ is weak* compact in $T_{x_0}(M)^*$. Consequently in Definition 2.3 we could have written “max” instead of “sup”.

Lemma 2.2. *Let $x_0 \in M, (O_\varphi, \varphi)$ be a chart with $x_0 \in O_\varphi, h_0 \in T_{x_0}(M)$ and $f : M \to \mathbb{R}$ be a locally Lipschitz function. Then*

$$f^0(x_0, h_0) = \limsup_{y \to \varphi(x_0)\ t \downarrow 0} \frac{f(\varphi^{-1}(y + t\varphi_*(x_0)(h_0))) - f(\varphi^{-1}(y))}{t}$$

P r o o f. We have that

$$\begin{aligned}
&f^0(x_0, h_0) = \max\{< x^*, h_0 >: x^* \in \partial f(x_0)\} = \\
&= \max\{< x^*, h_0 >: x^* = \varphi^*(x_0)(y^*) \ \text{ for some } \ y^* \in \partial(f \circ \varphi^{-1})(\varphi(x_0))\} = \\
&= \max\{< \varphi^*(x_0)(y^*), h_0 >: y^* \in \partial(f \circ \varphi^{-1})(\varphi(x_0))\} = \\
&= \max\{< y^*, \varphi_*(x_0)(h_0) >: y^* \in \partial(f \circ \varphi^{-1})(\varphi(x_0))\} = \\
&= (f \circ \varphi^{-1})^0(\varphi(x_0), \varphi_*(x_0)(h_0)) = \limsup_{y \to \varphi(x_0)\ t \downarrow 0} \frac{f(\varphi^{-1}(y + t\varphi_*(x_0)(h_0))) - f(\varphi^{-1}(y))}{t}
\end{aligned}$$

where the last equality is the definition of the Clarke derivative of the Lipschitz function $f \circ \varphi^{-1} : E \to \mathbb{R}$ at the point $\varphi(x_0) \in E$ in direction $\varphi_*(x_0)(h_0) \in E$ ([4], 3.2.1). This completes the proof. □

We proceed with recalling a basic notion for the present paper: the C^1-Finsler manifold (cf. [9], [7], [6]).

Definition 2.4. *Let M be a C^1-Banach manifold with $T(M)$ as tangent bundle and $T_x(M)$ as tangent space at the point x. A Finsler structure on $T(M)$ is a continuous function $\|.\| : T(M) \to [0, +\infty)$ such that:*

(a) *for each $x \in M$ the restriction $\|.\|_x$ of $\|.\|$ to $T_x(M)$ is a norm on the latter;*

(b) *for each $x_0 \in M$ and $k > 1$ there is a neighbourhood U of x_0 such that*

$$\frac{1}{k}\|.\|_x \leq \|.\|_{x_0} \leq k\|.\|_x$$

for all $x \in U$.

The C^1-Banach manifold M equipped with a Finsler structure is called C^1-Finsler manifold. If now $\sigma : [a, b] \to M$ is a C^1-path, the length of σ is defined by $L(\sigma) = \int_a^b \|\dot{\sigma}(t)\|dt$. The distance $\rho(x, y)$ between two points x and y in the same connected component of M is defined as the infimum of $L(\sigma)$ over all σ joining x and y. The function ρ is then a metric on each component of M (called the Finsler metric), and it generates the original topology of M.

The C^1-Banach manifold being endowed with a Finsler structure, it is natural to ask whether Definition 2.1 is consistent with the usual Lipschitz property on a metric space. The answer is "yes" and it is provided by the following

Lemma 2.3. *Let M be a C^1-Finsler manifold without boundary. Then $f : M \to \mathbb{R}$ is locally Lipschitz according to Definition* 2.1 *iff for every $x_0 \in M$ there exists an open neighbourhood U of x_0 and a positive K_U such that*

$$|f(x) - f(y)| \leq K_U \ . \ \rho(x, y)$$

for every x, y from U.

Proof. Let $f : M \to \mathbb{R}$ has the property from the formulation of this lemma. Let (O_φ, φ) be an arbitrary chart with $x_0 \in O_\varphi$ and $k > 1$. Let $U_1 \subset O_\varphi$ be an open neighbourhood of x_0 with the propery (b) of Definition 2.4. Let $U_2 \subset U_1 \cap U$ be an open neighbourhood of x_0 such that $\varphi(U_2)$ is convex in E. Let us fix x and y in U_2 and let $s(t) = \varphi^{-1}(\varphi(x) + t(\varphi(y) - \varphi(x)))$ for $t \in [0, 1]$. Since $s(t) \subset U_2$ and $\dot{s}(t) = \varphi_*^{-1}(\varphi(s(t)))(\varphi(y) - \varphi(x))$ we have

$$\begin{aligned} |f(\varphi^{-1}(\varphi(x))) - f(\varphi^{-1}(\varphi(y)))| = |f(x) - f(y)| \leq K_U \ . \ \rho(x, y) \leq \\ \leq K_U \int_0^1 \|\varphi_*^{-1}(\varphi(s(t)))(\varphi(y) - \varphi(x))\|_{T_{s(t)}(M)} dt \leq \\ \leq K_U.k \int_0^1 \|\varphi_*^{-1}(\varphi(x_0))(\varphi(y) - \varphi(x))\|_{T_{x_0}(M)} dt \leq \\ \leq K_U.k\|\varphi_*^{-1}(\varphi(x_0))\|_{\mathcal{L}(E, T_{x_0}(M))}\|\varphi(y) - \varphi(x)\|_E, \end{aligned}$$

i.e. $f \circ \varphi^{-1}$ is locally Lipschitz.

In order to prove the reverse implication we need the following

Claim. *Let M be a C^1-Finsler manifold, $x \in M, (O_\varphi, \varphi)$ be a chart such that $x_0 \in O_\varphi, k > 1$ and $U \subset O_\varphi$ be a neighbourhood of x_0 corresponding to (b) of Definition 2.4., i.e.*

$$\frac{1}{k}\|\varphi_*^{-1}(\varphi(x))(v)\|_{T_x(M)} \leq \|\varphi_*^{-1}(\varphi(x_0))(v)\|_{T_{x_0}(M)} \leq k\|\varphi_*^{-1}(\varphi(x))(v)\|_{T_x(M)}$$

for each $v \in E$ and each $x \in U$. Then

$$\|\varphi_*(x)\|_{\mathcal{L}(T_x(M),E)} \leq k\|\varphi_*(x_0)\|_{\mathcal{L}(T_{x_0}(M),E)}$$

holds true for every $x \in U$.

Proof of the claim. It is easy to check that if X and Y are two Banach spaces, $A \in \mathcal{L}(X,Y)$ is injective and surjective and $A^{-1} \in \mathcal{L}(Y,X)$ is its inverse, then

$$\|A^{-1}\|_{\mathcal{L}(Y,X)} = (\inf\{\|Ax\|_Y : \|x\|_X = 1\})^{-1}.$$

Let $x \in U$ be fixed. We have

$$\|\varphi_*(x)\|_{\mathcal{L}(T_x(M),E)} = (\inf\{\|\varphi_*^{-1}(\varphi(x))(v)\|_{T_x(M)} : \|v\|_E = 1\})^{-1} \leq$$
$$\leq k(\inf\{\|\varphi_*^{-1}(\varphi(x_0))(v)\|_{T_{x_0}(M)} : \|v\|_E = 1\})^{-1} = k\|\varphi_*(x_0)\|_{\mathcal{L}(T_{x_0}(M),E)}$$

which proves the claim.

Turning back to the proof of Lemma 2.3, let $f : M \to \mathbb{R}$ be locally Lipschitz according to Definition 2.1. Let $x_0 \in M$ be fixed and the chart (O_φ, φ) be such that $x_0 \in O_\varphi$. Let $k > 1, U \subset O_\varphi$ be an open neighbourhood of x_0 with the property (b) of Definition 2.4 and $f \circ \varphi^{-1}$ be Lipschitz on $\varphi(U)$ with some positive K_U. Let $r \in (0,1)$ be such that $\{z \in M : \rho(x_0, z) \leq 2r\} \subset U, V = \{z \in M : \rho(x_0, z) < \frac{r}{2}\}$ and $x, y \in V$ be arbitrary fixed. Let $\sigma_r : [0,1] \to M$ be a C^1-path connecting x and y and satisfying $\int_0^1 \|\dot{\sigma}_r(t)\|dt \leq \rho(x,y)(1+r)$. We claim that $\{\sigma_r(t) : t \in [0,1]\} \subset U$. If it was not the case, $\sigma_r(t_0) \notin U$ for some $t_0 \in (0,1)$. Then

$$\int_0^1 \|\dot{\sigma}_r(t)\|dt = \int_0^{t_0} \|\dot{\sigma}_r(t)\|dt + \int_{t_0}^1 \|\dot{\sigma}_r(t)\|dt \geq$$
$$\geq \rho(x, \sigma_r(t_0)) + \rho(\sigma_r(t_0), y) > 1.5r + 1.5r = 3r.$$

On the other hand

$$\int_0^1 \|\dot{\sigma}_r(t)\|dt \leq \rho(x,y)(1+r) < r(1+r) = r + r^2$$

which contradicts the above inequality. Finally, since $\frac{d}{dt}(\varphi \circ \sigma_r)(t) = \varphi_*(\sigma_r(t))(\dot{\sigma}_r(t))$ we have

$$|f(x) - f(y)| = |(f \circ \varphi^{-1})(\varphi(x)) - (f \circ \varphi^{-1})(\varphi(y))| \leq K_U \|\varphi(x) - \varphi(y)\| \leq$$
$$K_U \int_0^1 \|\frac{d}{dt}(\varphi \circ \sigma_r)(t)\|_E dt \leq K_U \int_0^1 \|\varphi_*(\sigma_r(t))\|_{\mathcal{L}(T_{\sigma_r(t)}(M),E)} \|\dot{\sigma}_r(t))\|_{T_{\sigma_r(t)}(M)} dt \leq$$
$$\leq K_U \; k \|\varphi_*(x_0)\|_{\mathcal{L}(T_{x_0}(M),E)}(1+r)\rho(x,y)$$

and Lemma 2.3 is proved. □

In Section 4 we shall need the equivalent of Proposition 2.1.1.a) from [4] for the case of locally Lipschitz function defined on C^1-Finsler manifold instead of on Banach space. This equivalent is

Lemma 2.4. *Let M be a C^1-Finsler manifold without boundary, $x_0 \in M, h_0 \in T_{x_0}(M)$ and $f : M \to \mathbb{R}$ be Lipschitz around x_0 with constant K. Then*

$$|f^0(x_0, h_0)| \leq K \|h_0\|_{T_{x_0}(M)}.$$

Proof. Let the chart (O_φ, φ) be such that $x_0 \in O_\varphi, k > 1$ and let $U \subset O_\varphi$ be an open neighbourhood of x_0 corresponding to (b) of definition 2.4 and having the properties:

— $\varphi(U)$ is convex in E;

— f is Lipschitz in U with constant K.

Let $y \in \varphi(U), t_0 > 0$ be such that $y + t_0\varphi_*(x_0)(h_0) \in \varphi(U)$ and $s(t) = \varphi^{-1}(y + t\varphi_*(x_0)(h_0))$ for $t \in [0, t_0]$. Then

$$\frac{|f(\varphi^{-1}(y + t\varphi_*(x_0)(h_0))) - f(\varphi^{-1}(y)|}{t} =$$
$$= \frac{|f(s(t)) - f(s(0))|}{t} \leq \frac{K}{t} \int_0^t \|\dot{s}(\tau))\|_{T_{s(\tau)}(M)} d\tau \leq$$
$$\leq \frac{K.k}{t} \int_0^t \|(\varphi_*^{-1}(\varphi(x_0))\varphi_*(x_0)(h_0)\|_{T_{x_0}(M)} = K.k \|h_0\|_{T_{x_0}(M)}.$$

Hence

$$|f^0(x_0, h_0)| = |\limsup_{\substack{y \to \varphi(x_0) \\ t \downarrow 0}} \frac{f(\varphi^{-1}(y + t\varphi_*(x_0)(h_0))) - f(\varphi^{-1}(y))}{t}| \leq$$
$$\leq \limsup_{\substack{y \to \varphi(x_0) \\ t \downarrow 0}} \frac{|f(\varphi^{-1}(y + t\varphi_*(x_0)(h_0))) - f(\varphi^{-1}(y))|}{t} \leq K.k \; \|h_0\|_{T_{x_0}(M)}.$$

Letting $k \downarrow 1$ we finish the proof. $\square$

A concluding definition remains to be given in these preliminaries. It introduces the notion "steepness" (for a locally Lipschitz function f) which, if negative at some point x, means that there is a direction in which we can "go down", starting from $(x, f(x))$ and following the graph of f.

Definition 2.5. *Let M be a C^1-Finsler manifold, $x_0 \in M$ and $f : M \to \mathbb{R}$ be locally Lipschitz. The number*

$$\inf\{f^0(x_0, h) : h \in T_{x_0}(M), \|h\|_{T_{x_0}(M)} = 1\}$$

is called steepness of f at x_0 and is denoted by $stf(x_0)$.

The steepness for locally Lipschitz functions defined on a Banach space was first introduced in [11], where its relation to a similar notion introduced by K. C. Chang in [2] was discussed. The steepness for locally Lipschitz functions defined on a C^1-Finsler manifold was introduced in [12].

3. Deformation lemma. In this section we establish the existence of deformations of a C^1-Finsler manifold which generalize the well known ones concerning Banach spaces or manifolds of smoothness at least C^{2-}.

The following notation will be used in the sequel:

For any subset S of a metric space X with metric ρ and for every positive α, $S_\alpha = \{x \in M : \text{dist}(x, S) \le \alpha\}$ where $\text{dist}(x, S) = \inf\{\rho(x, y) : y \in S\}$.

Theorem 3.1 (Deformation lemma). *Let M be a complete C^1-Finsler manifold without boundary and $f : M \to \mathbb{R}$ be a locally Lipschitz function on it. Let S be a subset of M, c be a real number, ε and δ be positives and $k > 1$. We suppose that $stf(y) < -\dfrac{2\varepsilon}{\delta}$ for every y in an open neighbourhood Q of $f^{-1}([c-\varepsilon, c+\varepsilon]) \cap S_{k\delta}$. Then there exists $\eta \in C([0,1] \times M, M)$ with the following properties:*

(i) *$\eta(0, x) = x$ for every $x \in M$;*

(ii) *$\eta(t, x) = x$ for every $x \in M \setminus Q, t \in [0,1]$;*

(iii) *$\eta(1, f^{-1}((-\infty, c+\varepsilon]) \cap S) \subset f^{-1}((-\infty, c-\varepsilon]) \cap S_{k^2\delta}$;*

(iv) *$\rho(x, \eta(1, x)) \le k^2\delta$ for every $x \in M$, where ρ is the Finsler metric on each component of M.*

Proof. We first prove the deformation lemma for the case of connected M. Without loss of generality we can assume that $Q \subset f^{-1}((c - k\varepsilon, c + k\varepsilon))$.

Step 1. Construction of the open covering $\{U_\gamma\}_{\gamma\in\Gamma}$.

The Finsler structure of M and the steepness condition on f yield for every $x \in Q$ the existence of a neighbourhood U_x of it, a tangent unit vector $h_x \in T_x(M), \|h_x\|_{T_x(M)} = 1$ and a chart $(O_{\varphi_x}, \varphi_x)$ with $U_x \subset O_{\varphi_x}$ such that for every $y \in U_x$ we have

$$f(\varphi_x^{-1}(\varphi_x(y) + t\varphi_{x*}(x)(h_x))) - f(y) < -\frac{2\varepsilon}{\delta}t$$

for every $t \in (0, t_x), t_x > 0$ and

$$\frac{1}{k}\|\varphi_{x*}^{-1}(\varphi_x(y))(v)\|_{T_y(M)} \leq \|\varphi_{x*}^{-1}(\varphi_x(x))(v)\|_{T_x(M)} \leq k\|\varphi_{x*}^{-1}(\varphi_x(y))(v)\|_{T_y(M)}$$

for every v in the Banach space E. Getting a smaller U_x if necessary we can assume

$$f(\varphi_x^{-1}(\varphi_x(y) + t\varphi_{x*}(x)(h_x))) - f(y) < -\frac{2\varepsilon}{\delta}t$$

whenever y and $\varphi_x^{-1}(\varphi_x(y) + t\varphi_{x*}(x)(h_x))$ are in U_x.

The family $\{U_x\}_{x\in Q}$ is an open covering of Q. Let $\{U_\gamma\}_{\gamma\in\Gamma}$ be a locally finite refinement of it and $\{\alpha_\gamma\}_{\gamma\in\Gamma}$ be a Lipschitz partition of unity subordinate to $\{U_\gamma\}_{\gamma\in\Gamma}$, that is $\alpha_\gamma : M \to [0,1]$ are Lipschitz continuous functions, $\alpha_\gamma(x) > 0$ iff $x \in U_\gamma$ and $\sum_{\gamma\in\Gamma} \alpha_\gamma(x) = 1$ for every x in the closed set $\overline{S} \cap f^{-1}([c-\varepsilon, c+\varepsilon])$.

Let us fix $\gamma \in \Gamma$. Then there exists a point $x_\gamma \in Q$ with $U_\gamma \subset U_{x_\gamma}$, a chart φ_γ with $O_{\varphi_\gamma} \supset U_\gamma$ and a vector $h_\gamma = \varphi_{\gamma*}(x_\gamma)(h_{x_\gamma}) \in E$ satisfying

$$f(\varphi_\gamma^{-1}(\varphi_\gamma(y) + th_\gamma)) - f(y) < -\frac{2\varepsilon}{\delta}t \tag{1}$$

whenever y and $\varphi_\gamma^{-1}(\varphi_\gamma(y) + th_\gamma)$ are in U_γ.

The following step gives the basic small deformations of M.

Step 2. Construction of the "elementary deformations" η_γ.

For a fixed $\gamma \in \Gamma$ we define $\eta_\gamma \in C([0,1] \times M, M)$ in the following way :

$$\eta_\gamma(t,x) = \begin{cases} x & \text{if } x \notin U_\gamma \\ \varphi_\gamma^{-1}(z(t, \varphi_\gamma(x))) & \text{if } x \in U_\gamma \end{cases}$$

where $z(t,y)$ is the solution of the Cauchy problem

$$\dot{z} = \alpha_\gamma(\varphi_\gamma^{-1}(z)).h_\gamma \qquad z(0) = y$$

at the moment $t > 0$. Then for every $t \in [0,1]$ and every $x \in M$ we have:

(A) $\rho(x, \eta_\gamma(t,x)) \leq kt\int_0^1 \alpha_\gamma(\eta_\gamma(ts,x))ds$;

(B) $f(\eta_\gamma(t,x)) - f(x) \leq -\frac{2\varepsilon}{\delta}t\int_0^1 \alpha_\gamma(\eta_\gamma(ts,x))ds$;

(C) $\eta_\gamma(t,.) : M \to M$ is a diffeomorphism for each $t \in [0,1]$.

Indeed (A) is obvious for $x \notin U_\gamma$. Otherwise $\sigma(s) = \varphi_\gamma^{-1}(z(ts,\varphi_\gamma(x)))$ is a C^1-path between x and $\eta_\gamma(t,x)$ and hence (using the claim from the proof of Lemma 2.3)

$$\begin{aligned}\rho(x,\eta_\gamma(t,x)) &\leq \int_0^1 \|\dot{\sigma}(s)\|_{T_{\sigma(s)}(M)}ds = \\ &= \int_0^1 \alpha_\gamma(\varphi_\gamma^{-1}(z(ts,\varphi_\gamma(x)))).t\|\varphi_{\gamma *}^{-1}(z(ts,\varphi_\gamma(x)))(h_\gamma)\|_{T_{\sigma(s)}(M)}ds = \\ &= t\int_0^1 \alpha_\gamma(\eta_\gamma(ts,x)).\|\varphi_{\gamma *}^{-1}(\varphi_\gamma(\eta_\gamma(ts,x))) \circ \varphi_{\gamma *}(x_\gamma)(h_{x_\gamma}))\|_{T_{\sigma(s)}(M)}ds \leq \\ &\leq t\int_0^1 \alpha_\gamma(\eta_\gamma(ts,x)).k\|\varphi_{\gamma *}^{-1}(\varphi_\gamma(x_\gamma)) \circ \varphi_{\gamma *}(x_\gamma)(h_{x_\gamma}))\|_{T_{x_\gamma}(M)}ds = \\ &= kt\int_0^1 \alpha_\gamma(\eta_\gamma(ts,x)).\|h_{x_\gamma}\|_{T_{x_\gamma}(M)}ds = kt\int_0^1 \alpha_\gamma(\eta_\gamma(ts,x))ds.\end{aligned}$$

To prove (B) we see that in fact

$$z(t,\varphi_\gamma(x)) = \varphi_\gamma(x) + (\int_0^t \alpha_\gamma(\varphi_\gamma^{-1}(z(\tau,\varphi_\gamma(x))))d\tau)h_\gamma$$

and so

$$f(\eta_\gamma(t,x)) - f(x) < -\frac{2\varepsilon}{\delta}\int_0^t \alpha_\gamma(\varphi_\gamma^{-1}(z(\tau,\varphi_\gamma(x))))d\tau = -\frac{2\varepsilon}{\delta}t\int_0^1 \alpha_\gamma(\eta_\gamma(ts,x))ds$$

by (1) because $x \in U_\gamma$ yields $z(t,\varphi_\gamma(x)) \in \varphi_\gamma(U_\gamma)$ for every t. If $x \notin U_\gamma, \eta_\gamma(ts,x) = x$ and the equality holds. The assertion (C) needs no proof.

Step 3. Composition of all η_γ's.

Let us think of the set Γ as of the ordinal interval $[0,\gamma_0)$ (i.e. let Γ be well ordered). We will construct a family $\{\xi_\gamma : 0 \leq \gamma \leq \gamma_0\}$ of deformations of M as follows:

(a) ξ_0 is the identity map, i.e. $\xi_0(t,x) = x$ for every $x \in M, t \in [0,1]$;

(b) if γ is not a limit ordinal, $\xi_\gamma(t,x) = \eta_{\gamma-1}(t,\xi_{\gamma-1}(t,x))$ for every $x \in M, t \in [0,1]$;

(c) if γ is a limit ordinal, $\xi_\gamma(t,x) = \lim_{\beta<\gamma}\xi_\beta(t,x)$ for every $x \in M, t \in [0,1]$.

In order such an inductive definition to be correct we need to show that the limit in (c) exists.

Lemma 3.1. *Let $\xi_\beta(t,x)$ be well defined as above and $f(\xi_\beta(t,x)) \leq f(\xi_\alpha(t,x))$ for all $\beta < \gamma, \alpha \leq \beta$. Then $\xi_\gamma(t,x)$ is well defined and $f(\xi_\gamma(t,x)) \leq f(\xi_\beta(t,x))$ for all $\beta \leq \gamma$.*

P r o o f. There is nothing to prove when $x \notin Q$. Let $x \in Q$. If γ is not a limit ordinal,

$$f(\xi_\gamma(t,x)) = f(\eta_{\gamma-1}(t,\xi_{\gamma-1}(t,x))) \leq$$
$$\leq f(\xi_{\gamma-1}(t,x)) - \frac{2\varepsilon}{\delta} t \int_0^1 \alpha_{\gamma-1}(\eta_{\gamma-1}(ts,\xi_{\gamma-1}(t,x)))ds \leq f(\xi_{\gamma-1}(t,x))$$

because $\alpha_{\gamma-1}(y) \geq 0$ for each $y \in M$ and we are done.

If γ is a limit ordinal, the generalized sequence $\{f(\xi_\beta(t,x))\}_{\beta<\gamma} \subset [c-k\varepsilon, c+k\varepsilon]$ is decreasing by the induction assumption: $f(\xi_\alpha(t,x)) \leq f(\xi_\beta(t,x))$ whenever $\beta \leq \alpha < \gamma$. Hence it is convergent and so at most countably many terms of the series

$$\sum_{\beta<\gamma} [f(\xi_\beta(t,x)) - f(\xi_{\beta+1}(t,x))]$$

are non-zero, all of them are nonnegative and the series converges.

We will see that

$$\rho(\xi_{\beta_1}(t,x), \xi_{\beta_2}(t,x)) \leq \sum_{\beta_2 \leq \alpha < \beta_1} \rho(\xi_{\alpha+1}(t,x), \xi_\alpha(t,x))$$

whenever $\beta_2 < \beta_1 < \gamma$.

We will proceed by induction on β_1. Indeed, if β_1 is not a limit ordinal, then

$$\rho(\xi_{\beta_1}(t,x), \xi_{\beta_2}(t,x)) \leq \rho(\xi_{\beta_1}(t,x), \xi_{\beta_1-1}(t,x)) + \rho(\xi_{\beta_1-1}(t,x), \xi_{\beta_2}(t,x)) \leq$$
$$\leq \sum_{\beta_2 \leq \alpha < \beta_1} \rho(\xi_{\alpha+1}(t,x), \xi_\alpha(t,x)).$$

If β_1 is a limit ordinal, then $\xi_{\beta_1}(t,x) = \lim_{\beta<\beta_1} \xi_\beta(t,x)$ and hence

$$\rho(\xi_{\beta_1}(t,x), \xi_{\beta_2}(t,x)) = \lim_{\beta<\beta_1} \rho(\xi_\beta(t,x), \xi_{\beta_2}(t,x)) \leq$$
$$\leq \sum_{\beta_2 \leq \alpha < \beta_1} \rho(\xi_{\alpha+1}(t,x), \xi_\alpha(t,x)).$$

Now the inequalities (A) and (B) of step 2 yield

$$\rho(\xi_{\beta+1}(t,x), \xi_\beta(t,x)) \leq kt \int_0^1 \alpha_\beta(\eta_\beta(ts,\xi_\beta(t,x)))ds \leq$$
$$\leq \frac{k\delta}{2\varepsilon} [f(\xi_\beta(t,x)) - f(\xi_{\beta+1}(t,x))]$$

for every $\beta < \gamma$. Therefore

$$\rho(\xi_{\beta_1}(t,x), \xi_{\beta_2}(t,x)) \leq \frac{k\delta}{2\varepsilon} \sum_{\beta_2 \leq \alpha < \beta_1} [f(\xi_\alpha(t,x)) - f(\xi_{\alpha+1}(t,x))].$$

for each $\beta_2 < \beta_1 < \gamma$. But we know that the above series is convergent and hence the generalized sequence $\{\xi_\beta(t,x)\}_{\beta<\gamma}$ is a Cauchy one. As M is complete, its limit exists and the lemma is proved. $\square$

We saw that for every fixed $(t,x) \in [0,1] \times M$ our definition is correct. The following lemmas enable us to show that the so defined ξ_γ's are deformations.

Lemma 3.2. *Let $0 < \gamma \leq \gamma_0$ be a limit ordinal and $(t_0, x_0) \in [0,1] \times M$. Then there exist a non-limit ordinal number $\gamma^* < \gamma$ and a neighbourhood U^* of $\xi_\gamma(t_0,x_0)$ such that $\xi_{\gamma^*}(t,x) = \xi_\gamma(t,x)$ whenever $\xi_{\gamma^*}(t,x) \in U^*$.*

Proof. The family $\{U_\alpha\}_{\alpha<\gamma}$ is locally finite and so there exists a neighbourhood U^* of $\xi_\gamma(t_0,x_0)$ which intersects at most finitely many of its members. Let us denote them by $U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_p}$ and $\alpha^* = \max\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ (if no $U_\alpha, \alpha < \gamma$ intersects U^*, we set $\alpha^* = 0$). Now $\alpha^* < \gamma$ and as γ is a limit ordinal, the non-limit ordinal $\gamma^* = \alpha^* + 1$ is strictly less than γ as well. For the so defined γ^* and U^* we will show that $\xi_\beta(t,x) = \xi_{\gamma^*}(t,x)$ whenever $\xi_{\gamma^*}(t,x) \in U^*$ and $\beta \in [\gamma^*, \gamma]$. We will proceed by induction on β. Indeed, the initial step $\beta = \gamma^*$ is obvious. If β is a non-limit ordinal, then $\xi_\beta(t,x) = \eta_{\beta-1}(t, \xi_{\beta-1}(t,x)) = \eta_{\beta-1}(t, \xi_{\gamma^*}(t,x))$ for $\xi_{\gamma^*}(t,x) \in U^*$ by the induction assumption. But as $\beta - 1 \geq \gamma^* > \alpha^*$, we have $U_{\beta-1} \cap U^* = \emptyset$ and so $\eta_{\beta-1}(t,x) = x$ for every $x \in U^*$. Hence $\xi_\beta(t,x) = \xi_{\gamma^*}(t,x)$ if $\xi_{\gamma^*}(t,x) \in U^*$ and we are done.

If β is limit, $\xi_\beta(t,x) = \lim_{\alpha<\beta} \xi_\alpha(t,x) = \lim_{\alpha<\beta} \xi_{\gamma^*}(t,x) = \xi_{\gamma^*}(t,x)$ for $\xi_{\gamma^*}(t,x) \in U^*$ by the induction assumption.

Therefore $\xi_\gamma(t,x) = \xi_{\gamma^*}(t,x)$ if $\xi_{\gamma^*}(t,x) \in U^*$ and the lemma is proved. $\square$

Lemma 3.3. *Let $(t_0,x_0) \in [0,1] \times M, \gamma \leq \gamma_0$ and U be an arbitrary neighbourhood of $\xi_\gamma(t_0,x_0)$. Then there exist a neighbourhood V of x_0, a neighbourhood W of t_0 and finitely many ordinals $\gamma > \beta_1 > \beta_2 > \cdots > \beta_s \geq 0$ such that $\xi_\gamma(t,x) = \eta_{\beta_1}(t, \eta_{\beta_2}(t, \ldots, \eta_{\beta_{s-1}}(t, \eta_{\beta_s}(t,x)) \ldots))$ and $\xi_\gamma(t,x) \in U$ for every $(t,x) \in W \times V$, i.e. $\xi_\gamma : [0,1] \times M \to M$ is continuous and locally it is a composition of finitely many of the diffeomorphisms $\{\eta_\beta\}_{\beta<\gamma}$.*

Proof. To every ordinal number $\beta \in (0,\gamma]$ we can assign an ordinal number $\overline{\beta} < \beta$ in the following way: $\overline{\beta}$ is $\beta - 1$ if β is nonlimit and $\overline{\beta}$ is $\beta^* - 1$ if β is limit where β^* is the ordinal number assigned to β by the previous lemma. In such a way we obtain a strictly decreasing sequence

$$\gamma = \beta_0 > \overline{\beta}_0 = \beta_1 > \overline{\beta}_1 = \beta_2 > \dots$$

of ordinal numbers. Therefore this sequence is a finite one:

$$\gamma = \beta_0 > \overline{\beta}_0 = \beta_1 > \overline{\beta}_1 = \beta_2 > \dots > \overline{\beta}_{s-1} = \beta_s = 0.$$

We will denote by U_0 the set U.

Let us fix an arbitrary $i \in \{0, 1, 2, \dots, s-1\}$. Now $\xi_{\beta_i}(t, x) = \eta_{\beta_{i+1}}(t, \xi_{\beta_{i+1}}(t, x)) \in U^*$ everywhere if β_i is not limit, and when $\xi_{\beta_i}(t, x) \in U_i^*$ (where U_i^* is the neighbourhood of $\xi_{\beta_i}(t_0, x_0)$ obtained by the previous lemma), if β_i is limit.

Let $\tilde{U}_i$ be U_i itself or $U_i \cap U_i^*$ in the limit-ordinal case. The continuity of $\eta_{\beta_{i+1}} : [0,1] \times M \to M$ at the point $(t_0, \xi_{\beta_{i+1}}(t_0, x_0))$ yields the existence of a neighbourhood U_{i+1} of $\xi_{\beta_{i+1}}(t_0, x_0)$ and a neighbourhood W_i of t_0 such that $\eta_{\beta_{i+1}}(t, y) \in \tilde{U}_i$ whenever $t \in W_i$ and $y \in U_{i+1}$, i.e. $\xi_{\beta_i}(t, x) \in \tilde{U}_i$ for every (t, x) with $t \in W_i$ and $\xi_{\beta_{i+1}}(t, x) \in U_{i+1}$.

To finish the proof we set $V = U_s$ to be the desired neighbourhood of x_0 and $W = \bigcap_{i=0}^{s-1} W_i$ to be the desired neighbourhood of t_0.

Indeed, a back-going induction on $i \in \{0, 1, 2, \dots, s\}$ shows that if $(t, x) \in W \times V$, then $\xi_\gamma(t, x) \in U$ and $\xi_\gamma(t, x) = \eta_{\beta_1}(t, \eta_{\beta_2}(t, \dots (\eta_{\beta_{s-1}}(t, \eta_{\beta_s}(t, x)))))$ by the above paragraph. $\square$

In the notations from the proof of the above lemma we have the following properties of ξ_γ which correspond to the properties (A), (B) and (C) of the elementary deformations $\{\eta_\gamma\}_{\gamma < \gamma_0}$ in step 2:

$$(A') \quad \begin{aligned} \rho(x, \xi_\gamma(t, x)) &\le \sum_{i=0}^{s-1} \rho(\xi_{\beta_i}(t, x), \xi_{\beta_{i+1}}(t, x)) = \\ &= \sum_{i=0}^{s-1} \rho(\eta_{\beta_{i+1}}(t, \xi_{\beta_{i+1}}(t, x)), \xi_{\beta_{i+1}}(t, x)) \le kt \sum_{i=0}^{s-1} \int_0^1 \alpha_{\beta_{i+1}}(\eta_{\beta_{i+1}}(ts, \xi_{\beta_{i+1}}(t, x)))ds; \end{aligned}$$

$$(B') \quad \begin{aligned} f(\xi_\gamma(t, x)) - f(x) &= \sum_{i=0}^{s-1} [f(\xi_{\beta_i}(t, x)) - f(\xi_{\beta_{i+1}}(t, x))] = \\ &= \sum_{i=0}^{s-1} [f(\eta_{\beta_{i+1}}(t, \xi_{\beta_{i+1}}(t, x))) - f(\xi_{\beta_{i+1}}(t, x))] \le \\ &\le -\frac{2\varepsilon}{\delta} t \sum_{i=0}^{s-1} \int_0^1 \alpha_{\beta_{i+1}}(\eta_{\beta_{i+1}}(ts, \xi_{\beta_{i+1}}(t, x)))ds; \end{aligned}$$

(C') $\xi_\gamma(t,.)$ is a local diffeomorphism for every $(t,x)\in W\times V, \gamma\le\gamma_0$.

In particular, $\rho(\xi_\gamma(t,x),x)\le\frac{k\delta}{2\varepsilon}[f(x)-f(\xi_\gamma(t,x))]$ whenever $\gamma\le\gamma_0, x\in M, t\in[0,1]$.

Step 4. Construction of η.

Let us denote the "maximal" of the deformations constructed in the previous step by ξ (i.e. $\xi=\xi_{\gamma_0}$). We will define ξ^n inductively as follows:

$$\xi^0(t,x)=x;\quad \xi^{n+1}(t,x)=\xi(t,\xi^n(t,x))$$

for every $x\in M, t\in[0,1]$ and positive integer n.

To go ahead, one needs to know something about the behaviour of the deformation ξ with respect to the level of the function f.

Lemma 3.4. *Let $x_0\in\overline{S}\cap f^{-1}((-\infty,c+\varepsilon])$ and let $t_0\in(0,1]$ be positive. Then there exist a positive integer m, a neighbourhood U of x_0 and a neighbourhood W of t_0 such that $f(\xi^m(t,x))<c-\varepsilon$ for every $x\in U, t\in W$.*

Proof. Let us consider the sequences $\{\xi^n(t_0,x_0)\}_{n=1}^\infty\subset M$ and $\{f(\xi^n(t_0,x_0))\}_{n=1}^\infty\subset\mathbb{R}$. The second one is decreasing by the property (B') in step 3. Moreover, it is bounded below because either $x_0\notin Q$ and then $\xi^n(t_0,x_0)=x_0$ for every n, or $x_0\in Q$ and then $\eta_\gamma(t_0,x)\in Q$ for $x\in Q,\gamma\in\Gamma$ yield $\xi^n(t_0,x_0)\in Q$. Thus $f(\xi^n(t_0,x_0))\ge c-k\varepsilon$ for every n. Hence the sequence is convergent and so is the series

$$\sum_{n=0}^{\infty}[f(\xi^n(t_0,x_0))-f(\xi^{n+1}(t_0,x_0))].$$

But

$$\rho(\xi^{n+1}(t_0,x_0),\xi^n(t_0,x_0))=\rho(\xi(t_0,\xi^n(t_0,x_0)),\xi^n(t_0,x_0))\le$$
$$\le\frac{k\delta}{2\varepsilon}[f(\xi^n(t_0,x_0))-f(\xi^{n+1}(t_0,x_0))]$$

by the property at the end of the previous step. Therefore $\{\xi^n(t_0,x_0)\}_{n=1}^\infty$ is a Cauchy sequence. Let us denote its limit by z.

We will show that $z\notin Q$. Indeed, let us assume the contrary. Then there exists $\gamma^*\in\Gamma$ so that $z\in U_{\gamma^*}$ and $\alpha_{\gamma^*}(z)>0$. Let $\mu>0$, $r>0$ be two positive reals such that $\alpha_{\gamma^*}(x)\ge\mu$ for every x with $\rho(z,x)\le r$. The convergence of the sequence yields the existence of n_0 with $\rho(z,\xi^n(t_0,x_0))\le\dfrac{r}{2}$ whenever $n\ge n_0$. Let us fix an arbitrary positive integer $n\ge n_0$. By Lemma 3.3 there exists a neighbourhood of $\xi^n(t_0,x_0)$ so that $\xi(t_0,.)$ is a composition of finitely many elementary deformations on it, say

$$\xi=\eta_{\beta_1}\circ\eta_{\beta_2}\circ\ldots\circ\eta_{\beta_s}.$$

The decreasing property from Lemma 3.1 gives

$$f(\xi^{n+1}(t_0,x_0)) \le f(\xi_{\beta_i}(t_0,\xi^n(t_0,x_0))) \le f(\xi^n(t_0,x_0))$$

and hence

$$\rho(\xi^n(t_0,x_0),\xi_{\beta_i}(t_0,\xi^n(t_0,x_0))) \le \frac{k\delta}{2\varepsilon}[f(\xi^n(t_0,x_0)) - f(\xi_{\beta_i}(t_0,\xi^n(t_0,x_0))] \le$$
$$\le \frac{k\delta}{2\varepsilon}\left[f(\xi^n(t_0,x_0)) - f(\xi^{n+1}(t_0,x_0))\right].$$

Therefore

$$\rho(\xi^n(t_0,x_0),\xi_{\beta_i}(t_0,\xi^n(t_0,x_0))) \le \frac{r}{2}$$

for every $i = 0,1,2,...,s$ and for every n which is big enough, say $n \ge n_1 \ge n_0$. So we have $\alpha_{\gamma^*}(\xi_{\beta_i}(t_0,\xi^n(t_0,x_0))) \ge \mu$ for each $i \in \{0,1,2,\dots,s\}$. Moreover, every path $\{\eta_{\beta_{i+1}}(t_0 s,\xi_{\beta_{i+1}}(t_0,\xi^n(t_0,x_0))) : s \in [0,1]\}$ between $\xi_{\beta_{i+1}}(t_0,\xi^n(t_0,x_0))$ and $\xi_{\beta_i}(t_0,\xi^n(t_0,x_0))$ is in $\{x : \rho(x,z) \le r\}$ for $n \ge n_1$ too, because f decreases along the integral lines of $\eta_{\beta_{i+1}}$ and the above argument aplies.

As ξ is actually the composition of all $\{\eta_\gamma\}_{\gamma<\gamma_0}$ and every intermediate point $\xi_\beta(t_0,\xi^n(t_0,x_0))$, $\beta \le \gamma_0$ is in $U_{\gamma*}, \eta_{\gamma_*}$ must act nontrivially, i.e. there exists $i \in \{1,2,\dots,s\}$ with $\gamma^* = \beta_i$. Then

$$\sum_{i=0}^{s-1}\int_0^1 \alpha_{\beta_{i+1}}(\eta_{\beta_{i+1}}(t_0 s,\xi_{\beta_{i+1}}(t_0,\xi^n(t_0,x_0))))ds \ge$$
$$\ge \int_0^1 \alpha_{\gamma^*}(\eta_{\gamma^*}(t_0 s,\xi_{\gamma^*}(t_0,\xi^n(t_0,x_0))))ds.$$

So by (B') and $\eta_{\gamma^*}(t_0 s,\xi_{\gamma^*}(t_0,\xi^n(t_0,x_0))) \subset \{x : \rho(x,z) \le r\}$ we have

$$f(\xi^{n+1}(t_0,x_0)) - f(\xi^n(t_0,x_0)) \le$$
$$\le -\frac{2\varepsilon}{\delta}t_0\int_0^1 \alpha_{\gamma^*}(\eta_{\gamma^*}(t_0 s,\xi_{\gamma^*}(t_0,\xi^n(t_0,x_0))))ds \le -\frac{2\varepsilon}{\delta}t_0\mu.$$

Thus $f(\xi^n(t_0,x_0)) - f(\xi^{n+1}(t_0,x_0)) \ge \frac{2\varepsilon}{\delta}t_0\mu > 0$ for every $n \ge n_1$ which contradicts the convergence of the series $\sum_{n=0}^{\infty}[f(\xi^n(t_0,x_0)) - f(\xi^{n+1}(t_0,x_0))]$. So we proved that $z \notin Q$.

On the other hand

$$\rho(x_0,z) = \lim_{n\to\infty}\rho(x_0,\xi^n(t_0,x_0)) \le \sum_{n=0}^{\infty}\rho(\xi^n(t_0,x_0),\xi^{n+1}(t_0,x_0)) \le$$
$$\le \frac{k\delta}{2\varepsilon}\sum_{n=0}^{\infty}[f(\xi^n(t_0,x_0)) - f(\xi^{n+1}(t_0,x_0))] = \frac{k\delta}{2\varepsilon}(f(x_0) - f(z)) \le \frac{k\delta}{2\varepsilon}(c+\varepsilon-f(z)).$$

If we assume that $f(z) \geq c - \varepsilon$, we have

$$\rho(x_0, z) \leq \frac{k\delta}{2\varepsilon}(c + \varepsilon - f(z)) \leq \frac{k\delta}{2\varepsilon}(c + \varepsilon - c + \varepsilon) = k\delta.$$

Hence $z \in S_{k\delta}$ because $x_0 \in \overline{S}$, i.e. $\operatorname{dist}(x_0, S) = 0$. But $f(z) = \lim_{n\to\infty} f(\xi^n(t_0, x_0)) \leq f(x_0) \leq c + \varepsilon$ and so $z \in f^{-1}([c-\varepsilon, c+\varepsilon]) \cap S_{k\delta} \subset Q$. This contradicts our result $z \notin Q$. Thus $f(z) < c - \varepsilon$. This gives us the existence of a positive integer m with $f(\xi^m(t_0, x_0)) < c - \varepsilon$. But $\xi^m : [0,1] \times M \to M$ and $f : M \to \mathbb{R}$ are continuous mappings. Therefore there exist a neighbourhood W of t_0 and a neighbourhood U of x_0 such that $f(\xi^m(t, x)) < c - \varepsilon$ whenever $x \in U$ and $t \in W$. The lemma is proved. $\square$

We go back to the construction of the desired deformation η. We first define ξ^a for $a \geq 0$ by

$$\xi^a(t, x) = \xi((a - [a])t, \xi^{[a]}(t, x))$$

where $[a]$ is the largest integer not greater than a. If a is integer, then

$$\xi^a(t, x) = \xi(0.t, \xi^a(t, x)) = \xi^a(t, x)$$

and so this definition agrees with the one given at the beginning of this step. It is straightforward to check that the so defined $\xi^a(t, x)$ is a jointly continuous function for $(a, t, x) \in [0, \infty) \times [0,1] \times M$.

Lemma 3.4 assigns to every point $x \in \overline{S} \cap f^{-1}([c-\varepsilon, c+\varepsilon])$ an open neighbourhood $V_x \subset Q$ of x and a positive integer m_x so that $f(\xi^{m_x}(1, y)) < c - \varepsilon$ for every $y \in V_x$. Now $\{V_x\}_{x \in \overline{S} \cap f^{-1}([c-\varepsilon, c+\varepsilon])}$ is an open covering of $\overline{S} \cap f^{-1}([c-\varepsilon, c+\varepsilon])$. Let $\{V_\beta\}_{\beta \in B}$ be a locally finite refinement of it. We denote by m_β one of the positive integers m_x for which $V_\beta \subset V_x$. There exists a partition of unity $\{a_\beta\}_{\beta \in B}$ subordinated to $\{V_\beta\}_{\beta \in B}$, i.e. $a_\beta : M \to [0,1]$ are continuous, $a_\beta(x) > 0$ iff $x \in V_\beta$ and $\sum_{\beta \in B} a_\beta(x) = 1$ for every x in the closed set $\overline{S} \cap f^{-1}([c-\varepsilon, c+\varepsilon])$.

At last, we are ready to define

$$\eta(t, x) = \xi^{\sum_{\beta \in B} a_\beta(x) m_\beta}(t, x)$$

for every $t \in [0,1], x \in M$.

The continuity of a_β and of $\xi^a(t, x)$ as a function of (a, t, x) gives us the continuity of $\eta : [0,1] \times M \to M$ as a superposition of continuous functions. We will check that η satisfies the conditions $(i) \div (iv)$. For the rest of this section $a(x)$ will denote $\sum_{\beta \in B} a_\beta(x) m_\beta$:

(i) $\eta(0, x) = \xi^{a(x)}(0, x) = \xi((a(x) - [a(x)]).0, \xi^{[a(x)]}(0, x)) = \xi^{[a(x)]}(0, x) = x$;

(ii) Let $x \notin Q$. Then $a_\beta(x) = 0$ for every $\beta \in B$ and hence $\eta(t,x) = \xi^0(t,x) = x$ for every $t \in [0,1]$;

(iii) Let $x \in S \cap f^{-1}((-\infty, c+\varepsilon])$. We know that $f(\xi^{m_\beta}(1,x)) < c - \varepsilon$ whenever $x \in V_\beta$ from the choice of m_β. Let $\overline{m} = \min\{m_\beta : x \in V_\beta\}$. If $f(x) \geq c - \varepsilon$, then $\sum_{\beta \in B} a_\beta(x) = 1$ and so

$$a(x) = \sum_{\beta \in B} a_\beta(x).m_\beta \geq \overline{m}$$

yields $[a(x)] \geq \overline{m}$. Hence

$$\begin{aligned} f(\eta(1,x)) = & f(\xi^{a(x)}(1,x)) = f(\xi(a(x) - [a(x)], \xi^{[a(x)]}(1,x))) \leq \\ \leq & f(\xi^{[a(x)]}(1,x)) \leq f(\xi^{\overline{m}}(1,x)) < c - \varepsilon. \end{aligned}$$

If $f(x) < c - \varepsilon$, then $f(\eta(1,x)) = f(\xi^{a(x)}(1,x)) \leq f(x) < c - \varepsilon$. Therefore $\eta(1,x) \in f^{-1}((-\infty, c-\varepsilon])$.

From (iv) below $x \in S$ yields $\operatorname{dist}(\eta(1,x), S) \leq \operatorname{dist}(\eta(1,x), x) \leq k^2\delta$. Thus

$$\eta(1, S \cap f^{-1}((-\infty, c+\varepsilon])) \subset f^{-1}((-\infty, c-\varepsilon]) \cap S_{k^2\delta}.$$

(iv)
$$\begin{aligned} &\rho(x, \eta(t,x)) = \rho(x, \xi^{a(x)}(t,x)) = \rho(x, \xi((a(x) - [a(x)])t, \xi^{[a(x)]}(t,x))) \leq \\ &\leq \rho(x, \xi^{[a(x)]}(t,x)) + \rho(\xi^{[a(x)]}(t,x), \xi((a(x) - [a(x)])t, \xi^{[a(x)]}(t,x))) \leq \\ &\leq \frac{k\delta}{2\varepsilon}[f(\xi^{[a(x)]}(t,x)) - f(\eta(t,x))] + \sum_{i=0}^{[a(x)]-1} \rho(\xi^{i+1}(t,x), \xi^i(t,x)) \leq \\ &\leq \frac{k\delta}{2\varepsilon}[f(\xi^{[a(x)]}(t,x)) - f(\eta(t,x))] + \sum_{i=0}^{[a(x)]-1} \frac{k\delta}{2\varepsilon}[f(\xi^i(t,x)) - f(\xi^{i+1}(t,x))] = \\ &= \frac{k\delta}{2\varepsilon}[f(\xi^{[a(x)]}(t,x)) - f(\eta(t,x))] + \frac{k\delta}{2\varepsilon}[f(x) - f(\xi^{[a(x)]}(t,x))] = \\ &= \frac{k\delta}{2\varepsilon}[f(x) - f(\eta(t,x))] \end{aligned}$$

for every $x \in M, t \in [0,1]$.

If $x \in Q$, then $\eta(1,x) \in Q$ as well and hence $f(x)$ and $f(\eta(1,x))$ are in $[c - k\varepsilon, c + k\varepsilon]$. Therefore $\rho(x, \eta(t,x)) \leq \dfrac{k\delta}{2\varepsilon}[(c + k\varepsilon) - (c - k\varepsilon)] = \dfrac{k\delta}{2\varepsilon} 2k\varepsilon = k^2\delta$. If $x \notin Q$, then $\eta(1,x) = x$ and $\rho(x, \eta(1,x)) = 0 < k^2\delta$, i.e. the deformation lemma is proved in the case when M is connected.

Next, supposing that M is not connected, let M_α for α from some index set $\mathcal{A}$ be all components of M. Let $\mathcal{A}_1 \subset \mathcal{A}$ be the set of all α with

$$\{x \in M : x \in f^{-1}([c-\varepsilon, c+\varepsilon]) \cap S\} \cap M_\alpha \neq \emptyset.$$

Clearly $\mathcal{A}_1 \neq \emptyset$ because otherwise $f^{-1}([c-\varepsilon, c+\varepsilon]) \cap S = \emptyset$ and there is nothing to prove. Now $Q_\alpha = Q \cap M_\alpha$ is an open neighbourhood of $f^{-1}([c-\varepsilon, c+\varepsilon]) \cap S_{k\delta} \cap M_\alpha$ in M_α. Let $\eta_\alpha \in C([0,1] \times M_\alpha, M_\alpha)$ be the above constructed deformation of M_α. Defining $\eta(t,x) = \eta_\alpha(t,x)$ for $(t,x) \in [0,1] \times M_\alpha$ when $\alpha \in \mathcal{A}_1$ and $\eta(t,x) = x$ for all $t \in [0,1]$ and x from all other components of M we obtain $\eta \in C([0,1] \times M, M)$ satisfying (i)÷(iv). The deformation lemma is thus proved. $\square$

4. Ljusternik-Schnirelmann theory on C^1-Finsler manifolds. Here we apply the deformation lemma from the previous section for proving the result of R. S. Palais (Theorem 1.1) for a locally Lipschitz function defined on a C^1-Finsler manifold. Theorem 4.2 below includes as a particular case the respective result of A. Szulkin – Theorem 3.1 on p. 126 in [13].

We begin with recalling the necessary definitions:

Definition 4.1. *Let M be a Banach manifold. The mapping $\eta \in C([0,1] \times M, M)$ is called deformation of M if $\eta(0,x) = x$ for every $x \in M$.*

Definition 4.2. *Let $\mathcal{F}$ be a family of subsets of a Banach manifold M. We shall say that $\mathcal{F}$ is deformation invariant if, given $A \in \mathcal{F}$ and a deformation η of M, $\eta(1,A) \in \mathcal{F}$ holds true, where $\eta(1,A) = \{x : x = \eta(1,y), y \in A\}$.*

Definition 4.3. *Let M be a Banach manifold, $f : M \to \mathbb{R}$ and $\mathcal{F}$ be a family of subsets of M. We denote by minimax$(f, \mathcal{F})$ the number* $\inf\{\sup\{f(x) : x \in A\} : A \in \mathcal{F}\}$.

Definition 4.4. *Let M be a C^1-Banach manifold and $f : M \to \mathbb{R}$ be locally Lipschitz. The real number c is said to be a critical value of f iff there exists $x \in M$ (called critical point of f) such that $c = f(x)$ and $0 \in \partial f(x)$.*

When proving existence of critical points, one imposes some kind of compactness condition (of Palais-Smale type) on the considered function. In this section we shall need such a condition which is stronger than the respective one used in the next section.

Definition 4.5. ([13], Remark 3.4 on p. 131) *Let M be a C^1-Finsler manifold, $c \in \mathbb{R}$ and $f : M \to \mathbb{R}$ be locally Lipschitz. We say that f satisfies the* *condition* $(sPS)_c$ *if, whenever a sequence* $\{x_n\}_{n=1}^{\infty}$ *is such that* $c = \lim_{n\to\infty} f(x_n)$ *and* $\liminf stf(x_n) \geq 0$*, then* c *is critical value of* f *and*

$$\inf \ \{\inf \ \{\rho(x_n, z) : z \in M_{x_n} \cap K_c\} : n \geq 1\} = 0$$

where M_x *is the component of* M *containing* $x \in M$*,* $K_c = \{x \in M : f(x) = c \text{ and } 0 \in \partial f(x)\}$ *is the set of the critical points at level* c *and* ρ *is the Finsler metric on each component of* M*.*

The next theorem will be applied in the proof of Theorem 4.2 below.

Theorem 4.1 (minimax theorem). *Let* M *be a complete* C^1-*Finsler manifold without boundary and* $f : M \to \mathbb{R}$ *be a locally Lipschitz function. Let* $\mathcal{F}$ *be a deformation invariant family of subsets of* M *such that*

$$-\infty < minimax(f, \mathcal{F}) < +\infty.$$

Let $c = minimax(f, \mathcal{F})$ *and let* f *satisfy the* $(sPS)_c$ *condition. Then* c *is a critical value of* f*.*

P r o o f. Let us assume the contrary, i.e. that c is not a critical value. We claim that in this case there exist $\varepsilon > 0$ and $\beta > 0$ such that $stf(x) < -\beta$ for each $x \in f^{-1}((c-2\varepsilon, c+2\varepsilon))$. If this claim was not true, we could find a sequence $\{x_n\}_{n=1}^{\infty} \subset M$ such that $c - \frac{1}{n} < f(x_n) < c + \frac{1}{n}$ and $stf(x_n) \geq -\dfrac{1}{n}$ for each natural n. Because of the $(sPS)_c$ condition this means that c is a critical value of f which contradicts our assumption.

Applying the deformation lemma with $S = M, Q = f^{-1}((c - 2\varepsilon, c + 2\varepsilon))$ and $\delta = 2\varepsilon/\beta$ we obtain a deformation $\eta \in C([0,1] \times M, M)$ such that $\eta(1, f^{-1}((-\infty, c + \varepsilon])) \subset f^{-1}((-\infty, c - \varepsilon])$. By the definition of c there is a nonempty $A \in \mathcal{F}$ with $A \subset f^{-1}((-\infty, c + \varepsilon])$. Then $\eta(1, A) \subset f^{-1}((-\infty, c - \varepsilon])$. Since $\eta(1, A) \in \mathcal{F}$ we have

$$c = minimax(f, \mathcal{F}) \leq \sup\{f(x) : x \in \eta(1, A)\} \leq c - \varepsilon$$

which is a contradiction. This completes the proof. □

Remark 4.1. For proving Theorem 4.1 it is sufficient to impose on f the weaker Palais-Smale condition from definition 5.2 below.

In what follows we shall need the notions of Ljusternik-Schnirelmann category.

Definition 4.6. ([13], p.124). *Let* M *be a topological space. A set* $A \subset M$ *is said to be of category* k *in* M *(denoted* $cat_M(A) = k$*) if it can be covered by* k *but not* *by $k-1$ closed sets which are contractible to a point in M. If such k does not exist, $cat_M(A) = +\infty$.*

The next two properties of the category follow directly from the definition.

Proposition 4.1. (a) *If $A \subset B$, then $cat_M(A) \leq cat_M(B)$;*
(b) *$cat_M(A \cup B) \leq cat_M(A) + cat_M(B)$.*

The main result in this section is

Theorem 4.2. *Let M be a complete C^1-Finsler manifold without boundary and $f : M \to \mathbb{R}$ be a locally Lipschitz function which is bounded below. Let $cat_M(M) = k$, where k is a natural number or $k = +\infty$. Let $\Lambda_j = \{A \subset M : A$ is closed and $cat_M(A) \geq j\}$ and $c_j = \inf\{\sup\{f(x) : x \in A\} : A \in \Lambda_j\}$ for $j = 1 \div k$. If f satisfies the $(sPS)_c$ condition for all $c = c_j, j = 1 \div k$ and for all $c > \sup\{f(x) : x \in K\}$, where K is the set of all critical points of f, then f has at least k distinct critical points.*

P r o o f. If $\sup\{f(x) : x \in K\} = +\infty$ then K is infinite and the assertion of the theorem holds true.

Let $\sup\{f(x) : x \in K\} < +\infty$. Then we shall prove that $c_j \leq \sup\{f(x) : x \in K\}$ for every $j = 1, \dots, k$. Indeed, if $c > \sup\{f(x) : x \in K\}$ we obtain that the set $f^{-1}([c, +\infty))$ does not contain any critical values of f. Because of the condition $(sPS)_c$, for every $d \geq c$ there exist $\varepsilon_d > 0$ and $\alpha_d > 0$ such that $st\, f(x) \leq -\alpha_d$ for every $x \in f^{-1}((d - 2\varepsilon_d, d + 2\varepsilon_d))$. Applying the deformation lemma we obtain the existence of a deformation η_d such that $\eta_d(1, f^{-1}((-\infty, d+\varepsilon_d]) \subset f^{-1}((-\infty, d-\varepsilon_d])$.

So, we can find countably many numbers d_n, $\varepsilon_n > 0$, and deformations η_n, $n = 1, 2, \dots$ such that

$$\bigcup_{n=1}^{\infty} (d_n - \varepsilon_n, d_n + \varepsilon_n) \supset [c, +\infty),$$

(and every compact interval $[c, d]$ is covered by finitely many intervals $(d_n - \varepsilon_n, d_n + \varepsilon_n)$) $d_n + \varepsilon_n > d_{n+1} - \varepsilon_{n+1}$, $c \leq d_n < d_{n+1}$,

$$\eta_n(1, f^{-1}((-\infty, d_n + \varepsilon_n)) \subset f^{-1}((-\infty, d_n - \varepsilon_n)),\ n = 1, 2, \dots$$

Next we define inductively the deformations ξ_n, $n = 1, 2, \dots$, as follows: $\xi_0(t, x) = x$ and $\xi_{n+1}(t, x) = \xi_n(t, \eta_{n+1}(t, x))$ for all $x \in M$ and $t \in [0, 1]$.

By setting $\xi(t, x) = \lim_{n\to\infty} \xi_n(t, x)$ we define a deformation for which $\xi(1, M) \subset f^{-1}((-\infty, c - \varepsilon])$, where $\varepsilon = c - d_1 + \varepsilon_1 > 0$.

Indeed, let x be an arbitrary point from M. If $f(x) \leq c - \varepsilon = d_1 - \varepsilon_1$, then $f(\xi(1,x)) \leq f(x)$. Let $x \in f^{-1}([c-\varepsilon,+\infty))$. Then there exists a positive integer k such that $f(x) < \inf\ \{f(y)|\ y \in \cup_{n=k+1}^{\infty}\ f^{-1}(d_n - 3\varepsilon_n, d_n + 2\varepsilon_n)\}$. Because of the deformation lemma we have that $\eta_n(t,y) = y$ for every $n \geq k+1$, $t \in [0,1]$ and every point $y \in f^{-1}((-\infty, f(x)+\varepsilon_{k+1}))$. Therefore $\xi(t,y) = \xi_k(t,y)$ for $t \in [0,1]$ and $y \in f^{-1}((-\infty, f(x)+\varepsilon_{k+1}))$. This shows that ξ depends continuously on its arguments.

Next we prove by induction on n that if $f(x) \leq d_n + \varepsilon_n$ then

$$f(\xi_n(1,x)) \leq d_1 - \varepsilon_1.$$

Let $n = 1$ and $f(x) \leq d_1 + \varepsilon_1$. Then $\xi_1(1,x) = \eta(1,x)$ and hence $\xi_1(1,x) \leq d_1 - \varepsilon_1$. Let us assume that this is true for n. For $n+1$ we have that if $f(x) \leq d_{n+1} + \varepsilon_{n+1}$ then $f(\eta_{n+1}(1,x)) \leq d_{n+1} - \varepsilon_{n+1} \leq d_n + \varepsilon_n$. Applying our induction assumption and the equality $\xi_{n+1}(1,x) = \xi_n(1,\eta_{n+1}(1,x))$ we obtain that $f(\xi_{n+1}(1,x)) \leq d_1 - \varepsilon_1$.

We obtained already that $\xi(1,x) = \xi_k(1,x)$, and hence $f(\xi(1,x)) = f(\xi_k(1,x)) \leq d_1 - \varepsilon_1 = c - \varepsilon$. Since x was arbitrarily fixed point from the set $f^{-1}([c-\varepsilon,+\infty))$, we proved in such a way that $\xi(1,M) \subset f^{-1}((-\infty, c-\varepsilon])$.

So, if $j \leq cat_M\ (M)$ then because of the relations

$$cat_M\ (M) \leq cat_M\ (\xi(1,M)) \leq cat_M\ (f^{-1}((-\infty, c-\varepsilon]),$$

we obtain that $c_j \leq c - \varepsilon < c$. Since c was an arbitrary real number greater than $\sup\{f(x)|\ x \in K\}$, we have that $c_j \leq \sup\{f(x)|\ x \in K\}$.

Since $\Lambda_{j+1} \subset \Lambda_j$, we have $c_j \leq c_{j+1}$. We already proved that $c_k \leq \sup\{f(x)|\ x \in K\} < +\infty$. Moreover

$$-\infty < \inf\{f(x) : x \in M\} \leq \inf\{f(x) : x \in A\}$$

for each $A \in \Lambda_1$ and, hence, $-\infty < c_1$. We thus proved that

$$-\infty < c_1 \leq c_2 \leq \ldots \leq c_k < +\infty.$$

Let $K_c = \{x \in M : f(x) = c \text{ and } 0 \in \partial f(x)\}$ for $c \in \mathbb{R}$. According to Theorem 6.2.(3) in [9] Λ_j is deformation invariant for all $j = 1, 2, \ldots, k$. Then Theorem 4.1 implies that $K_c \neq \emptyset$ for $c = c_j, j = 1 \div k$.

Given j, suppose $c_j = c_{j+1} = \ldots = c_{j+p}$ for some $p \geq 0$. If K_c is noncompact we are done.

Let K_c be a compact subset of M. Then by Theorem 5 in [8] and Theorem 6.3 in [9] there are $\mu > 0$ and a 2μ-neighbourhood $U_{2\mu}$ of K_c such that $cat_M(U_{2\mu}) = cat_M(K_c)$.

We next prove that there are $\beta > 0$ and $\varepsilon_0 > 0$ such that $stf(x) < -\beta$ for each $x \in f^{-1}((c-2\varepsilon_0, c+2\varepsilon_0)) \setminus \overline{U}_\mu$. Indeed, if it was not the case, then there would be a sequence $\{x_n\}_{n=1}^{\infty} \subset M \setminus \overline{U}_\mu$ such that $c - \frac{1}{n} \leq f(x_n) \leq c + \frac{1}{n}$ and $stf(x_n) \geq -\frac{1}{n}$. Then the $(sPS)_c$ condition yields a point $x_0 \in K_c \cap (M \setminus U_\mu)$ which is a contradiction because $K_c \subset U_\mu$.

Let $\overline{k} > 1$ be the constant in the formulation of the deformation lemma and $\varepsilon \in (0, \varepsilon_0)$ be such that $2\varepsilon/\beta < \mu/\overline{k}$. Now we apply the deformation lemma with this $\varepsilon, \delta \in (2\varepsilon/\beta, \mu/\overline{k}), S = M \setminus U_{2\mu}$ and $Q = f^{-1}((c-2\varepsilon_0, c+2\varepsilon_0)) \setminus \overline{U}_\mu$. Hence $\eta(1, f^{-1}((-\infty, c+\varepsilon]) \setminus U_{2\mu}) \subset f^{-1}((-\infty, c-\varepsilon])$ and therefore (using Theorem 6.2.(3) in [9])

$$\begin{aligned} cat_M(f^{-1}((-\infty, c-\varepsilon]) \geq & cat_M(\eta(1, f^{-1}((-\infty, c+\varepsilon]) \setminus U_{2\mu}) \geq \\ \geq & cat_M(f^{-1}((-\infty, c+\varepsilon]) \setminus U_{2\mu}). \end{aligned}$$

By the Proposition 4.1 we have

$$\begin{aligned} cat_M f^{-1}((-\infty, c+\varepsilon]) \leq & cat_M(f^{-1}((-\infty, c+\varepsilon]) \cup U_{2\mu}) \leq \\ \leq & cat_M(f^{-1}((-\infty, c+\varepsilon]) \setminus U_{2\mu}) + cat_M(U_{2\mu}) \leq \\ \leq & cat_M(f^{-1}((-\infty, c-\varepsilon])) + cat_M(K_c). \end{aligned}$$

Hence

$$cat_M(K_c) \geq cat_M(f^{-1}((-\infty, c+\varepsilon])) - cat_M(f^{-1}((-\infty, c-\varepsilon])).$$

Since $c = c_j = \ldots = c_{j+p}$, we obtain $cat_M(f^{-1}((-\infty, c+\varepsilon])) \geq j+p$ and $cat_M(f^{-1}((-\infty, c-\varepsilon])) \leq j-1$. Finally $cat_M(K_c) \geq j+p-(j-1) = p+1$ which means that the set K_c contains at least $p+1$ points. This completes the proof. $\square$

Remark 4.2. When M is compact the assumption that f satisfies $(sPS)_c$ for all $c > \sup\{f(x) : x \in K\}$ is not needed because in this case $c_j < +\infty$ for $j = 1 \div k$.

5. Min-max principle and mountain pass theorem. In this section we establish a general min-max principle (Theorem 5.1) using the deformations constructed in Section 3. As a corollary we obtain a general version of the mountain pass theorem (Theorem 5.2).

Here is the setting: Let M be a complete C^1–Finsler manifold without boundary and $f : M \to \mathbb{R}$ be locally Lipschitz. Let $\mathcal{F}$ be a family of subsets of M and F be a subset of M. Throughout this section we shall denote

$$c(F, f, \mathcal{F}) = \inf\{\sup\{f(x) : x \in A \cap F\} : A \in \mathcal{F}\}$$

and in particular

$$c = c(M, f, \mathcal{F}) = \inf\ \{\sup\ \{f(x) : x \in A\} : A \in \mathcal{F}\}.$$

We shall also use the conventional notations

$$\operatorname{dist}(F, B) = \inf\ \{\rho(x, y) : x \in F, y \in B\}$$

and

$$\operatorname{dist}(x, B) = \inf\{\rho(x, y) : y \in B\}$$

for $x \in M, F \subset M, B \subset M, M$ being a metric space with metric ρ.

Definition 5.1 (cf. [5]) *Let $B \subset M$. We shall say that a class $\mathcal{F}$ of subsets of M is a homotopy stable family with boundary B if*
(a) *every set in $\mathcal{F}$ contains B;*
(b) *for any set A in $\mathcal{F}$ and any $\eta \in C([0,1] \times M, M)$ verifying $\eta(t, x) = x$ for all (t, x) in $(\{0\} \times M) \cup ([0,1] \times B)$ we have $\eta(1, A) \in \mathcal{F}$, where*

$$\eta(1, A) = \{x \in M : x = \eta(1, y) \ \ \textit{for some} \ \ y \in A\}.$$

Theorem 5.1. *Let M be a complete connected C^1-Finsler manifold without boundary, $f : M \to \mathbb{R}$ be locally Lipschitz, $k \in (1, \frac{5}{4}), \mathcal{F}$ be a homotopy stable family of subsets of M with boundary B and F be a subset of M verifying*

(2) $$\operatorname{dist}(F, B) > 0 \ \ \textit{and} \ \ F \cap A \neq \emptyset \ \ \textit{for all} \ \ A \in \mathcal{F}$$

and

(3) $$\inf\{f(x) : x \in F\} \geq c.$$

Let $\varepsilon \in (0, \operatorname{dist}(F, B)/2)$. Then for every $A \in \mathcal{F}$ satisfying

$$\sup\{f(x) : x \in A \cap F_{k^2\varepsilon/3}\} < c + \frac{\varepsilon^2}{12}$$

there exists $x_\varepsilon \in M$ *with the properties:*

(i) $c - \dfrac{\varepsilon^2}{6} \le f(x_\varepsilon) \le c + \dfrac{5\varepsilon^2}{12}$

(ii) $stf(x_\varepsilon) \ge -\varepsilon$

(iii) $\mathrm{dist}(x_\varepsilon, F) \le \varepsilon$

(iv) $\mathrm{dist}(x_\varepsilon, A) \le \varepsilon$.

P r o o f. We first note that (2) and (3) imply $c(F, f, \mathcal{F}) = c(M, f, \mathcal{F}) = c$. Hence there is $A \in \mathcal{F}$ (appearing in the formulation of the theorem) satisfying

$$\sup\{f(x) : x \in A \cap F_{k^2\varepsilon/3}\} < c + \frac{\varepsilon^2}{12}$$

because

$$c = c(F, f, \mathcal{F}) \le c(F_{k^2\varepsilon/3}, f, \mathcal{F}) \le c(M, f, \mathcal{F}) = c.$$

We set

$$\psi_\varepsilon(x) = \max\{0, \frac{\varepsilon^2}{4} - \frac{\varepsilon}{2}.\mathrm{dist}(x, F)\}$$

$$f_\varepsilon(x) = f(x) + \psi_\varepsilon(x)$$

for $x \in M$, and $c_\varepsilon = c + \dfrac{\varepsilon^2}{4}$. It is easy to check that $c_\varepsilon \le c(F, f_\varepsilon, \mathcal{F}) \le c(M, f_\varepsilon, \mathcal{F}) \le c_\varepsilon$. Since $0 \le \psi_\varepsilon(x) \le \dfrac{\varepsilon^2}{4}$ for each $x \in M$,

$$\sup\{f_\varepsilon(x) : x \in A \cap F_{\frac{k^2\varepsilon}{3}}\} < c_\varepsilon + \frac{\varepsilon^2}{12}$$

holds true.

Let us choose S to be the set $A \cap F_{k^2\varepsilon/3}$. Then

$$S_{k\varepsilon/3} = (F_{k^2\varepsilon/3} \cap A)_{k\varepsilon/3} \subset F_{(k^2+k)\varepsilon/3} \cap A_{k\varepsilon/3}.$$

Since $k < \dfrac{5}{4}$, the set

$$Q_\varepsilon = int(f_\varepsilon^{-1}([c_\varepsilon - \frac{\varepsilon^2}{6}, c_\varepsilon + \frac{\varepsilon^2}{6}]) \cap F_\varepsilon \cap A_\varepsilon)$$

is an open neighbourhood of $f_\varepsilon^{-1}([c_\varepsilon - \dfrac{\varepsilon^2}{12}, c_\varepsilon + \dfrac{\varepsilon^2}{12}]) \cap S_{k\varepsilon/3}$.

We claim that there exists

$$x_\varepsilon \in f_\varepsilon^{-1}([c_\varepsilon - \frac{\varepsilon^2}{6}, c_\varepsilon + \frac{\varepsilon^2}{6}]) \cap F_\varepsilon \cap A_\varepsilon$$

such that $stf_\varepsilon(x_\varepsilon) \geq -\frac{\varepsilon}{2}$. Supposing the contrary, we apply the deformation lemma for the following choice of $f, S, c, \varepsilon, \delta, k$ and Q respectively: $f_\varepsilon, A \cap F_{k^2\varepsilon/3}, c_\varepsilon, \varepsilon^2/12, \varepsilon/3, k$ and Q_ε. We thus obtain $\eta \in C([0,1] \times M, M)$ satisfying the following properties:
(a) $\eta(0,x) = x$ for each $x \in M$;
(b) $\eta(t,x) = x$ for each $(t,x) \in ([0,1] \times M \setminus Q_\varepsilon)$;
(c) $\eta(1, f_\varepsilon^{-1}((-\infty, c_\varepsilon + \varepsilon^2/12]) \cap A \cap F_{k^2\varepsilon/3}) \subset f_\varepsilon^{-1}((-\infty, c_\varepsilon - \varepsilon^2/12])$;
(d) $\rho(x, \eta(1,x)) \leq k^2\varepsilon/3$ for each $x \in M$.

Let $A_1 = \eta(1, A)$. Since $Q_\varepsilon \subset F_\varepsilon$ we have $B \cap Q_\varepsilon = \emptyset$ and, hence, $\eta(t,x) = x$ for all $(t,x) \in ([0,1] \times B)$. Since $\mathcal{F}$ is homotopy stable with boundary B, we have $A_1 \in \mathcal{F}$. It follows from (d) that $A_1 \cap F \subset \eta(1, A \cap F_{k^2\varepsilon/3})$. But because of $A \cap F_{k^2\varepsilon/3} \subset f_\varepsilon^{-1}((-\infty, c_\varepsilon + \varepsilon^2/12])$, (c) implies $\eta(1, A \cap F_{\frac{k^2\varepsilon}{3}}) \subset f_\varepsilon^{-1}((-\infty, c_\varepsilon - \varepsilon^2/12])$. Then

$$c_\varepsilon = c(F, f_\varepsilon, \mathcal{F}) \leq \sup\{f_\varepsilon(x) : x \in A_1 \cap F\} \leq c_\varepsilon - \frac{\varepsilon^2}{12}$$

which is a contradiction. Hence there is

$$x_\varepsilon \in f_\varepsilon^{-1}([c_\varepsilon - \frac{\varepsilon^2}{6}, c_\varepsilon + \frac{\varepsilon^2}{6}]) \cap F_\varepsilon \cap A_\varepsilon$$

with $stf_\varepsilon(x_\varepsilon) \geq -\varepsilon/2$, i. e. x_ε satisfies (iii) and (iv). Moreover,

$$f(x_\varepsilon) = f_\varepsilon(x_\varepsilon) - \psi_\varepsilon(x_\varepsilon) \leq c_\varepsilon + \frac{\varepsilon^2}{6} = c + \frac{\varepsilon^2}{4} + \frac{\varepsilon^2}{6} = c + \frac{5\varepsilon^2}{12}$$

and

$$f(x_\varepsilon) = f_\varepsilon(x_\varepsilon) - \psi_\varepsilon(x_\varepsilon) \geq c_\varepsilon - \frac{\varepsilon^2}{4} - \frac{\varepsilon^2}{6} = c - \frac{\varepsilon^2}{6},$$

i.e. x_ε satisfies (i).

Finally, since $f_\varepsilon = f + \psi_\varepsilon$, using Lemma 2.2 we get $f_\varepsilon^\circ(x,h) \leq f^\circ(x,h) + \psi_\varepsilon^\circ(x,h)$ for every $x \in M$ and $h \in T_x(M)$ with $\|h\|_{T_x(M)} = 1$. Hence $stf(x) \geq stf_\varepsilon(x) + \inf\{-\psi_\varepsilon^\circ(x,h) : \|h\|_{T_x(M)} = 1\}$. But ψ_ε is a globally Lipschitz function with Lipschitz costant $\frac{\varepsilon}{2}$. It follows then from Lemma 2.4 that $|-\psi_\varepsilon^\circ(x,h)| \leq \varepsilon/2$ for every $x \in M$ and $h \in T_x(M)$ with $\|h\|_{T_x(M)} = 1$. So we obtain

$$st\ f(x_\varepsilon) \geq st\ f_\varepsilon(x_\varepsilon) - \frac{\varepsilon}{2} \geq -\frac{\varepsilon}{2} - \frac{\varepsilon}{2} = -\varepsilon.$$

and the proof of the theorem is complete. □

Remark 5.1. Our Theorem 5.1 includes as a special case Theorem 1.ter in [5] (and, hence, Theorem 1 in [5]) when the compact Lie group appearing in their formulation is the identity. Here we drop the compactness assumption on the elements of $\mathcal{F}$ and consider locally Lipschitz functions instead of C^1 ones. Theorem 2.2 in [3] is also a corollary of Theorem 5.1.

The Palais-Smale condition we shall need in the mountain pass theorem below is weaker than the one given in Definition 4.5.

Definition 5.2. *Let M be a C^1-Finsler manifold, $c \in \mathbb{R}$ and $f : M \to \mathbb{R}$ be locally Lipschitz. We say that f satisfies the condition $(PS)_c$ if, whenever a sequence $\{x_n\}_{n=1}^{\infty}$ is such that $c = \lim_{n\to\infty} f(x_n)$ and $\liminf st\ f(x_n) \geq 0$, then c is a critical value of f.*

Next we introduce the final necessary notation: Let u, v be two distinct points of the connected C^1-Finsler manifold M. We denote

$$\Gamma = \{g \in C([0,1], M) : g(0) = u, g(1) = v\}$$

(the set of paths connecting u and v).

Theorem 5.2. *Let M be a complete connected C^1-Finsler manifold without boundary, $f : M \to \mathbb{R}$ be locally Lipschitz, D be a closed subset of M and u, v be two points from M belonging to disjoint components of $M \setminus D$. Assume $c(M, f, \Gamma) = c(D, f, \Gamma) = c$. If f verifies $(PS)_c$ then c is a critical value of f.*

Proof. The assumption $c(M, f, \Gamma) = c(D, f, \Gamma) = c$ and the compactness of all $g \in \Gamma$ yield that u and v belong to disjoint components of $M \setminus D_1$, where $D_1 = \{x \in D : f(x) \geq c\}$. We apply Theorem 5.1 with $\mathcal{F} = \Gamma, B = \{u, v\}$ and $F = D_1$. Since D_1 is closed and B is compact, $D_1 \cap B = \emptyset$ implies that $\mathrm{dist}(D_1, B) > 0$. Since u, v belong to disjoint components of $M \setminus D_1, D_1 \cap g \neq \emptyset$ for all $g \in \Gamma$ and thus (2) is satisfied. The definition of D_1 implies (3). The family Γ is clearly homotopy stable. Next we combine (i) and (ii) with the $(PS)_c$ condition to end the proof. $\square$

Remark 5.2. It is obvious that (iii) and (iv) from Theorem 5.1 combined with the usual stronger Palais-Smaile conditions, if imposed on f, will yield information about the location of the established critical points (cf. Theorem 1 (ii) in [11]). Theorem 5.2 includes as special cases Theorem 1 (i) in [11] where f is defined on a Banach space, as well as Theorem 1 in [12] where f is defined on a $C^{2-}-$Finsler manifold.

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