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GENERALIZATIONS OF COLE'S SYSTEMS

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Abstract. There are four resolvable Steiner triple systems on fifteen elements. Some generalizations of these systems are presented here.

0. Introduction. The following definition is standard (see [2]).

Definition. *Let V be a finite set, $v = \#(V)$ be the number of elements of V, let $(V/3)$ (resp. $(V/2)$) be the set of all unordered triads (resp. pairs) of distinct elements of V. A Steiner triple system on V (or of order v) is a subset S of $(V/3)$ such that any pair from $(V/2)$ is contained in one and only one triad of S. Such a system is said to be resolvable if there exists a partition*

$$S = S_1 \cup S_2 \cup \ldots \cup S_n \tag{1}$$

of S into n disjoint subsets S_i, where $n = (v-1)/2$, $\#(S_i) = v/3$, all the triads inside each S_i are disjoint (i.e. each S_i presents a partition of V into disjoint triads), $i = 1, 2, \ldots, n$. Such a partition (1) is said to be Kirkman structure (or resolution) on the set V.

Frank Nelson Cole found in [1] four resolvable Steiner triple systems of order 15. Any resolvable system of order 15 is isomorphic to one and only one of these Cole's

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systems. The fourth Cole's system has one Kirkman structure, but each of three others admits two different resolutions. The first of these systems is isomorphic to the set PG(3,2) of all the lines in the 3-dimensional projective space $\mathbb{P}^3(\mathbb{F}_2)$ over the simple field $\mathbb{F}_2$ of two elements, but three other systems have not a nature so clear and evident. Our paper presents some general constructions of Steiner triple systems such that Cole's systems are special cases of ours.

A base of all the generalizations is an application of the operation S from the construction 1 below, this operation is applicable to finite abelian 2-groups of exponent 4 or 2, it is a functor from a category of the mentioned abelian groups (where the category morphisms are injective ones only) to a category of Steiner triple systems (where morphisms are also injective). An application of S to some subgroups produces all the Fano subplanes in examples 5 and 6. We do not consider an important problem concerning a resolvability of our Steiner triple systems, but we show that there are examples of nonresolvable ones as well as resolvable ones. The problem of resolvability of the systems from the constructions 3 and 4 is open. We touch slightly a question on analoga of the main theorem of projective geometry (see remarks 2, 3 and examples 5, 6, 10, 11). Note that the last of our constructions needs prime Mersenne numbers.

1. A simultaneous generalization of the first two Cole's systems.

Construction 1. Let G be an additive abelian finite group of exponent 4 or 2 (i.e. G is isomorphic to a direct sum of cyclic groups of orders 4 or 2). Let $S(G)$ be a subset of $((G-\{0\})/3)$ consisting of the triads $\{a,b,c\}$ such that either

$$a+b+c=0, \tag{2}$$

or

$$a+2b=0,\ a+2c=0,\ b+c=0, \tag{3}$$

or

$$b+2a=0,\ b+2c=0,\ a+c=0, \tag{4}$$

or

$$c+2a=0,\ c+2b=0,\ a+b=0. \tag{5}$$

Theorem 1. *$S(G)$ is a Steiner triple system on the set $G-\{0\}$.*

P r o o f. Let a, b be two distinct nonzero elements of G. If

$$a+2b \neq 0,\ b+2a \neq 0,\ a+b \neq 0,$$

then

$$\{a,b\} \subset \{a,b,-(a+b)\} \in S(G).$$

If

$$a+2b \neq 0,\ b+2a \neq 0,\ a+b = 0,$$

then

$$\{a,b\} \subset \{a,b,2b\} \in S(G).$$

Simultaneous equalities

$$a+2b = 0,\ b+2a = 0$$

are impossible for nonvanishing distinct elements a,b. If

$$a+2b = 0,\ b+2a \neq 0,$$

then

$$\{a,b\} \subset \{a,b,-b\} \in S(G).$$

If

$$b+2a = 0,\ a+2b \neq 0,$$

then

$$\{a,b\} \subset \{a,b,-a\} \in S(G).$$

Proof of the uniqueness (mentioned in the above definition) of the triple containing a given pair $\{a,b\}$. Let $\{a,b,p\}$ and $\{a,b,q\}$ be distinct triads from $S(G)$, i.e. $p \neq q$. Equalities of the form (2) are impossible for these triads simultaneously. If (3) holds for the first triad and (1) holds for the second, i.e.

$$a+2b = a+2p = p+b = a+b+q = 0,$$

then

$$q = -a-b = -(a+2b)+b = b,$$

that is a contradiction. Consequently, either

$$a+2b = 0,\ a+2p = 0,\ a+2q = 0,\ p+b = 0,\ q+b = 0,$$

or

$$b+2a = 0,\ b+2p = 0,\ b+2q = 0,\ p+a = 0,\ q+a = 0,$$

or

$$p + 2a = 0,\ p + 2b = 0,\ q + 2a = 0,\ q + 2b = 0,\ a + b = 0.$$

Each the case implies $p = q$. The theorem is proven. □

Remark 1. If H is a subgroup of G, then $S(H)$ is a subset of $S(G)$.

Remark 2. It is clear that $Aut(G)$ is a subgroup of $AutS(G)$. In the case of coincidence of these groups of automorphisms, we shall say that an analogue of the main theorem of projective geometry holds for G (cf. [3]).

Example 1. If $\#(G) = 2$, then $S(G) = \emptyset$.

Example 2. If $\#(G) = 4$, then $S(G)$ consists of one triad.

Example 3. If $\#(G) = 8$, then $S(G)$ is isomorphic to the Fano plane, i.e. $S(G)$ =PG(2,2) because of the uniqueness of Steiner triple system of order 7.

Example 4. If G is an elementary abelian 2-group, i.e. G is isomorphic to a direct sum of cyclic groups of order 2, $\#(G) = 2^{m+1}$, then $S(G) =$ PG(m,2), i.e. $S(G)$ is isomorphic to the set of all collinear triads of points of the m-dimensional projective space over the simple field $\mathbb{F}_2$ of two elements.

Example 5. Let G be a direct sum of two cyclic groups of order four, i.e. $G = (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$. Elements of G are pairs (x, y), where x, y are elements of $\mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$. Let such a pair be marked by the integer $4x + y$. Thus

(6) $$G - \{0\} = \{1, 2, 3, ..., 15\}.$$

The list of all the triads of $S(G)$ is presented by the following table

{1,2,3}	{1,4,15}	{1,5,14}	{1,6,13}	{1,7,12}	{1,8.11}	{1,9,10}
{4,8,12}	{2,9,11}	{2,7,15}	{2,8,10}	{2,4,14}	{2,5,13}	{2,6,12}
{5,10,15}	{3,5,12}	{3,8,9}	{3,7,14}	{3,10,11}	{3,6,15}	{3,4,13}
{6,7,11}	{6,8,14}	{4,6,10}	{4,5,11}	{5,6,9}	{4,7,9}	{5,7,8}
{9,13,14}	{7,10,13}	{11,12,13}	{9,12,15}	{8,13,15}	{10,12,14}	{11,14,15}

This system is isomorphic to the second Cole's one, more exactly, the following substitution (written in cyclic decomposition)

$$(1, 2, 3)(4, 5, 7, 12)(6, 11, 10, 9, 13, 15, 8)$$

maps $S(G)$ on the second Cole's system of [1]. The group $Aut(G)$ coincides with GL(2,$\mathbb{Z}/4\mathbb{Z}$). The above mentioned (see Remark 2) analogue of the main theorem of the projective geometry is false here, because Cole's substitutions (cf. [1])

$$(3, 5, 14)(1, 13, 6)(2, 8, 10)(12, 11, 7)(5, 14, 9)$$

and

$$(1)(5)(14)(3,11,9)(2,8,10)(4,6,12)(7,5,13)$$

are in $Aut(S(G)) - Aut(G)$.

The system $S(G)$ is resolvable, the columns of the above table fix a Kirkman structure mentioned in (1) (there are two possible such structures on $S(G)$, see [1]). $S(G)$ contains three Fano planes (to be precise, three families of lines of Fano planes). One plane is

$$S((2\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})) = S(\{0,1,2,3,8,9,10,11\}).$$

Other plane is

$$S((\mathbb{Z}/4\mathbb{Z}) \oplus (2\mathbb{Z}/4\mathbb{Z})) = S(\{0,2,4,6,8,10,12,14\}).$$

The third plane is $S(H)$, where $H \subset G$, the subgroup H is specified by the condition $x + y \equiv 0 \pmod 2$, i.e.

$$H = \{0,2,5,7,8,10,13,15\}.$$

Example 6. Let G be the direct sum $(\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$. Its elements are (x,y,z), where $x \in \{0,1,2,3\}$, $y \in \{0,1\}$, $z \in \{0,1\}$. Let the integer $4x + 2y + z$ be mark for the element (x,y,z). Then (6) holds and the list of triads (ordered lexicographically) of $S(G)$ is the following

{1,2,3}	{1,4,13}	{1,5,12}	{1,6,15}	{1,7,14}	{1,8,9}	{1,10,11}
{2,4,14}	{2,5,15}	{2,6,12}	{2,7,13}	{2,8,10}	{2,9,11}	{3,4,15}
{3,5,14}	{3,6,13}	{3,7,12}	{3,8,11}	{3,9,10}	{4,5,9}	{4,6,10}
{4,7,11}	{4,8,12}	{5,6,11}	{5,7,10}	{5,8,13}	{6,7,9}	{6,8,14}
{7,8,15}	{9,12,13}	{9,14,15}	{10,12,14}	{10,13,15}	{11,12,15}	{11,13,14}

This system is not resolvable (because the triad {1,2,3} does not belong to none family of five mutually skew triads of $S(G)$), but some properties of the system are similar to the properties of the system of the preceding example. $S(G)$ contains four Fano planes. They are the following

$$S((2\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) = S(\{0,1,2,3,8,9,10,11\}),$$

$$S((\mathbb{Z}/4\mathbb{Z}) \oplus (0) \oplus (\mathbb{Z}/2\mathbb{Z}) = S(\{0,1,4,5,8,9,12,13\}),$$

$$S((\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (0)) = S(\{0,2,4,6,8,10,12,14\}),$$

$$S(\{(x,y,z)|\ x + y \equiv 0 \pmod 2\}) = S(\{0,1,6,7,8,9,14,15\}).$$

$Aut(S(G))$ contains the following comparatively large subgroup of transformations

$$\begin{aligned} x' &\equiv ax + 2by + 2cz \pmod 4, \\ y' &\equiv nx + ly + mz \pmod 2, \\ z' &\equiv rx + py + qz \pmod 2, \end{aligned}$$

where a is 1 or 3, other coefficients are from {0,1}, $lq - mp \equiv 1 \pmod 2$.

2. Supplement to 1. A variation of the first construction for abelian 3-groups.

Construction 2. Let A be an additive abelian finite 3-group of exponent 9 or 3, i.e. cyclic direct constituents of A are of orders 9 or 3. Let $S(A)$ consist of the triads $\{a, b, c\} \in (A/3)$ such that either

(7) $$a + b + c = 0,$$

or

(8) $$a + 2b = 0,\ b + 2c = 0,\ c + 2a = 0,$$

or

(9) $$a + 2c = 0,\ b + 2a = 0,\ c + 2b = 0.$$

Theorem 2. *$S(A)$ is a Steiner triple system on the set A.*

Proof. Let a, b be different elements of A. If $a + 2b \neq 0$ and $b + 2a \neq 0$, then $\{a, b, c\}$ is the unique triple of $S(A)$ containing $\{a, b\}$. If $a + 2b = 0$, then $3a \neq 0$ (otherwise $3b = 0$, $a = b$), hence $a \neq -2a$, $b \neq -2a$ and $\{a, b\} \subset \{a, b, -2a\} \in S(A)$. By analogy, if $b + 2a = 0$, then $\{a, b\} \subset \{a, b, -2b\} \in S(A)$.

The case $a + 2b = b + 2a = 0$ is impossible, because $a \neq b$.

Let us prove the uniqueness of the triad $\{a, b, c\} \in S(A)$ containing $\{a, b\}$. Assume that $\{a, b, d\} \in S(A)$. Then either

(10) $$a + b + d = 0,$$

or

(11) $$a + 2b = 0,\ b + 2d = 0,\ d + 2a = 0,$$

or

$$a + 2d = 0,\ b + 2a = 0,\ d + 2b = 0. \tag{12}$$

If (7) and (10) hold, then $d = c$.
If (7) and (11) hold, then $c = b$.
If (7) and (12) hold, then $c = a$.
If (8) and (11) hold, then $c = d$.
If (8) and (12) hold, then $d = a$.
If (9) and (11) hold, then $b = d$.
If (9) and (12) hold, then $d = c$.
The theorem is proven. □

Example 7. $S((\mathbb{Z}/3\mathbb{Z})^n)$ is isomorphic to the system AG(n,3) of all the lines of the n-dimensional affine space over the simple field $\mathbb{F}_3$ of three elements.

Example 8. $S(\mathbb{Z}/9\mathbb{Z})$ is isomorphic to AG(2,3) because of the uniqueness of the Steiner triple system of order 9.

Example 9. $S((\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/9\mathbb{Z}))$ is not isomorphic to AG(3,3), because there are three elements in this direct sum (for instance $(0,0)$, $(0,1)$, $(1,1)$) for which does not exist an affine plane containing them.

The systems of the last three examples are resolvable.

3. A generalization of the fourth Cole's system. Let q be a power of 2, $q = 2^n$, $n \geq 2$, let $F = \mathbb{F}_q$ be a finite field consisting of q elements. Let G be a duplicate of the multiplicative set $F^* = F - \{0\}$, i.e. $G \cap F = \emptyset$ and a bijective correspondence $F^* \to G$ is fixed, let $[a]$ denote the image of a by this correspondence, where $a \in F^*, [a] \in G$. Let $V = F \cup G$.

Construction 3. Let a triple system $S^*(F)$ on the set V be specified by the following three conditions.

(i) $S(F) \subset S^*(F)$, where we consider F as an additive abelian 2-group (see the first construction),

(ii) $S^*(F)$ contains all the triads of the form $\{[a], [b], ab\}$, where $a \in F^*$, $b \in F^*$, $a \neq b$,

(iii) $S^*(F)$ contains all the triads of the form $\{0, [a], a^2\}$.

Theorem 3. *$S^*(F)$ is a Steiner triple system on the set V.*

P r o o f. Let $\{a, b\}$ belong to $((F \cup G)/2)$.

If $a \in F^*,\ b \in F^*$, then the pair $\{a, b\}$ enters into the unique triad $\{a, b, a + b\}$.

If $a = 0,\ b \in F^*$, then $\{0, b\}$ enters into the unique triad $\{0, b, [\sqrt{b}]\}$.

If $a = 0,\ b \in G,\ b = [c]$, then the pair enters into the unique triad $\{0, [c], c^2\}$.

If $a \in G,\ b \in G,\ a = [u],\ b = [v]$, then the pair $\{a, b\}$ enters into the unique triad $\{[u], [v], uv\}$.

If $a = [u] \in G,\ b \in F^*$, then $\{[u], b\} \subset \{[u], [b/u], b\} \in S^*(F)$.

The theorem is proven. □

Remark 3. The group $Aut(F)$ (coinciding with $\mathrm{Gal}(F/\mathbb{F}_2)$) acts by the following natural way on V : for $g \in Aut(F)$ and $[a] \in G$, element $g([a])$ is $[g(a)]$. This group preservs $S^*(F)$. Moreover, the multiplicative group F^* acts on V : for $c \in F^*$ and $[a] \in G$, element $c([a])$ is $[ca]$. The multiplicative group preservs $S^*(F)$. Hence, the group $H(F)$ defined by the extension

$$\{1\} \to F^* \to H(F) \to Aut(F) \to \{1\}$$

(where $Aut(F)$ acts by the natural way on F^*) is a subgroup of $Aut(S^*(F))$.

Example 10. The Fano plane (and the set of all its lines) is a special case of the third construction. Let $q = 4,\ F = \{0, 1, a, b\}$, where $a = b^2,\ b = a^2,\ 1 + a + b = 0$,

$$V = F \cup G = \{0, 1, a, b, [1], [a], [b]\}$$

The list of the triads of $S^*(F)$ is the following

$$\{0, 1, [1]\},\ \{0, [a], b\},\ \{0, [b], a\},$$

$$\{[1], a, [a]\},\ \{[1], b, [b]\}\ \{1, [a], [b]\},\ \{1, a, b\}$$

The group $H(F)$ is less than $Aut(F)$.

Example 11. Let

$$q = 8, \quad F = \{0, 1, e, e^2, e^3, e^4, e^5, e^6\},$$

where $1 + e + e^3 = 0$. Then

$$G = \{[1], [e], [e^2], [e^3], [e^4], [e^5], [e^6]\}.$$

Each element of the field F has a form $c_0 + c_1e + c_2e^2$, where the coefficients c_0, c_1, c_2 are from $\{0, 1\}$. Let the nonzero element of this form be marked by the natural number $c_0 + 2c_1 + 4c_2$, let zero element of F be marked by 8, let the element $[c_0 + c_1e + c_2e^2]$ be marked by $8+c_0+2c_1+4c_2$. More explicitly, this marking is specified by the following substitution

$$\begin{pmatrix} 1 & e & e^2 & e^3 & e^4 & e^5 & e^6 & 0 & [1] & [e] & [e^2] & [e^3] & [e^4] & [e^5] & [e^6] \\ 1 & 2 & 4 & 3 & 6 & 7 & 5 & 8 & 9 & 10 & 12 & 11 & 14 & 15 & 13 \end{pmatrix}$$

The following table is the list of all the triads of $S^*(F)$.

{1,2,3}	{1,4,5}	{1,6,7}	{1,8,9}	{1,10,13}	{1,11,14}	{1,12,15}
{4,9,12}	{2,11,15}	{2,9,10}	{2,4,6}	{2,5.7}	{2,12,13}	{2,8,14}
{5,8,11}	{3,10,12}	{3,8,15}	{3,13,14}	{3,9,11}	{3,5,6}	{3,4,7}
{6,13,15}	{6,9,14}	{4,11,13}	{5,10,15}	{4,14,15}	{4,8,10}	{5,9,13}
{7,10,14}	{7,8,13}	{5,12,14}	{7,11,12}	{6,8,12}	{7,9,15}	{6,10,11}

The columns of the table introduce a Kirkman srtucture in $S^*(F)$ (see (1)). This system is isomorphic to the last of the four Cole's, an isomorphism is established by the following substitution

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 1 & 2 & 3 & 12 & 14 & 7 & 4 & 13 & 10 & 5 & 15 & 11 & 9 & 6 & 8 \end{pmatrix}$$

Cole's description [1] of automorphisms of this system implies the coincidence of $H(F)$ with $Aut(S^*(F))$. More exactly, Cole wrote that the group of automorphisms of his fourth system is of order 21, but the exact sequence from remark 3 yields

$$\#(H(F)) = (\#(Aut(F)))(\#(F^*))$$

what equals 21 in our example, because

$$\#(Aut(F)) = \#(Gal(\mathbb{F}_8/\mathbb{F}_2)) = 3,\ \#(\mathbb{F}_8^*) = 7$$

4. A generalizations of the third Cole's system. Let our notations $q, n, F, G, V, [.]$ be the same as before the third construction. Moreover, assume that $p = q-1$, p is prime, i.e. $p = 2^n - 1$ *is a prime Mersenne number.* Let

$$\chi : (\mathbb{Z}/p\mathbb{Z}) \to \{+1, -1\}$$

be the following function

$$\chi(0) = 1,\ \chi(i) = \left(\frac{i}{p}\right)$$

is Legendre's symbol, i.e. for $i \not\equiv 0 \pmod p$ $\chi(i) = 1$ if i is a quadratic residue modulo p, $\chi(i) = -1$ otherwise. Note that $\chi(2) = 1$, because $p^2 \equiv 1 \pmod{4^2}$.

We fix a generator e of the multiplicative group F^*, thus

$$F = \{1, e, e^2, ..., e^{p-1}\},$$

this element e allows to define a logarithmic function

$$\log : F^* \to \mathbb{Z}/p\mathbb{Z}, \quad \log(e^i) = i.$$

Let σ and τ be the following maps of F^* into itself:

if $\chi(\log(a)) = 1$, then $\sigma(a) = a$, $\tau(a) = a^2$,

if $\chi(\log(a)) = -1$, then $\sigma(a) = \sqrt{a}$, $\tau(a) = a$.

Lemma. *The maps σ and τ are bijective.*

Proof. Assume $\sigma(a) = \sigma(b)$ or $\tau(a) = \tau(b)$.

If $\chi(\log(a)) = \chi(\log(b))$, then obviously $a = b$.

If $\chi(\log(a)) = +1$, $\chi(\log(b)) = -1$, then $b = a^2$,

$$\chi(\log(b)) = \chi(2\log(a)) = \chi(2)\chi(\log(a)) = \chi(\log(a)),$$

that is a contradiction.

The lemma is proven. □

Construction 4. We specify a triple system $S^0(F)$ on the set V by the following four conditions

(i) $\sigma^{-1}(S(F)) \subset S^0(F)$, in other words, if a, b, c are three distinct elements of F^* and

$$\sigma(a) + \sigma(b) + \sigma(c) = 0,$$

then $\{a, b, c\} \in S^0(F)$,

(ii) if $a \in F^*$, then $\{0, \tau(a), [a]\} \in S^0(F)$,

(iii) if $a \in F^*$, $a \neq 1$, then $\{[1], \sigma(a), [a]\} \in S^0(F)$,

(iv) if $a \in F^*$, $b \in F^*$, $a \neq 1$, $b \neq 1$, then $\{[a], [b], ab\} \in S^0(F)$.

Theorem 4. *$S^0(F)$ is a Steiner triple system on $V = F \cup G$.*

Proof. Let $\{x, y\} \in (V/2)$.

If $x \in F^*$, $y \in F^*$, then

$$\{x, y\} \subset \{x, y, \sigma^{-1}(\sigma(x) + \sigma(y))\} \in S^0(F).$$

If $x = 0$, $y \in F^*$, then

$$\{x, y\} \subset \{0, y, [\tau^{-1}(y)]\} \in S^0(F).$$

If $x = 0,\ y = [z] \in G,$ then

$$\{x, y\} \subset \{0, [z], \tau(z)\} \in S^0(F).$$

If $x = [1],\ y \in F^*,$ then

$$\{x, y\} \subset \{[1], y, [\sigma^{-1}(y)]\} \in S^0(F).$$

If $x = [1],\ y = [z] \in G,$ then

$$\{x, y\} \subset \{[1], [z], \sigma(z)\} \in S^0(F).$$

If $x = [u] \in G,\ y = [v] \in G,$ then

$$\{x, y\} \subset \{[u], [v], uv\} \in S^0(F).$$

An uniqueness of the triad containing the pair $\{x, y\}$ is clear in all the cases. The theorem is proven. □

Remark 4. The group $Aut(F)$ acts by a natural way on $S^0(F)$ (cf. remark 3 about the third construction).

Example 12. Let $p = 7$ and notations be the same as in example 11.

The following table presents a list of all triads of $S^0(F)$, the columns define a Kirkman stucture.

{1,2,5}	{1,3,6}	{1,4,7}	{1,8,9}	{1,10,13}	{1,11,14}	{1,12,15}
{3,8,11}	{2,9,10}	{2,11,15}	{2,4,6}	{2,3,7}	{2,12,13}	{2,8,14}
{4,9,12}	{4,11,13}	{3,10,12}	{3,13,14}	{4,14,15}	{3,9,15}	{3,4,5}
{6,13,15}	{5,12,14}	{5,8,13}	{7,11,12}	{5,9,11}	{4,8,10}	{6,10,11}
{7,10,14}	{7,8,15}	{6,9,14}	{5,10,15}	{6,8,12}	{5,6,7}	{7,9,13}

This set of triplets is isomorphic to Cole's third system. The substitution

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 14 & 1 & 4 & 2 & 12 & 3 & 7 & 13 & 15 & 11 & 9 & 8 & 6 & 5 & 10 \end{pmatrix}$$

establishes an isomorphism. Frobenius' automorphism on V is described by the following substitution

$$(1)(8)(9)(2, 4, 6)(3, 5, 7)(10, 12, 14)(11, 13, 15)$$

The whole group $Aut(S^0(F))$ is larger than $Aut(F)$, because we have $\#Aut(F) = 3$, but Cole remarked that $Aut(S^0(F))$ is the tetrahedral group.

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