Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **22** (1996), 117-124

Serdica
Mathematical Journal

Institute of Mathematics
and Informatics, Bulgarian
Academy of Sciences

A RANDOM EVOLUTION INCLUSION OF SUBDIFFERENTIAL TYPE IN HILBERT SPACES

D. Kravvaritis, G. Pantelidis

Communicated by N. M. Yanev

Abstract. In this paper we study a nonlinear evolution inclusion of subdifferential type in Hilbert spaces. The perturbation term is Hausdorff continuous in the state variable and has closed but not necessarily convex values. Our result is a stochastic generalization of an existence theorem proved by Kravvaritis and Papageorgiou in [6].

1. Introduction. In this paper we study a nonlinear random multivalued evolution inclusion of the form

$$(*)\qquad \left\{ \begin{array}{rcl} -\dot{x}(\omega,t) & \in & \partial\varphi(\omega,x(\omega,t)) + F(\omega,t,x(\omega,t)) \\ x(\omega,0) & = & x_0(\omega) \end{array} \right\},$$

where $F(\omega,t,x)$ is a random multivalued perturbation term with closed but not necessarily convex values.

Random differential inclusions have been studied by many authors (cf. Itoh [5], Papageorgiou [10], Kravvaritis and Papageorgiou [7], Nowak [9] and their references). Our result generalizes to the random case corresponding deterministic result proved by Kravvaritis and Papageorgiou in [6].

1991 *Mathematics Subject Classification:* 47H06, 47H15

Key words: subdifferential, evolution inclusions, Hausdorff continuity, measurable multifunctions

2. Mathematical preliminaries. Let (Ω, Σ, μ) be a complete probability space and H a separable Hilbert space. By $P_f(H)$ we will denote the nonempty closed subsets of H. A multifunction $F : \Omega \to P_f(H)$ is said to be *measurable* if, for all $x \in H$,

$$\omega \to d(x, F(\omega)) = \inf\{\|z - x\|,\ z \in F(\omega)\}$$

is measurable. By S_F^2, we will denote the set of measurable selectors of $F(\cdot)$, that belong in the Lebesgue-Bochner space $L^2(H)$, i.e.,

$$S_F^2 = \{f \in L^2(H)\ :\ f(\omega) \in F(\omega)\ \ \mu - \text{a.e.}\}.$$

It is easy to check that this set is closed and it is nonempty if and only if $\inf\{\|x\| : x \in F(\omega)\} \in L^2$.

A function $\varphi : \Omega \times H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a *normal integrand* if and only if $(\omega, x) \to \varphi(\omega, x)$ is measurable and, for all $\omega \in \Omega$, $x \to \varphi(\omega, x)$ is l.s.c. and proper. If, in addition, $\varphi(\omega, \cdot)$ is convex, then we say that φ is a *convex normal integrand.* Recall that for a proper, convex function $\varphi : H \to \overline{\mathbb{R}}$ the *subdifferential* at x is defined by

$$\partial\varphi(x) = \{y \in H\ :\ \varphi(z) - \varphi(x) \geq (y, z - x)\ \text{ for all }\ z \in H\}.$$

We say that $\varphi(\cdot)$ is of compact type, if for every $\lambda \in \mathbb{R}_+$, the level set $\{x \in H : \|x\|^2 + \varphi(x) \leq \lambda\}$ is compact.

The generalized Hausdorff metric on $P_f(H)$, is defined by

$$h(A, B) = \max\left[\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right].$$

We say that $F : H \to P_f(H)$ is Hausdorff continuous (h-*continuous*), if it is continuous from H into the metric space $(P_f(H), h)$.

3. The main result. Let $T = [0, b]$ be a bounded, closed interval in $\mathbb{R}_+$. Consider the following initial value problem

$$(**) \qquad \left\{ \begin{array}{rcl} -\dot{x}(t) & \in & \partial\varphi(x(t)) + f(t) \\ x(0) & = & x_0 \end{array} \right\}$$

where $f \in L^2(T, H)$ and $x_0 \in \overline{D(\partial\varphi)}$. A continuous function $x : T \to H$ is a strong solution of $(**)$ if $x(0) = x_0$, $x(\cdot)$ is absolutely continuous on every compact subset of $(0, b)$ and

$$x(t) \in D(\partial\varphi)\ \text{ and }\ \dot{x}(t) \in \partial\varphi(x(t)) + f(t)\ \text{ a.e. on }\ (0, b).$$

It is known [3] that $(**)$ has a unique strong solution. By a *random strong solution* of $(*)$ we understand a stochastic process $x : \Omega \times T \to H$ such that for every $\omega \in \Omega$, $x(\omega, \cdot)$ is a strong solution of

$$\begin{aligned} -\dot{x}(\omega, t) &\in \partial\varphi(\omega, x(\omega, t)) + f(\omega, t) \\ x(\omega, 0) &= x_0(\omega) \end{aligned}$$

for some $f(\omega, \cdot) \in S^2_{F(\omega,\cdot,x(\omega,\cdot))}$.

We will make the following hypotheses:

$H(\varphi)$: $\varphi : \Omega \times H \to \overline{\mathbb{R}}$ is a convex normal integrand, which is of compact type in the x-variable and the multifunction

$$D(\omega) = D(\partial\varphi(\omega, \cdot)) = \{x \in H : \partial\varphi(\omega, x) \neq \emptyset\}$$

has a bounded selector.

$H(F)$: $F : \Omega \times T \times H \to P_f(H)$ is a multifunction such that

(i) for all $x \in H$, $(\omega, t) \to F(\omega, t, x)$ is measurable,

(ii) for all $(\omega, t) \in \Omega \times T$, $x \to F(\omega, t, x)$ is h-continuous,

(iii) $|F(\omega, t, x)| = \sup\{\|z\| : z \in F(\omega, t, x)\} \leq a(t) + b(t)\|x\|$ a.e. for every $\omega \in \Omega$, with $a(\cdot)$, $b(\cdot) \in L^2_+$.

H_0 : $x_0 : \Omega \to H$ is measurable such that

$$\sup\{\|x_0(\omega)\|, \omega \in \Omega\} < \infty \quad \text{and} \quad \sup\{\varphi(\omega, x_0(\omega)), \omega \in \Omega\} < \infty.$$

Theorem. *If the hypotheses $H(\varphi)$, $H(F)$ and H_0 hold, then $(*)$ admits a random strong solution.*

Proof. Let $z(\cdot)$ be a bounded selector of $D(\cdot)$ and let $u(\omega) \in \partial\varphi(\omega, z(\omega))$. If we set

$$\hat{\varphi}(\omega, x) = \varphi(\omega, x) - \varphi(\omega, z(\omega)) - (u(\omega), x - z(\omega)),$$

then $(*)$ is equivalent to

$$\begin{aligned} -\dot{x}(\omega, t) &\in \partial\hat{\varphi}(\omega, x(\omega, t)) + F(\omega, t, x(\omega, t)) + u(\omega) \\ x(\omega, 0) &= x_0(\omega). \end{aligned}$$

So there is no loss of generality in assuming that

$$\min\{\varphi(\omega,x)\ :\ x\in H\}=\varphi(\omega,z(\omega))=0.$$

Therefore

$$(z(\omega),0)\in \mathrm{Gr}\,\partial\varphi(\omega,\cdot)\quad \text{for all}\quad \omega\in\Omega.$$

First let us obtain an a priori bound for the solutions of $(*)$. So let $x(\cdot,\cdot)$ be a random strong solution. By definition then there exists $h(\omega,\cdot)\in S^2_{F(\omega,\cdot,x(\omega,\cdot))}$ such that

$$\begin{aligned} -\dot{x}(\omega,t) &\in \partial\varphi(\omega,x(\omega,t))+h(\omega,t)\\ x(\omega,0) &= x_0(\omega). \end{aligned}$$

From Lemma 3.1 (p. 64) of Brezis [3] we know that

$$\|x(\omega,t)-z(\omega)\|^2\le \|x_0(\omega)-z(\omega)\|^2+2\int_0^t\|h(\omega,s)\|.\|x(\omega,s)-z(\omega)\|ds.$$

Invoking Lemma A.5, p. 157, of [3], we get

$$\|x(\omega,t)-z(\omega)\|\le \|x_0(\omega)-z(\omega)\|+\int_0^t\|h(\omega,s)\|ds\quad \text{for all}\quad \omega\in\Omega,\ t\in T.$$

Now, using the growth hypothesis $H(F)$ (iii) we have

$$\|x(\omega,t)-z(\omega)\|\le \|x_0(\omega)-z(\omega)\|+\int_0^t\left[a(s)+b(s)\|x(\omega,s)\|\right]ds,$$

and by Gronwall's inequality we get

$$\|x(\omega,t)\|\le L\cdot\exp\|b(\cdot)\|_1=M\quad \text{for all}\quad \omega\in\Omega,\ t\in T,$$

where

$$L=\sup\{\|x_0(\omega)-z(\omega)\|+\|z(\omega)\|\ :\ \omega\in\Omega\}+\|a(\cdot)\|_1.$$

Then consider the multifunction $\hat{F}:\Omega\times T\times H\to P_f(H)$ defined by

$$\hat{F}(\omega,t,x)=\begin{cases} F(\omega,t,x) & \text{if}\quad \|x\|\le M\\ F\left(\omega,t,\dfrac{Mx}{\|x\|}\right) & \text{if}\quad \|x\|>M. \end{cases}$$

We see that

$$\hat{F}(\omega,t,x)=F(\omega,t,p_M(x)),$$

where $p_M(\cdot)$ is the M-radial retraction for which we know that it is continuous. Hence $(\omega,t,x)\to\hat{F}(\omega,t,x)$ is measurable. Furthermore note that

$$|\hat{F}(\omega,t,x)|\leq a(t)+Mb(t)=\gamma(t),\ \gamma(\cdot)\in L^2(T).$$

We set

$$B(\gamma)=\{h\in L^2(T,H)\ :\ \|h(t)\|\leq\gamma(t)\text{ a.e.}\}.$$

Let now $q:\Omega\times L^2(T,H)\to C(T,H)$ be the map that to each $(\omega,h)\in\Omega\times L^2(T,H)$ assigns the unique solution of

$$(*)'\qquad -\dot{x}(t)\in\partial\varphi(\omega,x(t))+h(t)\ \text{ a.e.},\ \ x(0)=x_0(\omega).$$

We claim that q is a Caratheodory map. Fix $h\in L^2(T,H)$. From Lemma 2.1 of [2] we know that

$$q(\omega,h)=\lim_{\lambda\to 0}q_\lambda(\omega,h),$$

where $q_\lambda(\omega,h)$ is the unique solution of

$$\dot{x}_\lambda(t)=\nabla\varphi_\lambda(\omega,x_\lambda(t))+h(t),\ x_\lambda(0)=x_0(\omega)$$

with

$$\varphi_\lambda(\omega,x)=\inf\left[\frac{1}{2\lambda}\|x-z\|^2+\varphi(\omega,z)\ :\ z\in H\right].$$

Since $\nabla\varphi_\lambda(\omega,x)$ is measurable in ω (see Theorem 2.3 of [1]), $q_\lambda(\cdot,h)$ is also measurable and so $q(\cdot,h)$ is measurable. From Lemma 3.1 of Brezis [3], we know that, for each $\omega\in\Omega$, $q(\omega,\cdot)$ is continuous from $L^2(T,H)$ into $C(T,H)$. So $q(\cdot,\cdot)$ is a Caratheodory map. Let

$$W=\{q(\omega,h)=x:\text{strong solution of }\ (*)',\ \omega\in\Omega,\ h\in B(\gamma)\}.$$

Then for any $x(\cdot)\in W$ and $t,t'\in T$, $t<t'$, we have

$$\|x(t')-x(t)\|=\left\|\int_t^{t'}\dot{x}(s)ds\right\|\leq\int_t^{t'}\|\dot{x}(s)\|ds=\int_0^b\|X_{[t,t']}(s)\dot{x}(s)\|ds$$

$$\leq\left[\int_0^b\|X_{[t,t']}(s)\|^2ds\right]^{\frac{1}{2}}\left[\int_0^b\|\dot{x}(s)\|^2ds\right]^{\frac{1}{2}}.$$

From the estimates provided by Theorem 3.6 of Brezis [3], we know that

$$\left[\int_0^b\|\dot{x}(s)\|^2ds\right]^{\frac{1}{2}}\leq\|h\|_2+[\varphi(\omega,x_0(\omega))]^{\frac{1}{2}}\leq\|\gamma\|_2+\sup\left\{[\varphi(\omega,x_0(\omega))]^{\frac{1}{2}}\ :\ \omega\in\Omega\right\}=M_1.$$

So finally we have

$$\|x(t') - x(t)\| \le M_1 (t' - t)^{\frac{1}{2}},$$

which implies that W is equicontinuous.
Also again from Theorem 3.6 of [3] we know that

$$\|\dot{x}(t)\|^2 + \frac{d}{dt}\varphi(\omega, x(t)) = (h(t), \dot{x}(t)) \ \text{ a.e.},$$

so

$$\begin{aligned}\varphi(\omega, x(t)) &\le \varphi(\omega, x_0(\omega)) + \int_0^t (h(s), \dot{x}(s))ds \le \varphi(\omega, x_0(\omega)) + \|h\|_2 \|\dot{x}\|_2 \\ &\le \sup\{\varphi(\omega, x_0(\omega)) \ : \ \omega \in \Omega\} + \|\gamma\|_2 M_1 = M_2\end{aligned}$$

$$\Rightarrow \varphi(\omega, x(t)) \le M_2 \ \text{ for all } \ t \in T, \quad \omega \in \Omega \ \text{ and all } \ x(\cdot) \in W.$$

Recalling that $\varphi(\omega, \cdot)$ is of compact type, we deduce that $\overline{W(t)}$ is compact for all $t \in T$. Invoking the Arzela-Ascoli theorem, we conclude that W is relatively compact in $C(T, H)$. As in the proof of Theorem 3.1 in [6] we can prove that W is closed, hence compact in $C(T, H)$. It then follows that $\hat{W} = \overline{\text{conv}} W$ is compact in $C(T, H)$.

Now consider the multifunction $R : \Omega \times \hat{W} \to P_f(L^2(T, H))$ defined by

$$R(\omega, y) = S^2_{F(\omega, \cdot, y(\cdot))}.$$

For each $y \in \hat{W}$, the multifunction $\omega \to R(\omega, y)$ is measurable. Indeed, since $(\omega, t) \to F(\omega, t, x)$ is measurable and $x \to F(\omega, t, x)$ is h-continuous $F(\cdot, \cdot, \cdot)$ is jointly measurable ([11, Theorem 3.3]). Then for every $x \in H$,

$$(\omega, t, y) \to d(x, F(\omega, t, y))$$

is measurable. Since the distance function is continuous in x, for each $h \in L^2(T, H)$,

$$(\omega, t) \to d(h(t), F(\omega, t, y(t)))$$

is measurable. From Fubini's theorem we get that

$$\omega \to \int_0^b d(h(t), F(\omega, t, y(t)))dt = d(h, S^2_{F(\omega, \cdot, y(\cdot))})$$

is measurable, which implies that $\omega \to R(\omega, y)$ is measurable.

Next consider the multifunction $G : \Omega \to P_f(C(\hat{W}, L^2(T, H)))$ defined by

$$G(\omega) = \{r \in C(\hat{W}, L^2(T, H)) \ : \ r(y) \in R(\omega, y) \ \text{ for all } \ y \in \hat{W}\}.$$

From Fryszkowski's selection theorem [4], we know that, for all $\omega \in \Omega$, $G(\omega) \neq \emptyset$. We have

$$G(\omega) = \{r \in C(\hat{W}, L^2(T,H)) \ : \ d(r(y), R(\omega, y)) = 0 \ \text{ for all } \ y \in \hat{W}\}.$$

Since $x \to F(\omega, t, x)$ is h-continuous, the multifunction $y \to R(\omega, y)$ is also h-continuous. It then follows that $y \to d(r(y), R(\omega, y))$ is continuous. Thus if $\{y_n\}$ is dense in $\hat{W}$, then

$$G(\omega) = \bigcap_{n=1}^{\infty} \{r \in C(\hat{W}, L^2(T,H)) \ : \ d(r(y_n), R(\omega, y_n)) = 0\}.$$

Now, for fixed $y \in \hat{W}$, $(\omega, r) \to d(r(y), R(\omega, y))$ is a Caratheodory map. So the multifunction $\omega \to \{r \in C(\hat{W}, L^2(T,H)) \ : \ d(r(y), R(\omega, y)) = 0\}$ is measurable, which in turn implies that G is measurable. By the selection theorem of Kuratowski and Ryll-Nardzewski [8], there exists $r : \Omega \to C(\hat{W}, L^2(T,H))$ measurable such that $r(\omega) \in G(\omega)$ for all $\omega \in \Omega$. Observe that, for every $\omega \in \Omega$, $q(\omega, r(\omega)(\cdot)) : \hat{W} \to \hat{W}$. Applying Schauder's fixed point theorem, we get $x \in W$ such that $q(\omega, r(\omega)(x)) = x$. Set $S(\omega) = \{x \in \hat{W} \ : \ q(\omega, r(\omega)(x)) = x\}$. Then there exists $s : \Omega \to C(T,H)$ measurable such that $s(\omega) \in S(\omega)$ for all $\omega \in \Omega$. It then follows that

$$x(\omega, t) = q(\omega, r(\omega)(s(\omega)))(t)$$

is the desired random solution of $(*)$.

REFERENCES

[1] H. Attouch. Families d'operateurs maximaux monotones et mesurabilite. *Ann. Mat. Pura Appl.* **120** (1979), 35-111.

[2] V. Barbu. Nonlinear Semigroups and Differential Equations in Banach Spaces. Noordhoff Inter. Publishing, Leyden, The Netherlands, 1976.

[3] H. Brezis. Operateurs Maximaux Monotones et Semigroups de Contractions dans les Espaces de Hilbert. North Holland, Amsterdam, 1973.

[4] A. Fryszkowski. Continuous selections for a class of nonconvex multivalued maps. *Studia Math.* **76** (1983), 163-174.

[5] S. Itoh. Random fixed point theorems with an application to random differential equations in Banach spaces. *J. Math. Anal. Appl.* **67**, (1979), 261-273.

[6] D. Kravvaritis and N. S. Papageorgiou. Multivalued perturbations of subdifferential type evolution equation in Hilbert spaces. *J. Differential Equations* **76** (1988), 238-255.

[7] D. Kravvaritis and N. S. Papageorgiou. A random nonlinear multivalued evolution equation in Hilbert space. *Colloquium Math.* **LVIII**, (1989), 129-141.

[8] K. Kuratowski and C. Ryll-Nardzewski. A general theorem on selectors. *Bull. Acad. Polon. Sci. Math. Astronom. Phys.* **13** (1965), 397-403.

[9] A. Nowak. Random differential inclusions: Measurable selection approach. *Ann. Polon. Math.* **XLIX** (1989), 291-296.

[10] N. S. Papageorgiou. Random differential inclusions in Banach spaces. *J. Differential Equations* **65** (1986), 287-303.

[11] N. S. Papageorgiou. On measurable multifunctions with applications to random multi-valued equations. *Math. Japon.* **32** (1987), 701-727.

D. Kravvaritis, G. Pantelidis
Department of Mathematics
National Technical University of Athens
Zografou Campus
15780 Athens, Greece

Received January 15, 1996