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A PERTURBED ITERATIVE METHOD FOR A GENERAL CLASS OF VARIATIONAL INEQUALITIES

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Abstract. The generalized Wiener-Hopf equation and the approximation methods are used to propose a perturbed iterative method to compute the solutions of a general class of nonlinear variational inequalities.

1. Introduction, preliminaries and formulation. The field of inequality problems has seen a considerable development in mathematics and unilateral mechanics. Particularly, the theory of variational inequalities is now a well-developed theory in mathematics. The mechanical meaning of a variational inequality is given by the formulation of the principle of virtual work when a monotone stress-strain or reaction-displacement condition hold. Equally important is the study of the random equations involving the random operators in view of their need in dealing with probabilistic models in applied sciences. Motivated by a recent work of P. Shi [20] who established the equivalence between variational inequalities and Wiener-Hopf equation and inspired by a random version of this work which is due to Noor and Elsanousi [14], we investigate a general class of nonlinear variational inequalities for the deterministic case. Using the proximal technique, we show the equivalence between such variational inequalities and

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a generalized version of the so called Wiener-Hopf equation. This equivalence, along with the concept of epiconvergence, allows us to suggest and study a new perturbed iterative method for solving our general problem, which consists of coupling an iterative scheme with a data perturbation.

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $|\cdot|$ respectively. For a given nonlinear operator $A : H \to H$ and a given lower semicontinuous proper and convex function $\varphi : H \to \mathbb{R} \cup \{+\infty\}$, we shall consider the following general variational inequality:

$$\text{Find } u \in H \text{ such that} \tag{1}$$
$$\langle Au - Bu, v - u\rangle \geq \varphi(u) - \varphi(v),\ \forall v \in H,$$

where B is a nonlinear continuous mapping on H.

We recall that the subdifferential operator $\partial\varphi$ is defined by

$$(x, y) \in \partial\varphi \Longleftrightarrow \varphi(\xi) \geq \varphi(x) + \langle y, \xi - x\rangle \text{ for all } \xi \in H.$$

The original problem (1) has an equivalent formulation in terms of generalized equation:

$$\text{Find } u \in H \text{ such that} \tag{2}$$
$$0 \in (A - B)u + \partial\varphi(u).$$

Let us give some examples of problems which give raise to inequality (1) or equivalently to (2).

(i) Let K be a nonempty closed convex subset of H. Note that if the operator B is independent of u, that is, $B(u) = f$ for all $u \in H$ and $\varphi = I_K$ the indicator function of the subset K, then (1) is equivalent to the following problem:

$$\text{Find } u \in K \text{ such that} \tag{3}$$
$$\langle Au - f, v - u\rangle \geq 0,\ \ \forall v \in K.$$

Inequalities like (3) are known as the classical variational inequalities and have been extensively studied in the literature (see for instance [19] and references quoted therein).

(ii) Let K be a closed convex cone of H, and let K^* denote its positive polar, i.e.

$$K^* := \{u^* \in H \mid \langle u^*, u\rangle \geq 0\ \ \forall u \in K\}.$$

If $\varphi := I_K$ and B is identically null, then problem (1) reduces to the so called explicit complementarity problem given by

$$\begin{aligned} &\text{Find } u \in K \text{ such that} \\ &Au \in K^* \text{ and } \langle Au, u\rangle = 0. \end{aligned} \tag{4}$$

Such problems were introduced by Karamardian [10]. For further details we refer, for example, to [9]. Nevertheless, such problems are encountered frequently in several fields of applied mathematics such as for instance, mechanics, economic equilibrium theory, elasticity theory.

(iii) Let $J : H \to \mathbb{R}$ be a Gâteaux differentiable convex function such that $\nabla J(u) = Au$ $\forall u \in H$ and B identically null. Then (1) is equivalent to the convex optimisation problem

$$(J + \varphi)(\overline{u}) = \inf_{v\in H} (J + \varphi)(v). \tag{5}$$

This problem has been studied for example in [11] among others.

A large number of equilibrium problems arising in economics and transportation sciences can be formulated as

$$A(P_K(z)) + (z - P_K(z)) = f \tag{6}$$

where P_K stands for the projection operator of H on the convex set K.

As quoted by S. Robinson [17] or [18] and P. Shi [20], equation (6) can be derived from the variational inclusion

$$f \in Au + \partial I_K(u), \tag{7}$$

which is equivalent to problem (3). For the applications and more details of this type of equation we refer to [22], [20] and references cited therein.

More generally, let us consider the following problem

$$\begin{aligned} &\text{Find } z \in H \text{ such that} \\ &(A - B)J_\lambda^\varphi(z) + \frac{1}{\lambda}(I - J_\lambda^\varphi)(z) = 0, \end{aligned} \tag{8}$$

where $\lambda > 0$ is a constant, $J_\lambda^\varphi := (I + \lambda\partial\varphi)^{-1}$ is the so called proximal mapping and I stands for the identity operator on H.

Equation (8) is called the *generalized Wiener-Hopf equation.*

Let us now give a characterization of the proximal mapping J_λ^φ and let us state its nonexpansiveness property.

Lemma 1.1. [6]
(i) $u = J_\lambda^\varphi(z) \iff \lambda^{-1}\langle z-u, v-u\rangle \le \varphi(v)-\varphi(u) \;\; \forall v \in H.$
(ii) The proximal mapping J_λ^φ is nonexpansive, that is:

$$|J_\lambda^\varphi(u) - J_\lambda^\varphi(v)| \le |u-v| \quad \forall u, \, : v \in H.$$

We also need the following standard concepts.

Definition 1.1. *An operator $T: H \to H$ is said to be:*

1. *strongly monotone, if there exists a constant $\alpha > 0$ such that*

$$\langle Tu - Tv, u-v\rangle \ge \alpha|u-v|^2.$$

2. *Lipschitz continuous, if there exists a constant $k > 0$ such that*

$$|Tu - Tv| \le k|u-v|.$$

2. Equivalence. Using the general abstract duality principle of Attouch & Théra [4], we show the equivalence between problem (1) and the generalized Wiener-Hopf equation (8).

Theorem 2.1. *The general variational inequality* (1) *has a solution $u \in H$ if and only if the generalized Wiener-Hopf equation* (8) *has a solution $z \in H$, where*

$$z = u - \lambda(Au - Bu), \tag{9}$$

and

$$u = J_\lambda^\varphi(z). \tag{10}$$

P r o o f. Let us consider the equivalent formulation of problem (1):

$$\begin{gathered}\text{Find } u \in H \text{ such that}\\ 0 \in (A-B)(u) + \partial\varphi(u).\end{gathered} \tag{11}$$

For $\lambda > 0$, we have

$$0 \in \lambda(A - B)(u) + \lambda\partial\varphi(u).$$

By adding and subtracting u to equation (12), we obtain

$$0 \in -u + \lambda(A - B)(u) + u + \lambda\partial\varphi(u). \tag{12}$$

By setting,

$$\mathcal{A}u := (-I + \lambda(A - B))u,$$

$$\mathcal{T}u := (I + \lambda\partial\varphi)(u),$$

equation (12) becomes,

$$\mathcal{A}u + \mathcal{T}u \ni 0.$$

By applying the general abstract duality principle of H. Attouch and M. Théra [4], we get

$$z + \mathcal{A}\mathcal{T}^{-1}(z) \ni 0, \quad \text{with } z \in \mathcal{T}u.$$

Noticing that $\mathcal{T}^{-1}(z) = (I + \lambda\partial\varphi)^{-1}(z) = J_\lambda^\varphi(z)$ we finally obtain

$$z + (-I + \lambda(A - B))J_\lambda^\varphi(z) = 0,$$

or equivalently,

$$(A - B)J_\lambda^\varphi(z) + \frac{1}{\lambda}(I - J_\lambda^\varphi)(z) = 0. \tag{13}$$

Since $z \in \mathcal{T}(u)$, we have

$$u = J_\lambda^\varphi(z).$$

Which completes the proof. □

Remark 2.1. This theorem generalized results obtained by P. Shi [19] and A. M. Noor [13] in the case where $\varphi = I_K$ and $B(u) = f$ for all $u \in H$.

3. A perturbed iterative method. The equivalence established above plays an important role from numerical and approximation point of views and will be used in what follows to obtain algorithms for solving the general variational inequality (1).

Adopting the point of view of variational convergence, we perturb problem (1), at each iteration $n \in \mathbb{N}$, by replacing the original function φ by an approximate function φ^n to get a new problem

$$\begin{aligned} &\text{Find } u_n \in H \text{ such that} \\ &\langle Au_n - Bu_n, v - u_n\rangle \geq \varphi^n(u_n) - \varphi^n(v), \ \forall v \in H. \end{aligned} \tag{14}$$

We assume that the sequence $\{\varphi^n \mid n \in \mathbb{N}\}$ of lower semicontinuous convex and proper functions converges to φ in the sense of Mosco, that is

$\forall u \in H,\ \forall \{u_n | n \in \mathbb{N}\}$ such that $u_n \xrightarrow{w} u$, then $\varphi(u) \leq \liminf_{n\to+\infty} \varphi^n(u_n)$,

$\forall u \in H,\ \exists \{u_n | n \in \mathbb{N}\}$ such that $u_n \xrightarrow{s} u$ and $\varphi(u) \geq \limsup_{n\to+\infty} \varphi^n(u_n)$.

We adopt the notation $\varphi^n \xrightarrow{M} \varphi$, to denote the Mosco-epiconvergence of φ^n to φ, and we recall [3] that if $\varphi^n \xrightarrow{M} \varphi$, then

$$J_\lambda^{\varphi^n}(u) \xrightarrow{s} J_\lambda^{\varphi}(u) \text{ for all } \lambda > 0 \text{ and } u \in H.$$

For more details concerning the Mosco-epiconvergence we refer to the book of H. Attouch [3].

We illustrate the perturbation scheme (14) by the following examples.

Example 3.1. Penalty for Constrained Variational Inequalities. Let us consider the problem (3) with $K := \{u \in H \mid h(u) \leq 0\}$, where $h : H \to \mathbb{R}$ is a differentiable and convex function. Let $p : H \to \mathbb{R}$ be a penalty function, that is p is lower semicontinuous and convex function satisfying

$$p(u) \geq 0 \ \ \forall u \in H \text{ and } p(u) = 0 \iff u \in K.$$

Consider the sequence of functions $\{\varphi^n \mid n \in \mathbb{N}\}$ defined by

$$\varphi^n(u) = r_n p(u) \ \ \forall u \in H.$$

It has been shown [5] that, if $0 < r_n < r_{n+1}$ and $r_n \to +\infty$, then $\varphi^n \xrightarrow{M} \varphi$.

In this case, the perturbed problem (14) becomes the penalty variational inequality with parameter r_n. We note that such problems have been studied in [7].

Example 3.2. Nonlinear Complementarity Problem and Galerkin Method. Let us consider the explicit complementarity problem (4).

If K is a Galerkin cone, that is there exists a countable family of convex subcones $\{K_n | n \in \mathbb{N}\}$ of K such that:

(i) K_n is locally compact for every $n \in \mathbb{N}$

(ii) if $n \leq m$ then $K_n \subseteq K_m$

(iii) $K = \overline{\bigcup_{n \geq 0} K_n}$, then by taking the perturbed function $\varphi^n := I_{K_n}$, we have $\varphi^n \overset{M}{\to} \varphi = I_K$. Such approximation scheme has been studied in [9].

One can see that by using this notion of perturbation, the approximate variational inequality may have quite different computational properties.

Coupling formulations (9) and (10) with the concept of epi-convergence, we suggest the following perturbed iterative algorithm for solving our general problem (1).

The General Algorithm:

(i) At iteration $n = 0$, start with some initial point $z_0 \in H$.

(ii) At iteration n, compute the new point z_{n+1} by the iterative scheme:

$$u_n = J_\lambda^{\varphi^n}(z_n), \tag{15}$$

$$z_{n+1} = u_n - \lambda(Au_n - Bu_n). \tag{16}$$

(iii) If $|z_{n+1} - z_n| \leq \varepsilon$, for a given $\varepsilon > 0$, then stop. Otherwise, repeat (ii).

Remark 3.1. If the operator A is linear and A^{-1} exists ($B(u) = f$ for all $u \in H$), then the generalized Wiener-Hopf equation (8) becomes:

$$z = (I - \lambda^{-1}A^{-1})(I - J_\lambda^{\varphi})(z) + A^{-1}f. \tag{17}$$

Indeed by (8), we have

$$\lambda A J_\lambda^{\varphi} z = J_\lambda^{\varphi}(z) - z + \lambda f,$$

which is equivalent to

$$J_\lambda^{\varphi}(z) = \lambda^{-1}A^{-1}J_\lambda^{\varphi}(z) - \lambda^{-1}A^{-1}z + A^{-1}f.$$

Hence

$$z = (I - \lambda^{-1}A^{-1})(I - J_\lambda^{\varphi})(z) + A^{-1}f.$$

Using the fixed point formulation (17), we can replace step (ii) in the general algorithm by the following:

$$z_{n+1} := (I - \lambda^{-1}A^{-1})(I - J_\lambda^{\varphi^n})(z_n) + A^{-1}f. \tag{ii}'$$

Pitonyak, Shi and Shillor [21] have presented some numerical examples for solutions to obstacle problems by using algorithm (ii)′ with $\varphi^n = I_K$. The results obtained are encouraging.

Remark 3.2. In the case of the convex optimization problem (5), the iteration procedure takes the form:

$$\begin{cases} u_n & = \operatorname{argmin} \left\{ \varphi^n + \dfrac{1}{2\lambda} |\cdot - z_n|^2 \right\} \\ z_{n+1} & = u_n - \lambda \nabla J(u_n) \end{cases}$$

As a particular case, if we set $\varphi^n = I_{K^n}$ where $\{K^n \,|\, n \in \mathbb{N}\}$ is a family of closed convex subset approximating the nonempty closed convex subset K. Then the iterative scheme reduces to

$$\begin{cases} u_n & = P_{K^n}(z_n) \\ z_{n+1} & = u_n - \lambda \nabla J(u_n) \end{cases}$$

Now, let us state the convergence result for the general algorithm.

Theorem 3.1. *Let $A : H \to H$ be a strongly monotone and Lipschitz continuous operator. Assume that $\varphi^n \overset{M}{\to} \varphi$ and B is Lipschitz continuous. Then the sequence $\{z_n | n \in \mathbb{N}\}$ generated by the general algorithm converges strongly to the exact solution z of* (8)*, for*

$$\lambda < \frac{2(\alpha - \mu)}{\beta^2 - \mu^2} \text{ and } \mu < \alpha,$$

where α is the strong monotonicity constant of A and β, μ are the Lipschitz constants of A and B respectively.

P r o o f. Let $z \in H$ satisfy the generalized Wiener-Hopf equation (8). From (9) and (15), we get

$$\begin{aligned} |z_{n+1} - z| &= |u_n - u - \lambda(Au_n - Bu_n) + \lambda(Au - Bu)| \\ &= |u_n - u - \lambda(Au_n - Au) + \lambda(Bu_n - Bu)| \\ &\leq |u_n - u - \lambda(Au_n - Au)| + \lambda|Bu_n - Bu|. \end{aligned}$$

Since A is strongly monotone and Lipshitz continuous, we have

$$|u_n - u - \lambda(Au_n - Au)| \leq \sqrt{1 - 2\lambda\alpha + \lambda^2\beta^2}|u_n - u|.$$

By setting

$$t(\lambda) := \sqrt{1 - 2\lambda\alpha + \lambda^2\beta^2} + \lambda\mu,$$

where μ is the Lipschitz constant of B, we have

$$|z_{n+1} - z| \leq t(\lambda)|u_n - u|. \tag{18}$$

From (10) and (15), we get

$$|u_n - u| = |J_\lambda^{\varphi^n}(z_n) - J_\lambda^{\varphi}(z)|,$$

By introducing the term $J_\lambda^{\varphi^n}(z)$, we get

$$|u_n - u| \leq |J_\lambda^{\varphi^n}(z_n) - J_\lambda^{\varphi^n}(z)| + |J_\lambda^{\varphi^n}(z) - J_\lambda^{\varphi}(z)|.$$

Since $J_\lambda^{\varphi^n}$ is nonexpansive, we obtain

$$|u_n - u| \leq |z_n - z| + \varepsilon_n,$$

where $\varepsilon_n := |J_\lambda^{\varphi^n}(z) - J_\lambda^{\varphi}(z)|$ which converges to 0, since $\varphi^n \xrightarrow{M} \varphi$.

Hence

$$|u_n - u| \leq |z_n - z| + \varepsilon_n,$$

which combined with (18), yields

$$|z_{n+1} - z| \leq t(\lambda)|z_n - z| + t(\lambda)\varepsilon_n.$$

Thus

$$|z_{n+1} - z| \leq \theta|z_n - z| + \varepsilon'_n, \tag{19}$$

where $\theta := t(\lambda)$ and $\varepsilon'_n := t(\lambda)\varepsilon_n$.

The condition on the parameter λ implies that $\theta < 1$. From (19), we derive

$$|z_{n+1} - z| \leq \theta^{n+1}|z - z_0| + \sum_{j=1}^{n} \theta^j \varepsilon'_{n+1-j}.$$

The required result follows from [15], page 399. □

Remark 3.3. We note that when B is identically null, then the condition on the parameter λ in Theorem 3.1 reduces to

$$\lambda < \frac{2\alpha}{\beta^2},$$

where α and β are respectively the strong monotonicity and the Lipschitz constants of the operator A.

4. A weak convergence result. In this section, we consider the following problem

$$\begin{aligned} &\text{Find } u \in H \text{ such that} \\ &\langle Au - f, v - u\rangle \geq \varphi(u) - \varphi(v), \quad \forall v \in H. \end{aligned} \tag{20}$$

Using Theorem 2.1, problem (20) is equivalente to

$$AJ_\lambda^\varphi(z)+\frac{1}{\lambda}(I-J_\lambda^\varphi)(z)=f. \tag{21}$$

We introduce the following operator T defined by

$$T(z):=(I-\lambda A)J_\lambda^\varphi(z)+\lambda f.$$

We note that fixed points of the operator T are solutions of the equation (21).

In the following definition, we introduce the notion of *co-coercive* mappings as defined by Tseng [23] and also studied by Mataoui [12].

Definition 4.1. *The operator $A: H\to H$ is co-coercive if there exists $\Lambda>0$ such that*

$$\langle Au-Av,u-v\rangle\geq\frac{1}{\Lambda}|Au-Av|^2,$$

for all $u,\ v\in H$.

Remark 4.1.

(i) If A is strongly monotone with modulus α and Lipschitz continuous with constant β, then A is co-coercive with modulus $\Lambda=\dfrac{\beta^2}{\alpha}$.

(ii) A co-coercive operator is monotone and Lipschitz.

The following lemma will be usefull

Lemma 4.1. *If the operator A is co-coercive with modulus $\Lambda>0$ and $\lambda<\dfrac{2}{\Lambda}$, then the operator*

$$T(z):=(I-\lambda A)J_\lambda^\varphi(z)+\lambda f$$

is nonexpansive.

Proof. Let z_1 and z_2 in H, we have

$$\begin{aligned}|Tz_1-Tz_2|^2 &= |J_\lambda^\varphi z_1-J_\lambda^\varphi z_2-\lambda(AJ_\lambda^\varphi z_1-AJ_\lambda^\varphi z_2)|^2\\ &\leq |J_\lambda^\varphi z_1-J_\lambda^\varphi z_2|^2-\lambda(\frac{2}{\Lambda}-\lambda)|AJ_\lambda^\varphi z_1-AJ_\lambda^\varphi z_2|^2.\end{aligned}$$

Since the proximal mapping J_λ^φ is nonexpansive and $\lambda<\dfrac{2}{\Lambda}$, the result of the Lemma follows. □

Lemma 4.2. [Browder] *Let H be a Hilbert space and C a closed convex and bounded subset of H. Assume that an operator $T : C \to C$ is nonexpansive. For any $s \in]0,1[$ and $z_0 \in H$, define the following iterative method*

$$z_{n+1} = sz_n + (1-s)Tz_n.$$

Then the sequence $\{z_n | n \in \mathbb{N}\}$ converges weakly to a fixed point of T and $|z_{n+1} - z_n| \to 0$ as $n \to +\infty$.

We suggest the following iterative algorithm with relaxation:

ALG2:

(i) At iteration $n = 0$, start with $z_0 \in H$.

(ii) For $s \in]0,1[$, z_{n+1} is given by

$$z_{n+1} = sz_n + (1-s)(I - \lambda A)J_\lambda^\varphi(z_n) + (1-s)\lambda f.$$

(iii) If $|z_{n+1} - z_n| \leq \varepsilon$, for a given $\varepsilon > 0$, then stop. Otherwise, repeat (ii).

We have the following convergence result:

Theorem 4.3. *Suppose that* (20) *has a solution. If the operator A is co-coercive with modulus $\Lambda > 0$ and $\lambda < \dfrac{2}{\Lambda}$. Then the sequence $\{z_n \mid n \in \mathbb{N}\}$ generated by the algorithm* ALG2 *converges weakly to a solution z^* of the equation* (21).

P r o o f. Let $\overline{u}$ be a solution of (20). By using Theorem 2.1

$$\overline{z} = \overline{u} - \lambda A\overline{u} - \lambda f,$$

is a solution of the equation (21).
Set

$$C := \{v \in H \ : \ |v - \overline{z}| \leq |z_0 - \overline{z}|\}.$$

By Lemma 4.1 T is nonexpansive and thus T maps C into C.
Lemma 4.2 yields that the sequence $\{z_n\}$ weakly converges to a fixed point z^* of T, which is a solution of (21). This completes the proof. □

Example 4.1. Let us consider the following boundary value problem with discontinuous nonlinearities:

$$(\mathcal{P}) \begin{cases} \text{Find } u \in H^2(\Omega) \text{ such that} \\ 0 \in -\Delta u(x) + \partial J(x, u(x)) \text{ a.e. in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

Here Ω is an open subset of $\mathbb{R}^N$ with boundary $\partial\Omega$ sufficiently smooth and $J : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ is a continuous convex and subquadratic function with respect to the second variable, i.e.

$$J(x,s) \le \frac{a}{2}|s|^2 + b, \quad \forall (x,s) \in \overline{\Omega}\times\mathbb{R}, \quad a < \lambda_1, \tag{22}$$

where λ_1 is the first eigenvalue of the homogeneous Dirichlet problem for the operator $-\Delta$.

It is easy to see that $(\mathcal{P})$ can be rewritten in the form (20) with $H = H_0^1(\Omega)$ and $\langle Au, v\rangle = \int_\Omega \nabla u \nabla v dx$.

We choose $\lambda > \frac{1}{2\alpha}$ where α is determinated by the Poincaré's inequality.

In this case, the iterative algorithm ALG2 becomes

$$z_{n+1} = s z_n + (1-s)(I - \lambda^{-1}A^{-1})(I - (\partial J)_\lambda)(z_n), \tag{23}$$

where

$$(\partial J)_\lambda(u) = \text{argmin}\,\{J(\cdot, v) + \frac{1}{2\lambda}\|u - v\|_H^2\}.$$

From a computational point of view, the preceding algorithm (23) is not expensive since at each step, it only requires the calculus of the operator A^{-1} which is given by the Green representation

$$A^{-1}w(x) = \int_\Omega G(x,y)w(y)dy, \quad w \in L^2(\Omega).$$

This can be easily achieved if we perform previously a tabulation of the function $(\partial J)_\lambda$.

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