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A NOTE ON PRESERVED SMOOTHNESS

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Let X be a Banach space equipped with norm $\|\cdot\|$. We say that $\|\cdot\|$ is Gâteaux differentiable at x if for every $h \in S_X(\|\cdot\|)$,

$$(*) \qquad \lim_{t\to 0} \frac{\|x+th\| - \|x\|}{t}$$

exists. We say that the norm $\|\cdot\|$ is Gâteaux differentiable if $\|\cdot\|$ is Gâteaux differentiable at all $x \in S_X(\|\cdot\|)$. Suppose the limit in $(*)$ exists uniformly in $x \in S_X(\|\cdot\|)$ for every $h \in S_X(\|\cdot\|)$, we say that $\|\cdot\|$ is uniformly Gâteaux differentiable (UG for short). A point $x \in S_X(\|\cdot\|)$ is said to be a smooth point if the norm $\|\cdot\|$ is Gâteaux differentiable at x. A smooth point x is said to be a preserved smooth point if the bidual norm is also Gâteaux differentiable at x. The norm $\|\cdot\|$ is said to be octahedral if there exists a $u \in X^{**} \setminus \{0\}$ such that $\|x+u\| = \|x\| + \|u\|$ for all $x \in X$.

B. V. Godun [3] has shown that a separable Banach space is reflexive if and only if each smooth point is preserved in each equivalent norm. On the other hand, the dual version of the above result has also been obtained in another paper of B. V. Godun (cf., [4]), which says that a Banach space is reflexive if and only if for

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each equivalent norm the extreme points of the unit ball are preserved. P. Morris [6] has shown that a separable Banach space has a subspace isomorphic to c_0 if and only it admits an equivalent strictly convex norm in which no element on the sphere is a preserved extreme point. Some other related results on extreme points are also contained in [5].

It is clear that if a Gâteaux differentiable norm $\|\cdot\|$ is octahedral, then every point on the sphere $S_X(\|\cdot\|)$ is not a preserved smooth point. It is unknown whether a separable space that contains l_1 necessarily admits a Gâteaux differentiable octahedral norm. However, in this note, we show the following:

Theorem. *Let X be a separable Banach space containing an isomorphic copy of l_1. Then X admits a uniformly Gâteaux differentiable LUR norm such that its bidual norm is nowhere Gâteaux differentiable at points of X.*

At the end of the paper, we show by elementary methods (without using octahedrality of norms) that l_1 has also such a property. We refer the readers to [1] for some unexplained notions and results used in this note. We also refer to [2] for more related results.

Proof of Theorem. Since X contains l_1, it admits an octahedral norm $\|\cdot\|$ (cf. [2]). Let $\{x_n\}$ be a countable dense set of $S_X(\|\cdot\|)$. Define an equivalent dual norm $|\cdot|$ on X^* by $|f| = \|f\| + p(f)$, where $p(f) = \left(\sum_{i=0}^{\infty} \frac{f(x_i)^2}{2^i}\right)^{1/2}$. The norm $|\cdot|$ is w^*-lower semicontinuous and weak* uniformly rotund (W^*UR). Therefore its predual is uniformly Gâteaux differentiable (cf. [1, II.6.7]).

Let $x \in S_X(|\cdot|)$, $f = |\cdot|'(x) \in S_{X^*}(|\cdot|)$. By octahedrality of $\|\cdot\|$ there exists a $u \in X^{**}$ such that u has no point of continuity on $(\|f\| B_{X^*}(\|\cdot\|), w^*)$ (cf. [1, III.2.4]). Therefore, there exists a sequence $\{f_n\} \subset \|f\| B_{X^*}(\|\cdot\|)$ such that $f_n \to f$ in the w^*-topology but $u(f_n)$ does not converge to $u(f)$.

We note that $|f_n| \to |f|$. Indeed, since p is w^*-continuous, $p(f_n) \to p(f)$; furthermore $\|f_n\| \to \|f\|$ as $\|\cdot\|$ is w^*-lower semicontinuous. Hence, according to the Šmulyan's lemma (cf. [1, I.1.4]), the dual norm of $|\cdot|$ is not Gâteaux differentiable at x.

Finally, let $|\cdot|_1$ be an equivalent locally uniformly rotund (LUR), uniformly Gâteaux differentiable norm on X. The norm $|||\cdot||| = [|\cdot|_1^2 + |\cdot|^2]^{\frac{1}{2}}$ is LUR and uniformly Gâteaux differentiable on X but its bidual is not differentiable at points of X. $\square$

Using a different method, we show that l_1 admits a uniformly Gâteaux differentiable norm such that its bidual norm is nowhere differentiable at points of l_1.

P r o o f. We show that the norm in [7, p.86] is a required norm. Define $\|\cdot\|$ on l_∞ by $\|y\| = \|y\|_\infty + p(y)$ where $p(y) = \left(\sum \frac{y_i^2}{2^i}\right)^{\frac{1}{2}}$. The norm $\|\cdot\|$ is W^*UR, thus its predual is uniformly Gâteaux differentiable. Let $x \in S_{l_1}(\|\cdot\|)$ and $y = \|\cdot\|'(x) \in S_{l_\infty})(\|\cdot\|)$. We shall construct a sequence y^k such that $\|y^k\| \to \|y\|$, $y^k(x) \to y(x)$, but y^k does not converge to y weakly. We may write $y = (y_1, y_2, y_3, \dots)$ and consider two cases:

Case I. If $y_n \to 0$.

Then there exists an integer N such that $|y_n| < \frac{1}{4}$ for all $n > N$. We, define y^k for $k \geq N$:

$$y_n^k = \begin{cases} \frac{1}{2} & \text{if } \ n > m \\ y_n & \text{otherwise,} \end{cases}$$

We note the following:

(1) $(y^k, x) = (y, x) - \sum_{n=k}^{\infty} x_n(y_n - \frac{1}{2}) \to (y, x) = 1$ as $k \to \infty$.

(2) $p^2(y^k) = p^2(y) - \sum_{n=k}^{\infty} \left(\frac{y_n^2 - 1/4}{2^n}\right) \to p^2(y)$ as $k \to \infty$.

(3) $\|y^k\|_\infty = \|y\|_\infty$ as $1 = \|y\| \leq 2\|y\|_\infty$.

(2) and (3) imply that $\|y^k\| \to \|y\|$.

However, y^k does not converge weakly to y, since any convex combination of $\{y^k\}$ has distance at least $\frac{1}{4}$ from y. Therefore by the Šmulyan's lemma, $\|\cdot\|$ is not differentiable in l_1^{**} at x.

Case II. If y_n does not converge to zero.

Then there exists $\epsilon > 0$ and a subsequence $\{y_{n_k}\}$ such that $|y_{n_k}| > 2\epsilon$ for all k. Define y^m as follows:

$$y_n^m = \begin{cases} 0 & \text{if } \ n > m \\ y_n & \text{otherwise.} \end{cases}$$

As in case I, $\|y^m\| \to \|y\|$ as $m \to \infty$, but y^m does not coverge to y weakly. □

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