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EXISTENCE OF GLOBAL SOLUTIONS TO SUPERCRITICAL SEMILINEAR WAVE EQUATIONS

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1. Introduction. In this work we study the existence of global solution to the semilinear wave equation

(1.1) $$(\partial_t^2 - \Delta)u = F(u),$$

where $F(u) = O(|u|^\lambda)$ near $|u| = 0$ and $\lambda > 1$. Here and below Δ denotes the Laplace operator on $\mathbb{R}^n$.

The existence of solutions with small initial data, for the case of space dimensions $n = 3$ was studied by F. John in [13], where he established that for $1 < \lambda < 1+\sqrt{2}$ the solution of (1.1) blows-up in finite time, while for $\lambda > 1 + \sqrt{2}$ the solution exists globally in time. Therefore, the value $\lambda_0 = 1 + \sqrt{2}$ is critical for the semilinear wave equation (1.1).

To obtain the existence theorem in his work [13] F. John proved the following weighted $L^\infty-$ estimate for the wave equation $(\partial_t^2 - \Delta)u = F$ in $\mathbb{R}^{3+1}$ with zero initial data

(1.2) $$\|\tau_+^\alpha \tau_-^\beta u\|_{L^\infty} \le C\|\tau_+^\gamma \tau_-^\delta F\|_{L^\infty},$$

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where $\tau_\pm = 1 + |t \pm |x||$ are the weights associated with the characteristic surfaces of the wave equation and the non-negative parameters $\alpha, \beta, \gamma, \delta$ satisfy the conditions

$$\alpha = 1 \quad , \quad \beta = \gamma - 2 < 1 \quad , \quad \delta > 1. \tag{1.3}$$

For general space dimensions it was W. Strauss who proposed in [28] the conjecture that the critical value for the nonlinearity is the positive root $\lambda_0(n)$ of the equation

$$(n-1)\lambda^2 - (n+1)\lambda - 2 = 0. \tag{1.4}$$

We shall make a brief review of the results concerning this conjecture.

For $n = 2$ a proof of the conjecture was given by R. Glassey ([10], [11]). A blow-up result for arbitrary space dimensions when $1 < \lambda < \lambda_0(n)$ was established by T. Sideris [27].

The critical values $\lambda = \lambda_0(n)$ were studied by J. Schaeffer in [26] for $n = 2, 3$. A simplified proof was found by H.Takamura [35].

Another interesting effect is the influence of the decay rate of the initial data on the existence of global solutions. In this case the solution might blow-up in finite time when the initial data decay very slowly at infinity even in the supercritical case $\lambda > \lambda_0(n)$. For the case $n = 3$ this effect was established by F. Asakura [3] for the supercritical case. The critical cases for $n = 2, 3$ were studied by K. Kubota ([22]) , K. Tsutaya ([36], [37], [38], R. Agemi and H. Takamura [2]. For the case $n \geq 4$ and supercritical nonlinearity the blow-up result for slowly decaying initial data is due to H. Takamura [34].

On the other hand, the existence part of the conjecture of W. Strauss for $n > 3$ is much less elucidated. A conformal transformation was used by Y. Choquet-Bruhat [5], [6] in order to obtain global existence result for the case when the nonlinearity λ is sufficiently large.

Recently, Y. Zhou [40] has found a complete answer for $n = 4$ by using suitable weighted Sobolev estimates and the method developed by S. Klainerman [15], [16], [17] for proving the existence of small amplitude solutions.

The existence of a global solution for the case $\lambda = (n+3)/(n-1)$ was established by W. Strauss [30] by the aid of the conformal methods and the classical Strichartz inequality.

Another partial answer was given by R. Agemi, K. Kubota, H. Takamura in [1] for a special class of integral nonlinearities in (1.1). The approach in this work follows the approach of F. John based on his estimate (1.2).

A complete proof of the conjecture of W. Strauss for spherically symmetric initial data and odd space dimensions was found by H. Kubo [21] (see also [19], [20]).

Different approach to establish the conjecture of Strauss for spherically symmetric initial data and arbitrary space dimensions was proposed in a recent work of H. Lindblad and C. Sogge [23]. They use a suitable generalizations of the classical Strichartz inequality involving mixed norms (i.e. L^p in time and L^q in space variables). Their approach enables one to treat even the case of non-spherically symmetric initial data and space dimensions $n \leq 8$.

Let us make a brief conclusions of the above review of results concerning the missing existence part in the conjecture of W. Strauss.

1. The methods based on the John estimate (1.2) enable one to control the $L^\infty-$norm of the solutions. They work very well when the Riemann function is nonnegative (i.e. for $n \leq 3$). A similar idea enables one to consider the case of spherically symmetric initial data.

2. The application of weighted Sobolev inequality in combination with the conformal energy estimate for the wave equation (as it was done in [40]) leads to a weaker restriction $n \leq 4$ (or may be $n \leq 7$ as it was mentioned in [40]) due to the singularity of the nonlinear function $F(u)$.

3. The application of the classical Strichartz inequality enables one to overcome the obstruction caused by the singularity of the nonlinear function, but leads only to local existence and uniqueness of the solution, when

$$1 < \lambda \leq \frac{n+3}{n-1}$$

(see [27]) or the global existence for $\lambda = (n+3)/(n-1)$(see [30]). Even the refined mixed norm Strichartz inequalities applied in [23] need some upper restriction for the space dimension for non-spherically symmetric initial data.

The main purpose of this work is twofold.

In order to overcome the above difficulties and to prove the existence of a small amplitude solution for the general case of arbitrary space dimensions, non-spherically symmetric initial data and

$$\lambda_0(n) < \lambda < \frac{n+3}{n-1},$$

we shall combine the approaches of F. John and R. Strichartz so that a more refined L^p-L^q estimate, taking into account the influence of the weights $\tau_\pm$, shall be established. Therefore, this estimate will enable us to use the advantages of the both previous estimates due to F. John and R. Strichartz. Actually, we shall have a precise information about the decay rate of the solution with respect to $\tau_\pm$ weights and we shall be able to avoid the loss of derivatives typical for the Sobolev estimates.

On the other hand, we shall be able to apply this estimate to the semilinear wave equation (1.1) and to establish that the existence part in the conjecture of W. Strauss is true for any space dimensions $n \geq 2$ and non-spherically symmetric initial data.

To state the weighted estimate we consider the Cauchy problem for the linear wave equation

$$(\partial_t^2 - \Delta)u = F, \tag{1.5}$$

with zero initial data. For simplicity we shall assume that the supports of u and F lie in the light cone, that is

$$\mathrm{supp} F(t,x) \subset \{|x| \leq t + R\}. \tag{1.6}$$

Our main weighted estimate has the following form

Theorem 1. *Suppose* $1 < p, q < \infty$ *satisfy*

$$\begin{aligned} &\frac{1}{q} < \frac{1}{p} \quad , \quad \frac{1}{q} + \frac{1}{p} \leq 1, \\ &\frac{n-3}{2} < \frac{n}{q} - \frac{1}{p}, \end{aligned} \tag{1.7}$$

while the nonnegative parameters $\alpha, \beta, \gamma, \delta$ *satisfy*

$$\begin{aligned} \alpha &< \frac{n-1}{2} - \frac{n}{q}, \\ \frac{n-1}{2p} - \frac{n+1}{2q} < \beta &= \gamma - \frac{n+1}{2} + \frac{n}{p} - \frac{1}{q} < \frac{n-1}{2} - \frac{n}{q}, \\ \delta &> 1 - \frac{1}{p}. \end{aligned} \tag{1.8}$$

Then the solution u *satisfies the estimate*

$$\|\tau_+^\alpha \tau_-^\beta u\|_{L^q(\mathbb{R}^{n+1}_+)} \leq C \|\tau_+^\gamma \tau_-^\delta F\|_{L^p(\mathbb{R}^{n+1}_+)}, \tag{1.9}$$

where $\tau_\pm = 1 + |t \pm |x||$ *and* $\mathbb{R}^{n+1}_+ = \{(t,x) \in \mathbb{R}^{n+1} : t \geq 0\}$.

Remark 1. The assumptions (1.7) in the above theorem determine a triangle $\triangle ABC$ in the plane of $1/q, 1/p-$coordinates with vertices

$$A\left(\frac{n-3}{2(n-1)}, \frac{n-3}{2(n-1)}\right), \quad B\left(\frac{1}{2}, \frac{1}{2}\right), \quad C\left(\frac{n-1}{2(n+1)}, \frac{n+3}{2(n+1)}\right).$$

The point A corresponds to the John estimate, while the point C corresponds to the Strichartz estimate.

Taking into account the estimate $\tau_+^\gamma \tau_-^\delta \leq C\tau_+^{\gamma+\theta}\tau_-^{\delta-\theta}$ for $\theta \geq 0$,we obtain the following.

Corollary 1. *Suppose* $1 < p, q < \infty$ *satisfy the assumptions* (1.7) *and the nonnegative real numbers* $\alpha, \beta, \gamma, \delta$ *satisfy the hypotheses*

$$\begin{aligned} \alpha &< \frac{n-1}{2} - \frac{n}{q}, \\ \frac{n-1}{2p} - \frac{n+1}{2q} < \beta &= \gamma - \frac{n+1}{2} + \frac{n}{p} - \frac{1}{q} - \theta, \\ \beta &< \frac{n-1}{2} - \frac{n}{q}, \\ \delta + \theta &> 1 - \frac{1}{p}. \end{aligned} \tag{1.10}$$

with some $\theta \geq 0$. *Then the solution* u *satisfies the estimate*

$$\|\tau_+^\alpha \tau_-^\beta u\|_{L^q(\mathbb{R}_+^{n+1})} \leq C\|\tau_+^\gamma \tau_-^\delta F\|_{L^p(\mathbb{R}_+^{n+1})}. \tag{1.11}$$

The application of this weighted estimate will give us the possibility to establish the conjecture of W. Strauss for the semilinear wave equation

$$\begin{aligned} &(\partial_t^2 - \Delta)u = F(u), \\ &u(0,x) = \varepsilon f, \quad \partial_t u(0,x) = \varepsilon g, \end{aligned} \tag{1.12}$$

where f, g are smooth compactly supported functions. and ε is a sufficiently small positive number. For the nonlinear function $F(u)$ we shall assume that $F(u) \in C^0$ near $u = 0$ and for some $\lambda > 1$ satisfies

$$\begin{aligned} |F(u)| &\leq C|u|^\lambda \ , \\ |F(u) - F(v)| &\leq C|u-v|(|u|^{\lambda-1} + |v|^{\lambda-1}) \end{aligned} \tag{1.13}$$

near $u, v = 0$.

The existence and the uniqueness of the local solution in $C([0,T]; L^q(\mathbb{R}^n))$ for $q = 2(n+1)/(n-1)$ and $1 < \lambda \leq (n+3)/(n-1)$ is established in [27] by using the Strichartz inequality and contraction mapping principle. A small improvement of the uniqueness result can be done by another variant of the Strichartz inequality (see [33])

$$\|u\|_{L^q(\mathbb{R}^{n+1})} \leq C\|(\partial_t^2 - \Delta)u\|_{L_p(\mathbb{R}^{n+1})}. \tag{1.14}$$

Then the uniqueness is fulfilled in the weaker space $L^q([0,T]\times\mathbb{R}^n)$. For $\lambda=(n+3)/(n-1)$ the existence of a global solution is established in [30] by the aid of a conformal method. Therefore, it remains to examine the existence of global solution to (1.12) for

$$\lambda_0(n)<\lambda<\frac{n+3}{n-1}, \tag{1.15}$$

where $\lambda_0(n)$ is the positive root of (1.4). For this case we have the following

Theorem 2. *Suppose $n\geq 2$ and the assumptions* (1.13), (1.15) *are fulfilled with $\lambda_0(n)$ being the positive root of the equation*

$$(n-1)\lambda^2-(n+1)\lambda-2=0. \tag{1.16}$$

Then there exists $\varepsilon_0>0$ so that for $0<\varepsilon<\varepsilon_0$ the Cauchy problem (1.12) *admits a global solution*

$$u\in L^q_{\alpha,\beta}(\mathbb{R}^{n+1}_+).$$

Here $L^q_{\alpha,\beta}(\mathbb{R}^{n+1}_+)$ denotes the Banach space of all measurable functions with finite norm

$$\|\tau_+^\alpha\tau_-^\beta u\|_{L^q(\mathbb{R}^{n+1}_+)}.$$

We shall explain the main idea to establish the weighted estimate of Theorem 1.

The solution of the Cauchy problem (1.5) can be represented by the aid of a Fourier transform

$$u(t,x)=(2\pi)^{-n}\int_0^t\int_{\mathbb{R}^n}\exp(ix\xi)\frac{\sin\left((t-s)|\xi|\right)}{|\xi|}\hat{F}(s,\xi)d\xi ds, \tag{1.17}$$

where $\hat{F}(s,\xi)=\int\exp(-iy\xi)F(s,y)dy$ is the partial Fourier transform of F. It is clear that $u(t,x)=\int_0^t U(F)(t,s,x)ds$, where

$$U(F)(t,s,x)=(2\pi)^{-n}\int_{\mathbb{R}^n}\exp(ix\xi)\frac{\sin\left((t-s)|\xi|\right)}{|\xi|}\hat{F}(s,\xi)d\xi. \tag{1.18}$$

The Fourier integral operator U can be imbedded into analytic family of operators U_z defined for $z\in\mathbb{C}$ and $F(s,y)\in C_0^\infty(\mathbb{R}^{n+1})$ as follows

$$U_z(F)(t,s,x)=c(n)\int_{\mathbb{R}^n}\exp(ix\xi)(t-s)^{\frac{n}{2}-z}|\xi|^{z-\frac{n}{2}}J_{\frac{n}{2}-z}((t-s)|\xi|)\hat{F}(s,\xi)d\xi, \tag{1.19}$$

where $J_\nu(s)$ is the Bessel function of order ν and $c(n) = \sqrt{\frac{2}{\pi}}(2\pi)^{-n}$. The above family was introduced by R. Strichartz [31] , [32] in order to obtain $L^p - L^q$ estimate for the wave equation. Integrating over s, we introduce the operator

$$W_z(F)(t,x) = \int_0^t U_z(F)(t,s,x)ds. \tag{1.20}$$

Since $J_{\frac{1}{2}}(s) = \sqrt{\frac{2}{\pi}}\frac{\sin(s)}{\sqrt{s}}$, we see that the solution u can be represented as

$$u(t,x) = W_{\frac{n-1}{2}}(F)(t,x). \tag{1.21}$$

Applying the formula (see [7])

$$\int_{\mathbb{R}^n} \exp(-iy\xi)(t-s)^{\frac{n}{2}-z}|\xi|^{z-\frac{n}{2}} J_{\frac{n}{2}-z}((t-s)|\xi|)d\xi = \\ = \frac{(2\pi)^{n/2}2^z}{\Gamma(1-z)}((t-s)^2 - |y|^2)_+^{-z} \tag{1.22}$$

with $s_+^{-z} = s^{-z}$ for $s > 0$ and $s_+^{-z} = 0$ for $s \le 0$, we get

$$U_z(F)(t,s,x) = \frac{(2\pi)^{n/2}2^z}{\Gamma(1-z)} \int ((t-s)^2 - |x-y|^2)_+^{-z} F(s,y)dy. \tag{1.23}$$

For $\mathbf{Re}\, z < 1$ the integral in (1.23) is a classical one, while for $\mathbf{Re}\, z \ge 1$ it is necessary to consider (1.23) as the action of the distribution

$$K_z(t,s,x,y) = \frac{(2\pi)^{n/2}2^z}{\Gamma(1-z)}((t-s)^2 - |x-y|^2)_+^{-z} \tag{1.24}$$

on the test function $F(s,y)$.

The possibility to apply a complex interpolation for the strip $0 \le \mathbf{Re}\, z \le (n+1)/2$ relies on a combined use of (1.19) and (1.23). More precisely, the proof of the well-known Strichartz estimate is based on the following L^∞ estimate on the line $\mathbf{Re}\, z = 0$

$$\|U_z(F)(t,s,.)\|_{L^\infty} \le C\|F(s,.)\|_{L^1} \tag{1.25}$$

and this is a direct consequence of (1.23). Making the observation that the representation formula (1.23) keeps its classical sense for $\mathbf{Re}\, z < 1$, we plan to use this classical representation for the larger semiplane $\mathbf{Re}\, z < 1$ and to prove a weighted L^∞ estimate for this semiplane. To be more precise, we shall follow the approach of F. John and we shall obtain $L^\infty-$ estimate with weights $\tau_\pm$ for $\mathbf{Re}\, z < 1$.

Our next step is to derive L^2- weighted estimate on the line $\mathbf{Re}\, z = (n+1)/2-\varepsilon$. For the purpose we shall use the representation (1.19). Then the kernel K_z can be represented by the oscillatory integral

$$(1.26)\quad K_z(t,s,x,y) = c(n)\int_{\mathbb{R}^n} \exp(i(x-y)\xi)(t-s)^{\frac{n}{2}-z}|\xi|^{z-\frac{n}{2}} J_{\frac{n}{2}-z}((t-s)|\xi|)d\xi.$$

We shall split the space of variables (t,s,x,y) into few complementary domains. The first domain is characterized by

$$|x| \le t-2, \quad |y| \le s-2, \quad s \le \frac{t-|x|}{4}.$$

For this domain we have

$$t-s-|x-y| \ge C(t-|x|)$$

so the distribution function in (1.24) is a classical function and we shall use this representation (1.24) to estimate W_z. For the second domain, defined by

$$|x| \le t-2, \quad |y| \le s-2, \quad s \ge \frac{t-|x|}{4}, \quad s-|y| \ge \delta(t-|x|), \delta > 0,$$

we can follow the classical approach of Strichartz and using the inequality $s-|y| \ge \delta(t-|x|)$ we can derive a weighted variant of the corresponding L^2- estimate.

The most difficult part is the estimate of the kernel of W_z restricted to the domain

$$\begin{gathered} |x| \le t-2, \quad |y| \le s-2, \quad s \ge \frac{t-|x|}{4}, \\ s-|y| \le \delta(t-|x|), \quad t-s \ge \delta(t-|x|) \end{gathered}$$

for $\delta > 0$ sufficiently small. For this domain we make the change of variable

$$(1.27)\qquad s \to \sigma = s-|y|$$

and use the Fourier representation (1.26) of the kernel K_z. In this case a more refine analysis based on the application of Fourier integral operators is needed.

The plan of the work is the following. In Section 2 we make a localization in (t,s,x,y)-coordinates. The next two subsections are devoted to the proof of $L^\infty-$ weighted estimates of the operator W_z for the semiplane $\mathbf{Re}\, z < 1$. In these two subsections we consider separately the corresponding interior and exterior regions for (t,x). In section 3 we reduce the L^2-weighted estimate of W_z on the line $\mathbf{Re}\, z = (n+1)/2-\varepsilon$ to the L^2-boundedness of a local Fourier integral operator. In section 4 we show that this is a Fourier integral operator whose canonical relation is local canonical graph. Using a complex interpolation we complete the proof of Theorem 1 in section 5. Suitable weighted estimates for the homogeneous wave equation are also obtained in section 5. The application to the semilinear wave equation and the proof of the conjecture of W. Strauss are given in section 6. Finally, in the appendix we recall some useful consequences of the Sobolev inequality.

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2. Analytic family of operators associated with the wave operator. Using the Fourier transform representation (1.19) and the following estimate for the Bessel functions (see [39])

$$|s^{-\nu}J_\nu(s)| \leq \frac{C(z)}{(1+s)^\varepsilon},\ s \geq 0, \tag{2.1}$$

for $\mathbf{Re}\,\nu = -1/2+\varepsilon, \varepsilon > 0, C(z) = C\exp(b|\mathbf{Im}\,z|^2)$, one obtains

$$\|U_z(F)(t,s,.)\|_{L^2(\mathbb{R}^n)} \leq \frac{C(z)}{(t-s)^{1-\varepsilon}}\||.|^{-\varepsilon}\hat{F}(s,.)\|_{L^2(\mathbb{R}^n)} \tag{2.2}$$

for $\mathbf{Re}\,z = (n+1)/2-\varepsilon, \varepsilon > 0$.

So the operator W_z in (1.20) satisfies on the line $\mathbf{Re}\,z = (n+1)/2-\varepsilon$ the estimate

$$\|W_z(F)(t,.)\|_{L^2(\mathbb{R}^n)} \leq C(z)\int_0^t (t-s)^{-1+\varepsilon}\||.|^{-\varepsilon}\hat{F}(s,.)\|_{L^2(\mathbb{R}^n)}ds, \tag{2.3}$$

where $C(z) = C\exp(b|\mathbf{Im}\,z|^2)$. The estimate (2.3) suggests us to consider the following.

Example. Consider the operator

$$g \in C_0^\infty(\mathbb{R}_+) \to I(g)(t) = \int_0^t (t-s)^{-1+\varepsilon}g(s)ds, \tag{2.4}$$

where $0 < \varepsilon < 1$. In order to derive weighted L^2-estimate our first step is to make a translation and reduce the situation to the case

$$\mathrm{supp}\,g \subset [2,\infty). \tag{2.5}$$

Further, we consider the truncated operator

$$I_T = H([T, 2T])I,$$

where here and below $H(A)$ will denote the characteristic function of the set A. Note that the assumption (2.5) assures that $I_T = 0$ for $T \leq 1$. Our goal is to derive for $a \in [0, 1/2)$ and $T > 1$ the following estimate

$$\| |.|^{a-\varepsilon} I(g)\|_{L^2([T,2T])} \leq C\| |.|^a g\|_{L^2([0,2T])} \tag{2.6}$$

with some constant C, independent of T.

Once this estimate is established, we make the decomposition

$$I = \sum_{k=1}^{\infty} I_k = \sum_{k=1}^{\infty} H([2^k, 2^{k+1}])I \tag{2.7}$$

and taking any $\delta > 0$, we get from (2.6)

$$\begin{aligned} \|(1+|.|)^{a-\varepsilon-\delta} I(g)\|_{L^2(\mathbb{R}_+)} &\leq C \sum_{k=1}^{\infty} 2^{-k\delta} \|(1+|.|)^{a-\varepsilon} I(g)\|_{L^2([2^k, 2^{k+1}])} \\ &\leq C_1 \sum_{k=1}^{\infty} 2^{-k\delta} \|(1+|.|)^a g\|_{L^2(\mathbb{R}_+)} \leq C_2 \|(1+|.|)^a g\|_{L^2(\mathbb{R}_+)}. \end{aligned}$$

Thus, the localized estimate (2.6) implies

$$\|(1+|.|)^{a-\varepsilon_1} I(g)\|_{L^2(\mathbb{R}_+)} \leq C\|(1+|.|)^a g\|_{L^2(\mathbb{R}_+)}$$

for $\varepsilon_1 > \varepsilon, a \in [0, 1/2]$.

Let $T > 1$. In this case the left side of (2.6) can be replaced by

$$T^{a-\varepsilon} \|I(g)\|_{L^2([T,2T])}. \tag{2.8}$$

Further, we decompose $I(g)$ as $I_1(g) + I_2(g)$, where

$$I_1(g)(t) = \int_{t/2}^{t} (t-s)^{-1+\varepsilon} g(s) ds$$

and

$$\begin{aligned} |I_2(g)(t)| &= \left| \int_0^{t/2} (t-s)^{-1+\varepsilon} g(s) ds \right| \leq \\ &\leq CT^{-1+\varepsilon} \int_0^{t/2} |g(s)| ds \end{aligned}$$

for $t \in [T, 2T]$. Applying the Cauchy inequality we find

$$\|I_2(g)\|_{L^2([T,2T])} \le CT^{-1/2+\varepsilon} T^{1/2-a} \||.|^a g\|_{L^2([0,T])}$$

for any $a \in [0, 1/2)$ so

$$T^{a-\varepsilon}\|I_2(g)\|_{L^2(A(T))} \le C\||.|^a g\|_{L^2([0,T])}. \tag{2.9}$$

For $I_1(g)$ we apply the Young inequality and obtain

$$\|I_1(g)\|_{L^2([T,2T])} \le CT^{\varepsilon}\|g\|_{L^2([T/2,2T])}.$$

It is obvious now that

$$T^{a-\varepsilon}\|I_1(g)\|_{L^2([T,2T])} \le C\||.|^a g\|_{L^2([T/2,2T])}. \tag{2.10}$$

From (2.9) and (2.10) we see that the quantity in (2.8) is dominated by the right side of (2.6) and this completes the proof of (2.6).

The example considered above suggests us to make a translation in time and to consider the case

$$\mathrm{supp} F \bigcup \mathrm{supp} W_z(F) \subseteq \{|x| \le t-2\}. \tag{2.11}$$

Further, we follow the construction from the above Example and consider the truncated operator

$$H(\tau \le t - |x| \le 2\tau) W_z. \tag{2.12}$$

First, we make the decomposition

$$\begin{aligned} H(\tau \le t-|x| \le 2\tau)W_z &= H(\tau \le t-|x| \le 2\tau)W_z H(s \le \tau/4) + \\ &+ H(\tau \le t-|x| \le 2\tau)W_z H(s > \tau/4). \end{aligned} \tag{2.13}$$

The operator $H(\tau \le t-|x| \le 2\tau)W_z H(s \le \tau/4)$ has kernel

$$H(\tau \le t-|x| \le 2\tau) K_z(t,s,x,y) H(s \le \tau/4). \tag{2.14}$$

From (1.24) we see that this kernel is a classical function with absolute value dominated by

$$Ct^{-\mathbf{Re}\, z}(t-|x|)^{-\mathbf{Re}\, z} \le Ct^{-\mathbf{Re}\, z}\tau^{-\mathbf{Re}\, z}.$$

In fact, if (t,s,x,y) is in the support of the kernel (2.14), then we have $t > t-|x| \ge \tau$ and

$$t-s-|x-y| \ge t-s-|x|-|y| \ge t-|x|-2s \ge \tau - 2\tau/4 = \tau/2.$$

This argument leads to the inequality

$$|H(\tau \leq t-|x| \leq 2\tau)W_z\left(H(s \leq \tau/4)F\right)(t,x)| \leq$$
$$\leq Ct^{-\mathbf{Re}\, z}\tau^{-\mathbf{Re}\, z}\int_0^{\tau/4}\int_{|y|\leq s-2}|F(s,y)|dyds.$$

The application of the Hölder inequality leads to

$$\begin{aligned}&\|H(\tau \leq t-|x| \leq 2\tau)W_z\left(H(s \leq \tau/4)F\right)\|_{L^\infty} \leq \\ &\leq Ct^{-\mathbf{Re}\, z}\tau^{-\mathbf{Re}\, z-\gamma+n-n/p}\|\tau_+^\gamma\tau_-^\delta F\|_{L^p}\end{aligned} \tag{2.15}$$

provided $\gamma < n-n/p$ and $\delta > 1-1/p$. Note that (2.15) holds even for any z with $\mathbf{Re}\, z \geq 0$. Moreover, for $\delta < 1-1/p, \gamma < n-n/p$ we have the following variant of (2.15)

$$\begin{aligned}&\|t^{\mathbf{Re} z}H(\tau \leq t-|x| \leq 2\tau)W_z\left(H(s \leq \tau/4)F\right)\|_{L^\infty} \leq \\ &\leq C\tau^{-\mathbf{Re} z-\gamma-\delta+n+1-(n+1)/p}\|\tau_+^\gamma\tau_-^\delta F\|_{L^p}.\end{aligned} \tag{2.16}$$

Using the estimate

$$\|f\|_{L^q(\mathbb{R}^{n+1})} \leq C\|\tau_+^a\tau_-^b f\|_{L^\infty(\mathbb{R}^{n+1})}$$

with $a > n/q, b = 1/q$, we derive from (2.15) the following.

Proposition 2.1. *For* $\mathbf{Re}\, z \geq 0$ *we have*

$$\|\tau_+^\alpha\tau_-^\beta\chi(\tau \leq t-|x| \leq 2\tau)W_z(\chi(s \leq \tau/4)F)\|_{L^q} \leq C\|\tau_+^\gamma\tau_-^\delta F\|_{L^p} \tag{2.17}$$

provided the nonnegative parameters $\alpha, \beta, \gamma, \delta$ *satisfy*

$$\alpha < \mathbf{Re}\, z - \frac{n}{q}, \tag{2.18}$$

$$\gamma = \beta - \mathbf{Re}\, z + n - \frac{n}{p} + \frac{1}{q} < n(1-\frac{1}{p}), \tag{2.19}$$

$$\delta > 1 - \frac{1}{p}. \tag{2.20}$$

In a similar way, from (2.16) we get

Proposition 2.2. *For* $\mathbf{Re}\, z \geq 0$ *the estimate* (2.17) *is valid also when the nonnegative parameters* $\alpha, \beta, \gamma, \delta$ *satisfy*

$$\alpha < \mathbf{Re}\, z - \frac{n}{q}, \tag{2.21}$$

(2.22) $$\gamma = \beta - \mathbf{Re}\, z - \delta + n + 1 - \frac{n+1}{p} + \frac{1}{q} < n(1 - \frac{1}{p}),$$

(2.23) $$\delta < 1 - \frac{1}{p}.$$

The remaining part of this section consists of few preliminary steps needed in the next two subsections to estimate the kernel of $H(\tau \le t - |x| \le 2\tau) W_z \chi(s \ge \tau/4)$. It is clear that it is sufficient to estimate $H(\tau \le t - |x| \le 2\tau) W_z(F)$ assuming

(2.24) $$\operatorname{supp} F(s,y) \subset \{s \ge \tau/4, |y| \le s - 2\}.$$

The application of the Hölder inequality when

(2.25) $$\mathbf{Re}\, z < \frac{1}{p'} = 1 - \frac{1}{p}$$

enables one to estimate the kernel (1.24) of the operator W_z. More precisely, we have

(2.26) $$|\chi(\tau \le t - |x| \le 2\tau) W_z F(t,x)| \le C \chi(\tau \le t - |x| \le 2\tau) I(t,x) \|\tau_+^\gamma \tau_-^\delta F\|_{L^p},$$

where

(2.27) $$I^{p'}(t,x) = \int\int_{\operatorname{supp} F(s,y)} ((t-s)^2 - |x-y|^2)_+^{-\mathbf{Re} z p'} (s + |y|)^{-\gamma p'} |s - |y||^{-\delta p'} dy ds.$$

Making the change of variables

$$\rho = |x - y| \quad , \quad \omega = (y - x)/\rho,$$

we see that

$$\begin{aligned} I^{p'}(t,x) \quad &\le \quad C \int_0^t \int_0^{t-s} \int_{\mathbb{S}^{n-1}} ((t-s)^2 - \rho^2)^{-\mathbf{Re} z p'} \times \\ &\times \quad (s + |x + \rho\omega|)^{-\gamma p'} |s - |x + \rho\omega||^{-\delta p'} \times \\ &\times \quad \chi(|x + \rho\omega| < s - 2) d\omega \rho^{n-1} d\rho ds. \end{aligned} \tag{2.28}$$

Now we are in situation to apply the following.

Lemma 2.1. (see [1])
We have

$$\int_{\mathbf{S}^{n-1}} f(|x + \rho\omega|) d\omega = \frac{c}{(\rho|x|)^{n-2}} \int_{|\rho - |x||}^{\rho + |x|} \lambda f(\lambda) h(\lambda, \rho, |x|) d\lambda,$$

where

$$h(\lambda,\rho,|x|) = (Q(\lambda,\rho,|x|))^{\frac{n-3}{2}},$$

$$Q(\lambda,\rho,|x|) = (\lambda^2 - (\rho-|x|)^2)((\rho+|x|)^2 - \lambda^2). \tag{2.29}$$

This Lemma enables one to get the following estimate for $I(t,x)$ in (2.28)

$$\begin{aligned} I^{p'}(t,x) &\leq \frac{C}{|x|^{(n-2)}}\Big(\int_0^t \int_0^{t-s} \int_{|\rho-|x||}^{\min(s-2,\rho+|x|)} ((t-s)^2-\rho^2)^{-\mathbf{Re}zp'} \times \\ &\times \ \rho(s+\lambda)^{-\gamma p'}|s-\lambda|^{-\delta p'}\lambda h(\lambda,\rho,|x|)d\lambda d\rho ds\Big). \end{aligned} \tag{2.30}$$

In the sequel we shall need the following estimate for Q (respectively $h = Q^{(n-3)/2}$).

Lemma 2.2. *Suppose the positive numbers λ,ρ,r satisfy $|\rho - r| \leq \lambda \leq \rho + r$. Then we have*

$$|\lambda - r| \leq \rho \leq \lambda + r \quad , \quad |\lambda-\rho| \leq r \leq \lambda+\rho \tag{2.31}$$

and the quantity $Q(\lambda,\rho,r) = (\lambda^2 - (\rho-r)^2)((\rho+r)^2-\lambda^2)$ satisfy the estimates

$$Q \leq 4\lambda\rho r^2, \tag{2.32}$$

$$Q \leq 4\lambda r \rho^2, \tag{2.33}$$

$$Q \leq 4\rho r \lambda^2, \tag{2.34}$$

$$Q \leq 4\lambda^2\rho^2, \tag{2.35}$$

$$Q \leq 4\lambda^2 r^2, \tag{2.36}$$

$$Q \leq 4r^2\rho^2. \tag{2.37}$$

P r o o f. The above estimates are essentially established in [1], but we shall prove them again for completeness. Our proof is based on the observation that

$$Q = \frac{S^2}{16}, \tag{2.38}$$

where S is the surface of the triangle with sides

$$a = \lambda\ ,\ b = \rho\ ,\ c = r.$$

This geometrical observation shows that (2.31) follow from the existence of this triangle guaranteed by $|\rho - r| \leq \lambda \leq \rho + r$. Then the well-known inequality $S \leq ab/2$ implies

(2.35). In a similar way we get (2.36) and (2.37). The inequality $S^2 \leq abc^2/4$ implies (2.32). In the same way we get (2.33) and (2.35).

This completes the proof of the lemma. □

2.1. Interior estimate. In this subsection we consider the interior domain

$$\{|x| \leq (1-\varepsilon)t\ ,\ t \geq 2\} \tag{2.1.1}$$

where $\varepsilon > 0$ is a sufficiently small number.

Combining (2.1.1) and (2.24) we see that

$$s > \frac{\varepsilon t}{8} \quad \text{on} \quad \text{supp} F. \tag{2.1.2}$$

To estimate the density h in (2.30) we use Lemma and find

$$h(\lambda, \rho, |x|) \leq C\lambda^{n-3}|x|^{n-3}. \tag{2.1.3}$$

Setting

$$\varphi(v, |x|) = \frac{1}{|x|}\int_{-|x|}^{|x|} (1+|v-\lambda|)^{-\delta p'} d\lambda, \tag{2.1.4}$$

we see that

$$\frac{1}{|x|}\int_{|\rho-|x||}^{\rho+|x|} (1+|s-\lambda|)^{-\delta p'} d\lambda \leq \varphi(s-\rho, |x|).$$

For $\delta p' > 1$ it is easy to see that

$$\|\varphi(., |x|)\|_{L^1} \leq C|x|^{-1}\int_{-|x|}^{|x|} 1 d\lambda = 2C. \tag{2.1.5}$$

Now we can estimate the integral in (2.30) as follows

$$|I(t,x)| \leq \frac{C}{(1+t)^{\gamma}}\left(\int_0^t \int_0^{t-s} \varphi(s-\rho, |x|)\rho s^{n-2}((t-s)^2-\rho^2)^{-\mathbf{Re}\, z\, p'} d\rho ds\right)^{1/p'}.$$

Since

$$((t-s)^2-\rho^2)^{-\mathbf{Re}\, z\, p'}\rho \leq C\rho^{1-\mathbf{Re}\, z\, p'}(t-s-\rho)^{-\mathbf{Re}\, z\, p'} \leq Ct^{1-\mathbf{Re}\, z\, p'}(t-s-\rho)^{-\mathbf{Re}\, z\, p'},$$

we get

$$|I(t,x)| \leq \frac{C}{t^{\gamma+\mathbf{Re} z-(n-1)/p'}}(\int_0^t \int_0^{t-s} \varphi(s-\rho, |x|)(t-s-\rho)^{-\mathbf{Re}\, z\, p'} d\rho ds)^{1/p'}.$$

Making the change of variables

$$v = s - \rho \quad , \quad u = s + \rho,$$

from (2.1.5) and (2.25) we separate the variables in the above double integral and get

$$|I(t,x)| \le Ct^{-\gamma - 2\mathbf{Re}\,z + n/p'} = Ct^{-\gamma - 2\mathbf{Re}\,z + n - n/p} \tag{2.1.6}$$

provided

$$\delta > 1 - \frac{1}{p} > \mathbf{Re}\,z. \tag{2.1.7}$$

2.2. Exterior estimate. In this subsection we continue the analysis of the quantity $I(t,x)$ represented in (2.30) for the exterior domain

$$\{t - 2 \ge |x| \ge (1-\varepsilon)t\}. \tag{2.2.1}$$

In addition to this assumption, we shall assume that (2.24) is fulfilled.

First, we shall consider the case

$$s \ge t - \delta(t - |x|) \ \text{ on } \ \mathrm{supp}_s F, \tag{2.2.2}$$

where $\delta > 0$ is a sufficiently small number. Then Lemma implies that the density $h = Q^{(n-3)/2}$ satisfies the estimate $h \le C\rho^{n-3}\lambda^{n-3}$ so from (2.30) we have

$$\begin{aligned} I^{p'}(t,x) \ &\le \ \frac{C}{t^{(n-2)}} \times \\ &\times \ \Big(\int_{t-\delta(t-|x|)}^{t} \int_0^{t-s} (t-s)^{-\mathbf{Re}\,z\,p'} (t-s-\rho)^{-\mathbf{Re}\,z\,p'} \rho^{n-2}\lambda^{n-2} \times \\ &\times \ \int_{|\rho-|x||}^{\min(s-2,\rho+|x|)} (s+\lambda)^{-\gamma\,p'} |s-\lambda|^{-\delta\,p'} d\lambda d\rho ds \Big). \end{aligned}$$

From (2.2.1) and (2.2.2) we see that

$$s \ge t(1 - \varepsilon\delta) \ge \frac{t}{2}$$

for $\varepsilon\delta < 1/2$ and we have

$$(1+s+\lambda)^{-\gamma\,p'}\lambda^{n-2} \le Ct^{-\gamma\,p' + n-2}.$$

So we arrive at

$$|I(t,x)| \le \frac{C(t-|x|)^{(n-2)/p'}}{t^{\gamma}} \left(\int_{t-\delta(t-|x|)}^{t} \int_0^{t-s} (t-s)^{-\mathbf{Re}\,z\,p'} (t-s-\rho)^{-\mathbf{Re}\,z\,p'} d\rho ds \right)^{1/p'}$$

in view of $\rho \le t-s \le \delta(t-|x|)$. A direct computation for $\mathbf{Re}\, z\, p' < 1$ shows that

$$\int_{t-\delta(t-|x|)}^{t} \int_0^{t-s} (t-s)^{-\mathbf{Re} z\, p'} (t-s-\rho)^{-\mathbf{Re} z\, p'} d\rho ds \le C(t-|x|)^{2-2\mathbf{Re} z\, p'}.$$

Hence, we get

$$I(t,x) \le \frac{C}{t^\gamma}(t-|x|)^{-2\mathbf{Re} z+n-n/p}. \tag{2.2.3}$$

Assuming that

$$\gamma \ge \mathbf{Re}\, z \tag{2.2.4}$$

from (2.2.3) we find

$$I(t,x) \le Ct^{-\mathbf{Re} z}(t-|x|)^{-\mathbf{Re} z-\gamma+n-n/p} \tag{2.2.5}$$

and this completes the study of the case (2.2.2).

It remains to consider the case

$$(t-|x|)/4 \le s \le t-2(t-|x|) \quad \text{on} \quad \mathrm{supp} F. \tag{2.2.6}$$

A more precise evaluation of the quantity

$$Q(\lambda,\rho,|x|) = (\lambda-\rho+|x|)(\lambda+\rho-|x|)(\lambda+\rho+|x|)(\rho+|x|-\lambda),$$

appearing in (2.30), is based on (2.2.6) and the estimates

$$\lambda \le s \quad , \quad t-s \ge \rho \ge |x|-\lambda \ge t-s-(t-|x|)$$

valid on the integration domain in (2.30). Thus, we obtain

$$\begin{aligned}
\lambda-\rho+|x| &\le s-(t-s)+(t-|x|)+|x| = 2s,\\
\lambda+\rho-|x| &\le s+(t-s)-|x| = t-|x|,\\
\lambda+\rho+|x| &\le Ct,\\
\rho+|x|-\lambda &\le (t-s)+|x|-s \le Ct.
\end{aligned}$$

Hence, we get

$$h(\lambda,\rho,|x|) \le Ct^{n-3}s^{(n-3)/2}(t-|x|)^{(n-3)/2}.$$

This estimate and (2.30) lead to

$$\begin{aligned}
|I(t,x)| \le\ & \frac{C(t-|x|)^{(n-3)/2p'}}{t^{1/p'}} \times\\
&\times \Big(\int_{(t-|x|)/4}^{t-2(t-|x|)} \int_{t-s-(t-|x|)}^{t-s} (t-s)^{-\mathbf{Re} z\, p'}(t-s-\rho)^{-\mathbf{Re} z\, p'} \rho s^{(n-1)/2} \times\\
&\times \int_{|\rho-|x||}^{\min(s-2,\rho+|x|)} (s+\lambda)^{-\gamma p'}|s-\lambda|^{-\delta p'} d\lambda d\rho ds\Big)^{1/p'}.
\end{aligned}$$

From $\delta p' > 1$ and $\gamma p' > \frac{n+1}{2}$ we get

$$|I(t,x)| \leq \frac{C(t-|x|)^{(n-3)/2p'}}{t^{1/p'}} \times$$
$$\times \left(\int_{(t-|x|)/4}^{t-2(t-|x|)} (t-s)^{1-\mathbf{Re}\, z\, p'}(t-|x|)^{1-\mathbf{Re}\, z\, p'} s^{-\gamma p'+(n-1)/2} ds\right)^{1/p'}$$

in view of

$$\int_{t-s-(t-|x|)}^{t-s} (t-s-\rho)^{-\mathbf{Re}\, z\, p'} \rho d\rho \leq C(t-s)(t-|x|)^{1-\mathbf{Re}\, z\, p'}.$$

Having in mind that for $\gamma p' > \frac{n+1}{2}$ we have

$$\int_{(t-|x|)/4}^{t-2(t-|x|)} (t-s)^{1-\mathbf{Re}\, z\, p'} s^{-\gamma p'+(n-1)/2} ds \leq C(t-|x|)^{-\gamma p'+(n+1)/2} t^{1-\mathbf{Re}\, z\, p'},$$

we obtain

$$|I(t,x)| \leq Ct^{-\mathbf{Re} z}(t-|x|)^{-\gamma+n/p'-\mathbf{Re} z}$$

so we get

$$|I(t,x)| \leq Ct^{-\mathbf{Re} z}(t-|x|)^{-\mathbf{Re} z-\gamma+n-n/p}.$$

Thus, summarizing the estimates of the two subsections we arrive at the following.

Proposition 2.3. *Suppose* $\mathbf{Re}\, z < 1 - 1/p$. *Then the operator* W_z *satisfies the estimate*

$$\|\tau_+^\alpha \tau_-^\beta W_z(F)\|_{L^\infty} \leq C\|\tau_+^\gamma \tau_-^\delta F\|_{L^p} \tag{2.2.7}$$

provided

$$\alpha = \mathbf{Re}\, z, \tag{2.2.8}$$

$$\frac{n+1}{2}\left(1-\frac{1}{p}\right) < \gamma = \beta - \mathbf{Re}\, z + n - \frac{n}{p} < n\left(1-\frac{1}{p}\right), \tag{2.2.9}$$

$$\delta > 1 - \frac{1}{p}. \tag{2.2.10}$$

3. Reduction of the global L^2 - estimate to a local one. Our goal is to derive the following estimate on the line $\mathbf{Re}\, z = (n+1)/2 - \varepsilon$

$$\|\tau_+^{1/2-\varepsilon_1} \tau_-^{1/2} W_z(F)\|_{L^2} \leq C(z)\|\tau_+^{1/2}\tau_-^{1/2} F\|_{L^2} \tag{3.1}$$

for $\varepsilon_1 > 2\varepsilon$.

Here and below $C(z) = C \exp(b|\mathbf{Im} z|^2)$ is the constant appearing in a natural way from the estimate (2.1) of the Bessel function.

Having in mind that on the support of the kernel K_z of the operator W_z we have (see (1.24))

$$t - s \geq |x - y|,$$

we see that $t - |x| \geq s - |y|$ so making the change of variables $s \to \sigma = s - |y|$, we arrive at the representation formula

$$W_z(F)(t,x) = \int_0^{t-|x|} A_{z,t,\sigma}(F(\sigma + |.|, .))d\sigma, \tag{3.2}$$

where $A_{z,t,\sigma}$ is the Fourier integral operator defined by

$$A_{z,t,\sigma}(h)(x) = \iint e^{i(x-y)\xi}(t-\sigma-|y|)^{\frac{n}{2}-z}|\xi|^{z-\frac{n}{2}}J_{\frac{n}{2}-z}((t-\sigma-|y|)|\xi|)h(y)dyd\xi. \tag{3.3}$$

Further, we follow the construction from the Example considered in the section 2 and consider the truncated operator

$$H(\tau \leq t - |x| \leq 2\tau)W_z. \tag{3.4}$$

Recall that $H(B)$ denotes the characteristic function of the set B.

Then we have the decomposition

$$\begin{aligned} H(\tau \leq t - |x| \leq 2\tau)W_z \quad &= \quad H(\tau \leq t - |x| \leq 2\tau)W_z H(s - |y| \leq \delta\tau) + \\ & \qquad H(\tau \leq t - |x| \leq 2\tau)W_z H(s - |y| \geq \delta\tau), \end{aligned} \tag{3.5}$$

where $\delta > 0$ is a sufficiently small number.

First, we shall estimate the second term in the right side of (3.5).

Lemma 3.1. *We have the estimate*

$$\begin{aligned} &\|H(\tau \leq t - |x| \leq 2\tau)W_z(H(s - |y| \geq \delta\tau)F)\|_{L^2(\mathbb{R}^n)} \leq \\ &\leq C(z)\int_0^t (t-s)^{-1+\varepsilon}t^{\varepsilon}\|H(s-|y| \geq \delta\tau)F)\|_{L^2(\mathbb{R}^n)}ds. \end{aligned}$$

P r o o f. The inequality follows directly from (2.3) and the estimate (A.3) from the Appendix (applied with $\gamma = 0$.)

Now we can apply the argument given in the Example of section 2. In this way we get

$$\begin{aligned} &\|\tau_+^{1/2-\varepsilon_1}H(\tau \leq t - |x| \leq 2\tau)W_z(H(s - |y| \geq \delta\tau)F)\|_{L^2(\mathbb{R}^{n+1})} \leq \\ &\qquad\qquad \leq C(z)\|\tau_+^{1/2}H(s - |y| \geq \delta\tau)F\|_{L^2(\mathbb{R}^{n+1})} \end{aligned}$$

for $\varepsilon_1 > 2\varepsilon$. This estimate implies

$$\|\tau_+^{1/2-\varepsilon_1}\tau_-^{1/2}H(\tau \le t-|x| \le 2\tau)W_z(H(s-|y| \ge \delta\tau)F)\|_{L^2(\mathbb{R}^{n+1})} \le$$
$$\le C(z)\|\tau_+^{1/2}\tau_-^{1/2}H(s-|y| \ge \delta\tau)F\|_{L^2(\mathbb{R}^{n+1})}.$$

Therefore, the decomposition (3.5) shows that it remains to estimate

$$W_{z,\tau} = H(\tau \le t-|x| \le 2\tau)W_z H(s-|y| \le \delta\tau). \tag{3.6}$$

The key to estimate this operator is the representation formula (3.2) and the following scale-invariant L^2 estimate

$$\|H(\tau \le t-|x| \le 2\tau)A_{z,t,\sigma}(H(L \le t-\sigma-|.| \le 2L)H(M \le |.| \le 2M)f)\|^2_{L^2(\mathbb{R}^n)} \le$$
$$\le C(z)^2\frac{t^{-1+2\varepsilon}M}{\tau}L^{-1+2\varepsilon}\|f\|^2_{L^2(\mathbb{R}^n)}. \tag{3.7}$$

It is important to underline that the constant $C(z)$ in (3.7) is independent of the parameters t,τ,σ,L,M. We shall see how the estimate (3.7) leads to the desired L^2- weighted estimate for the operator W_z.

The starting point is the representation of the kernel $K_{z,\tau}$ of the operator $W_{z,\tau}$ in (3.6) as a sum over integers j,k of the kernels of the form

$$K_{z,\tau,j,k} = H(2^j\tau\delta \le t-s < 2^{j+1}\tau\delta)K_{z,\tau}H(2^k\tau\delta \le s < 2^{k+1}\tau\delta). \tag{3.8}$$

for $k+2 \le j$ and of the form

$$K_{z,\tau,j,k,l} = K_{z,\tau,j,k}H(2^j l\tau\delta \le s < 2^j(l+1)\tau\delta) \tag{3.9}$$

for $k+2 > j, l = 0,...,2^k$.

Denote by $W_{z,\tau,j,k}$ the operator having kernel $K_{z,\tau,j,k}$. Respectively $W_{z,\tau,j,k,l}$ denotes the operator with kernel $K_{z,\tau,j,k,l}$ We can represent the above sum as

$$I + II + III + IV, \tag{3.10}$$

where

$$I = \sum_{j=2}^{\infty}\sum_{k=-\infty}^{-1} W_{z,\tau,j,k},$$
$$II = \sum_{j=2}^{\infty}\sum_{k=0}^{j-2} W_{z,\tau,j,k},$$

$$
\begin{aligned}
III &= \sum_{j=2}^{\infty}\sum_{k=j-1}^{\infty}\sum_{l} W_{z,\tau,j,k,l}, \\
IV &= \sum_{j=-\infty}^{1}\sum_{k} W_{z,\tau,j,k}.
\end{aligned} \tag{3.11}
$$

The form of the operator $W_{z,\tau}$ in (3.6) and the definition (3.8) guarantee that for the first term I in (3.11) we can apply the estimate (2.15) with $p = 2$. In this way we get

$$
\|\tau_+^{1/2-\varepsilon_1}\tau_-^{1/2} I(F)\|_{L^2} \le C(z)\|\tau_+^{1/2}\tau_-^{1/2}F\|_{L^2}
$$

for $\varepsilon_1 > 2\varepsilon$. To estimate the terms II and III we use the estimate (3.7) and get

$$
\|W_{z,\tau,j,k}(F)(t,.)\|_{L^2(\mathbb{R}^n)} \le \frac{C(z)}{\tau^{1-2\varepsilon}} H(C^{-1}2^j\tau\delta \le t < C2^{j+1}\tau\delta) \times
$$
$$
\times \int_0^{\delta\tau} 2^{-j+2j\varepsilon}2^{k/2}\|H(2^k\tau\delta \le \sigma + |.| < 2^{k+1}\tau\delta)F(\sigma + |.|,.)\|_{L^2} d\sigma.
$$

where $k + 2 \le j$ and C is a sufficiently large constant. Taking the L^2-norm with respect to t and setting $F_k = H(2^k\tau\delta \le s < 2^{k+1}\tau\delta)F$, we find

$$
\tau^{1-2\varepsilon}2^{j/2-2j\varepsilon}\|W_{z,\tau,j,k}(F)\|_{L^2} \le C(z)2^{k/2}\|\tau_-^{1/2+\varepsilon_0}F_k\|_{L^2}, \tag{3.12}
$$

where $\varepsilon > 0$. Since $2^j\tau$ is equivalent to $\tau_+(t,x)$ and $2^k\tau$ is equivalent to $\tau_+(s,y)$, we get

$$
\begin{aligned}
\|\tau_+^{1/2-\varepsilon_1}\tau_-^{1/2} II(F)\|_{L^2} &\le C(z)\sum_{j=2}^{\infty}\sum_{k=0}^{j-2} 2^{k/2}\tau^{1/2}2^{-j\varepsilon_1+2j\varepsilon}\|\tau_-^{1/2}F_k\|_{L^2} \le \\
&\le C(z)\|\tau_+^{1/2}\tau_-^{1/2}F_k\|_{L^2}
\end{aligned}
$$

for $\varepsilon_1 > 2\varepsilon$ and $\varepsilon_0 < \varepsilon_1 - 2\varepsilon$.

For the third term III we apply the inequality (3.7) and obtain.

$$
\|\tau_+^{-\varepsilon_1} III(F_k)\|_{L^2}^2 \le C(z)^2\frac{2^{-2k\varepsilon_1}}{\tau^{2\varepsilon_1}}\sum_l \left(\int_{2^j(l-1)\tau\delta}^{2^j(l+2)\tau\delta} \|\sum_{j\le k+1} W_{z,\tau,j,k,l}(F_k)(t,.)\|_{L^2(\mathbb{R}^n)}^2 dt\right)
$$

$$
\le \frac{C(z)^2}{\tau}\sum_{j\le k+1} 2^{k\varepsilon_0}\sum_l 2^{-2k\varepsilon_1+4k\varepsilon}\|H(2^j l\tau\delta \le s < 2^j(l+1)\tau\delta)\tau_-^{1/2}F\|_{L^2}^2 \le
$$

$$
\le \frac{C(z)^2}{\tau}\sum_k 2^{-k\varepsilon_1+2k\varepsilon}\|\tau_-^{1/2+\varepsilon_0}F\|_{L^2}^2.
$$

Note that $\tau_+(t,x)$ is equivalent to $\tau_+(s,y)$ in the kernel of the operator III. Taking the sum over k for $\varepsilon_1 > 2\varepsilon$ and $\varepsilon_0 < \varepsilon_1 - 2\varepsilon$, we get

$$\|\tau_+^{-\varepsilon_1}\tau_-^{1/2} III(F)\|_{L^2} \le C(z)\|\tau_-^{1/2}F\|_{L^2}.$$

Finally, the term IV in (3.11) is identically 0. Indeed, on the kernel of this operator we have

$$t - s \ge |x-y|, \tag{3.13}$$

$$s - |y| \le \delta\tau, \tag{3.14}$$

$$\tau \le t - |x| \le 2\tau, \tag{3.15}$$

$$t - s \le \delta\tau. \tag{3.16}$$

The inequalities (3.13), (3.15) and (3.16) imply

$$s - |y| = s - t + t - |x| + |x| - |y| \ge t - |x| - (t-s) - |x-y| \ge \tau - 2\delta\tau.$$

Choosing $\delta > 0$ sufficiently small, we see that this contradicts (3.14), so the term IV is identically zero. Therefore, we arrive at the following weighted estimate for the operator $W_{z,\tau}$

$$\|\tau_+^{1/2-\varepsilon_1}\tau_-^{1/2}W_{z,\tau}(F)\|_{L^2} \le C(z)\|\tau_+^{1/2}\tau_-^{1/2}F\|_{L^2}$$

for $\varepsilon_1 > 2\varepsilon$. The argument given in the Example considered in section 2 yields the desired estimate (3.1).

4. Local scale-invariant estimate. The main purpose of this section will be the proof of the estimate (3.7). For the sake of simplicity we shall omit the indices z, t, σ. Thus we have to verify the inequality

$$\begin{aligned} &\|H(\tau \le t - |x| \le 2\tau)A(H(L \le t - \sigma - |.| \le 2L)H(M \le |.| \le 2M)f)\|^2_{L^2(\mathbb{R}^n)} \le \\ &\le C(z)^2 \frac{t^{-1+2\varepsilon}ML^{-1+2\varepsilon}}{\tau}\|f\|^2_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{4.1}$$

After a rescaling we reduce the proof to the case $1 \le t \le 2$, $0 < \tau < t$. In this case the definition of the weights $\tau_\pm$ is $t \pm |x|$. To this end we can use the following asymptotic expansion of the Bessel function ([39])

$$\begin{aligned} s^{-\nu}J_\nu(s) &= a_0(s) \quad &&\text{if} \quad 0 \le s \le 1, \\ s^{-\nu}J_\nu(s) &= e^{is}a_+(s)/s^{\nu-1/2} + e^{-is}a_-(s)/s^{\nu-1/2} \quad &&\text{if} \quad s \ge 1, \end{aligned}$$

where $a_0(s)$ is a smooth compactly supported function, while $a_\pm(s)$ are smooth symbols of order 0. Given any smooth compactly supported function χ which is 1 near the origin, we make the following decomposition

$$\begin{aligned}(t-\sigma-|y|)^{-\nu}|\xi|^{-\nu}J_\nu((t-\sigma-|y|)|\xi|) &= a_0((t-\sigma-|y|)|\xi|)\chi(|\xi|L)\\ +e^{i(t-\sigma-|y|)|\xi|}a_+((t-\sigma-|y|)|\xi|)/((t-\sigma-|y|)|\xi|)^{\nu-1/2}(1-\chi(|\xi|L))\\ +e^{-i(t-\sigma-|y|)|\xi|}a_-((t-\sigma-|y|)|\xi|)/((t-\sigma-|y|)|\xi|)^{\nu-1/2}(1-\chi(|\xi|L).\end{aligned}$$

It is clear that the operator A can be decomposed correspondingly as a sum $A^0+A^++A^-$ of three Fourier integral operators. For the first one the support in ξ is in the ball of radius proportional to L^{-1}. Therefore, the kernel of this operator is a classical function satisfying the estimate

$$|K^0_{t,\sigma}(x,y)| \le \frac{C}{(t-\sigma-|y|)^{(n+1)-2\varepsilon}} \tag{4.2}$$

We shall need the following.

Lemma 4.1. *On the intersection of the support of the kernel of the operator A and the function $H(L \le t-\sigma-|.| \le 2L)H(M \le |.| \le 2M)f$ we have*

$$\left|\frac{x}{|x|}-\frac{y}{|y|}\right| \le Cd(t,\tau,L,M) \tag{4.3}$$

where

$$d(t,\tau,L,M)^2 = \frac{L\tau}{M}. \tag{4.4}$$

Proof. For $|x| \le t/2$ the quantity τ is equivalent to a constant. Then the quantity $d(t,\tau,L,M)$ is also equivalent to a constant and the desired estimate (4.3) is trivial. For $|x| \ge t/2$ we have

$$\begin{aligned}&\sigma \le \delta\tau,\\ &\delta\tau \le t-|x| \le t-\sigma-|y|\\ &\tau < t-|x| < 2\tau\\ &t-\sigma-|y| \ge |x-y|\end{aligned}$$

on the intersection of the supports of the kernel of A and $H(L \le t-\sigma-|.| \le 2L)H(M \le |.| \le 2M)f$. A continuity argument shows it is sufficient to establish (4.3) for $\delta=\sigma=0$. But then (4.3) follows from

$$(t-\sigma-|y|)^2 = (t-|y|)^2 \ge |x-y|^2$$

and the relation

$$|x-y|^2 = (|x|-|y|)^2 + \frac{|x||y|}{2}\left(\frac{x}{|x|}-\frac{y}{|y|}\right)^2$$

Now the fact that $|x| > t/2$ is equivalent to constant completes the proof of the lemma. $\square$

From (4.2) and (4.3) and the Cauchy inequality we get the estimate

$$\|H(\tau \le t-|x| \le 2\tau)A^0(H(L \le t-\sigma-|.| \le 2L)H(M \le |.| \le 2M)f)\|_{L^2(\mathbb{R}^n)} \le$$
$$\le Cd(t,\tau,L,M)^{n-1}L^{-n-1+2\varepsilon}\sqrt{\tau L} \times$$
$$\times \|H(L \le t-\sigma-|.| \le 2L)H(M \le |.| \le 2M)f\|_{L^2(\mathbb{R}^n)}.$$

Note that

$$\sqrt{\tau L}d(t,\tau,L,M)^{n-1}L^{-n-1+2\varepsilon} \le CL^{-\frac{1}{2}+2\varepsilon}\tau^{-\frac{1}{2}}.$$

This estimate yields

$$\|H(\tau \le t-|x| \le 2\tau)A^0(f)\|^2_{L^2(\mathbb{R}^n)} \le$$
$$\le \frac{C(z)^2}{\tau}\|(t-\sigma-|.|)^{-1/2+\varepsilon}f\|^2_{L^2(\mathbb{R}^n)}.$$

Each of the operators $A^{\pm}$ has symbol of type

$$a_{\pm}((t-\sigma-|y|)|\xi|)(t-\sigma-|y|)^{-\varepsilon}|\xi|^{-\varepsilon}(1-\chi(L|\xi|)). \tag{4.5}$$

Note that in the above symbol we use cut-off function only in ξ coordinates. The cut-off function $\chi(s)$ is 1 near the origin and 0 for $s \ge 1$. Since $t-\sigma-|y|$ is equivalent to L, we see that we can replace $t-\sigma-|y|$ by L in the symbol and in this way to freeze in y the symbol. In fact for $|\xi| \ge 1/L$ the symbol $a_{\pm}((t-\sigma-|y|)|\xi|)$ can be replaced by

$$\sum_{j=0}^{n}\frac{c_j}{(t-\sigma-|y|)^j|\xi|^j} + O\left(\frac{1}{(t-\sigma-|y|)^{n+1}|\xi|^{n+1}}\right).$$

Then we can attach the bounded factors $(t-\sigma-|y|)^jL^{-j}$ to the function f and in this way we see that without loss of generality we can assume that the symbol in (4.5) is replaced by

$$b(\xi) = b_{t,\sigma,\tau,L,M}(\xi) = a_{\pm}(L|\xi|)(t-\sigma-k\delta\tau))^{-\varepsilon}|\xi|^{-\varepsilon}(1-\chi(L|\xi|)). \tag{4.6}$$

Therefore, we have to estimate the operator

$$A(f)(x) = \int\int e^{i\phi_{t,\sigma}(x,y,\xi)}b(\xi)f(y)dyd\xi$$

with a phase function of type

$$\phi_{t,\sigma}(x, y, \xi) = (x - y)\xi - (t - \sigma - |y|)|\xi|.$$

Our main observation is that on the set of critical values

$$\{(x, y, \xi) : \nabla_\xi \phi(x, y, \xi) = 0\} \tag{4.7}$$

the second Hessian satisfies the estimate

$$\left|\partial^2_{y,\xi}\phi(x, y, \xi)\right| \geq C\frac{\tau}{|y|(t - \sigma - |y|)}. \tag{4.8}$$

In fact, the set (4.7) is determined by

$$x - y = (t - \sigma - |y|)\xi/|\xi|. \tag{4.9}$$

Since $\sigma \leq \delta\tau$ with $\delta > 0$ sufficiently small, we see it is sufficient to establish (4.8) for $\delta = \sigma = 0$. But then (4.9) implies that

$$\left|\partial^2_{y,\xi}\phi(x, y, \xi)\right| = \frac{1}{2}\left|\frac{y}{|y|} - \frac{\xi}{|\xi|}\right|^2 = \frac{1}{2}\left|\frac{y}{|y|} - \frac{x - y}{|x - y|}\right|^2. \tag{4.10}$$

Then the estimate (4.3) and the sin-theorem for the triangle with sides $|x|, |y|, |x - y|$ implies that (4.8) is valid.Having in mind that $|y|$ can be replaced by M, we see that the right side of (4.8) is equivalent to

$$D(t, \tau, L, M)^2 = \frac{\tau}{ML} \tag{4.11}$$

It is not difficult to see that the quantity in (4.11) is equivalent to positive constant in any of the following three cases

Case A: τ is equivalent to constant,

Case B: L is equivalent to τ and τ is sufficiently small,

Case C: M/τ is bounded.

In case the quantity in (4.11) is equivalent to positive constant the operator A is a local canonical graph so one can apply the well known result about the L^2- boundedness of the Fourier integral operators which are local canonical graph.

This observation shows that we lose no generality assuming

$$\tau \leq \delta, \quad L \geq N\tau, \quad M \geq N\tau, \tag{4.12}$$

where $\delta > 0$ is sufficiently small, while N is sufficiently large.

Further, we make a partition of unity on the sphere S^{n-1}

$$1 = \sum_m \psi_m\left(\frac{x}{|x|}\right)$$

where ψ_m are smooth functions on the unit sphere with diameter of the support proportional to d, where $d = d(t, \tau, L, M)$ is the quantity defined in (4.4). The partition of the unity is locally finite. Moreover, there exists a number N such that at most N supports of functions of the partition overlap and this number is independent of the diameter of the supports of the functions ψ_m. Then we represent the operator A as

$$\sum_{m,l} \psi_m\left(\frac{x}{|x|}\right) A\psi_l\left(\frac{y}{|y|}\right). \tag{4.13}$$

Lemma 4.1 implies that

$$\left|\frac{x}{|x|} - \frac{y}{|y|}\right| \leq Cd(t, \tau, L, M), \tag{4.14}$$

when (x, y) is in the intersection of the supports of the kernel of A and f. This estimate shows that $A(f)$ can be represented as a sum of terms

$$\psi_m\left(\frac{x}{|x|}\right) A\left(\psi_l\left(\frac{y}{|y|}\right) f\right)$$

with sum taken over m, l with $|m - l| \leq N_0$ and N_0 is independent of t, τ, σ, L, M. This observation and the fact that the norm $\|f\|^2_{L^2}$ is equivalent to

$$\sum_m \|\psi_m f\|^2_{L^2}$$

shows that it is sufficient to estimate the localized operator

$$A_{m,l} = \psi_m\left(\frac{x}{|x|}\right) A\psi_l\left(\frac{y}{|y|}\right) \tag{4.15}$$

for fixed m, l with $|m - l| \leq N_0$.

Assuming that the support of ψ_m is at distance δd from the fixed point $(1, 0, \ldots, 0)$ on the unit sphere, we introduce local coordinates $x' = (x_2, \ldots, x_n)$ on this support. Then for $\delta > 0$ sufficiently small we can use the same coordinates for the support of ψ_l. For example, we can take $\delta N_0 = 1$. Then we have

$$|x'| \leq \delta d x_1, \quad |y'| \leq d y_1 \tag{4.16}$$

on the support of the operator in (4.15). Then we make the following rescaling in x', y', ξ' coordinates

$$y' \to \overline{y'} = d^{-1}y' \tag{4.17}$$
$$x' \to \overline{x'} = \lambda^{-1}x', \tag{4.18}$$
$$\xi' \to \overline{\xi'} = \mu^{-1}\xi', \tag{4.19}$$

where λ, μ shall be chosen so that the second Hessians $\partial^2_{y,\xi}\phi$ and $\partial^2_{x,\xi}\phi$ are uniformly nondegenerate in the new coordinates. In fact, the phase function takes the form

$$\phi(\overline{x},\overline{y},\overline{\xi}) = (\lambda\overline{x}' - d\overline{y}')\mu\overline{\xi}' + (\overline{x}_1 - \overline{y}_1)\overline{\xi}_1 - \left(t - \sigma - \sqrt{\overline{y}_1^2 + |d\overline{y}'|^2}\right)\sqrt{\overline{\xi}_1^2 + |\mu\overline{\xi}'|^2}.$$

The second Hessian $|\partial^2_{\overline{x}\overline{\xi}}\phi|$ is proportional to $(\lambda\mu)^{n-1}$. So we can take

$$\lambda\mu = 1 \tag{4.20}$$

in order to assure uniform nondegeneracy of this Hessian. The other Hessian $|\partial^2_{\overline{y}\overline{\xi}}\phi|$ on the set

$$\{(\overline{x},\overline{y},\overline{\xi}) : \partial_{\overline{\xi}}\phi = 0\}$$

is equivalent to $(\mu d)^{n-1}D^2$, where $D = D(t,\tau,L,M)$ is defined according to (4.11). Now we make the choice

$$(\mu d)^{n-1}D^2 = 1 \tag{4.21}$$

so from this requirement and (4.21) we determine $\lambda\mu$ as follows

$$\lambda = dD^{2/(n-1)}, \quad \mu = 1/\lambda = d^{-1}D^{-2/(n-1)}. \tag{4.22}$$

The operator $A_{m,l}$ in the new coordinates will have the form

$$\overline{A_{m,l}}(h)(\overline{x}) = L^{-1+\varepsilon}\int\int e^{i\phi(\overline{x},\overline{y},\overline{\xi})}\psi_m b(\xi(\overline{\xi}))\psi_l h(\overline{y})d\overline{y}d\overline{\xi}.$$

This operator is a local canonical graph. Note that the $\overline{x}$ coordinates are not localized, but the independence of the symbol of the operator of $|x|$ assures the L^2 boundedness of thei operator, i.e.

$$\int\left|\overline{A_{m,l}}(h)(\overline{x}\right|^2 d\overline{x} \le CL^{-2+2\varepsilon}\int|\psi_l h(\overline{y})|^2 d\overline{y}.$$

Making the inverse change of variables in (4.19) we get

$$\int|A_{m,l}(h)(x)|^2 dx \le CL^{-2+2\varepsilon}\mu^{2n-2}(\lambda d)^{n-1}\int|\psi_l h(y)|^2 dy.$$

From (4.20) and (4.21) we get

$$\mu^{2n-2}(\lambda d)^{n-1} = 1/D^2$$

so

$$\|A_{m,l}(f)\|_{L^2} \leq C(t-\sigma-k\delta\tau)^{-1+\varepsilon} D^{-1} \|\psi_l f\|_{L^2}$$

From this estimate and (4.11) we arrive at

$$\|A_{m,l}(f)\|_{L^2} \leq C M^{1/2}(t-\sigma-k\delta\tau)^{-1/2+\varepsilon} \tau^{-1/2} \|\psi_l f\|_{L^2}$$

and this completes the proof of the estimate (4.1).

5. Interpolation. For $\mathbf{Re}\, z = a < 1$, the estimates of Section 2 guarantee that

$$\|\tau_+^{\alpha_0} \tau_-^{\beta_0} W_z(F)\|_{L^{q_0}} \leq C \|\tau_+^{\gamma_0} \tau_-^{\delta_0} F\|_{L^{p_0}}, \tag{5.1}$$

provided

$$q_0 = \infty, \tag{5.2}$$

$$\mathbf{Re}\, z = a < 1 - \frac{1}{p_0}, \tag{5.3}$$

$$\alpha_0 = \mathbf{Re}\, z, \tag{5.4}$$

$$\beta_0 = \mathbf{Re}\, z + \gamma_0 - n + \frac{n}{p_0}, \tag{5.5}$$

$$\frac{n+1}{2}\left(1 - \frac{1}{p_0}\right) < \gamma_0 < n\left(1 - \frac{1}{p_0}\right), \tag{5.6}$$

$$\delta_0 > 1 - \frac{1}{p_0} \tag{5.7}$$

and the supports of u, F are contained in the cone $\{|x| \leq t + R\}$. On the other hand, the estimates of section 3 assure that for $\mathbf{Re}\, z = (n+1)/2 - \varepsilon$ we have

$$\|\tau_+^{\alpha_1} \tau_-^{\beta_1} W_z(F)\|_{L^{q_1}} \leq C(z) \|\tau_+^{\gamma_1} \tau_-^{\delta_1} F\|_{L^{p_1}}, \tag{5.8}$$

provided $C(z) = C\exp(b|\mathbf{Im}\, z|^2)$ and the conditions

$$\alpha_1 + 2\varepsilon < \beta_1 = \gamma_1 = \delta_1 = \frac{1}{2} \tag{5.9}$$

are fulfilled. Our next goal is the application of the Stein interpolation theorem (see [25]). Recall that this interpolation theorem concerns an analytic family of operators $T(z)$ acting from $L^{p_0} + L^{p_1}$ into $L^{q_0} + L^{q_1}$. We shall denote by $\mathcal{L}(X;Y)$ the Banach space of linear operators acting from a Banach space X into a Banach space Y.

Theorem 5.1. (Stein interpolation theorem, see [25]) *Suppose* $1 \le p_0, p_1, q_0, q_1 \le \infty$, *$T(t)$ is a continuous function from the strip* $0 \le \mathbf{Re}\, z \le 1$ *into* $\mathcal{L}(L^{p_0} + L^{p_1}; L^{q_0} + L^{q_1})$, *analytic for* $0 < \mathbf{Re}\, z < 1$ *and satisfying the properties*

$$\|T(z)\|_{\mathcal{L}(L^{p_0};L^{q_0})} \le C\exp(b|\mathbf{Im}\, z|^2) \quad \text{for} \quad \mathbf{Re}\, z = 0, \tag{5.10}$$

$$\|T(z)\|_{\mathcal{L}(L^{p_1};L^{q_1})} \le C\exp(b|\mathbf{Im}\, z|^2) \quad \text{for} \quad \mathbf{Re}\, z = 1. \tag{5.11}$$

Then for any $\theta \in (0,1)$ *we have*

$$\|T(\theta)\|_{\mathcal{L}(L^p;L^q)} \le C,$$

where

$$\frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1} \quad , \quad \frac{1}{q} = (1-\theta)\frac{1}{q_0} + \theta\frac{1}{q_1}. \tag{5.12}$$

A minor modification of this result is valid for the case of weighted L^p- spaces.

Theorem 5.2. (weighted Stein interpolation theorem) *Suppose the assumptions of Theorem* 5.1 *are fulfilled and* $\varphi_0, \psi_0, \varphi_1, \psi_1 > 1$ *are positive weights , such that*

$$\|\varphi_0 T(z) f\|_{L^{q_0}} \le C\exp(b|\mathbf{Im}\, z|^2)\|\psi_0 f\|_{L^{p_0}} \quad \text{for} \quad \mathbf{Re}\, z = 0, \tag{5.13}$$

$$\|\varphi_1 T(z) f\|_{L^{q_1}} \le C\exp(b|\mathbf{Im}\, z|^2)\|\psi_1 f\|_{L^{p_1}} \quad \text{for} \quad \mathbf{Re}\, z = 1, \tag{5.14}$$

Then for any $\theta \in (0,1)$ *we have*

$$\|\varphi T(\theta) f\|_{L^q} \le C\|\psi f\|_{L^p},$$

where q, p *are defined by* (5.12) *and*

$$\varphi = \varphi_0^{1-\theta}\varphi_1^{\theta},$$

$$\psi = \psi_0^{1-\theta}\psi_1^{\theta}.$$

P r o o f. If we consider the family of operators

$$\varphi_0^{1-z}\varphi_1^z T(z)\psi_0^{-1+z}\psi_1^{-z},$$

then we can apply the classical Stein interpolation theorem from Theorem 5.1 and in this way we obtain the conclusion of the Theorem.

The application of the weighted Stein interpolation theorem gives for $z = (n-1)/2$

$$\|\tau_+^{\alpha}\tau_-^{\beta}W_z(F)\|_{L^q} \leq C\|\tau_+^{\gamma}\tau_-^{\delta}F\|_{L^p}, \tag{5.15}$$

where

$$\frac{n-1}{2} = (1-\theta)\alpha_0 + \theta(\frac{n+1}{2} - \varepsilon) \tag{5.16}$$

for some real number $\theta \in (0,1)$. The numbers p, q are determined by

$$\frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1}, \quad , \quad \frac{1}{q} = (1-\theta)\frac{1}{q_0} + \theta\frac{1}{q_1} \tag{5.17}$$

and the parameters $\alpha, \beta, \gamma, \delta$ are defined by

$$\begin{aligned} \alpha &= \alpha_0(1-\theta) + \alpha_1\theta, \\ \beta &= \beta_0(1-\theta) + \beta_1\theta, \\ \gamma &= \gamma_0(1-\theta) + \gamma_1\theta, \\ \delta &= \delta_0(1-\theta) + \delta_1\theta. \end{aligned} \tag{5.18}$$

Since $q_0 = \infty, p_1 = 1/2, q_1 = 1/2$, from (5.16) (5.17) we can express p_0, a, θ as functions of p, q.. More precisely, the needed expressions have the form

$$\begin{aligned} \theta &= \frac{2}{q}, \\ \frac{1}{p_0} &= \frac{\frac{1}{p} - \frac{1}{q}}{1 - \frac{2}{q}}, \\ \alpha_0 &= \frac{\frac{n-1}{2} - \frac{n+1}{q} + \frac{2\varepsilon}{q}}{1 - \frac{2}{q}}. \end{aligned} \tag{5.19}$$

The requirements $1 < p_0 < \infty, a = \alpha_0 < 1 - 1/p_0$ lead to the following restrictions on $1 < p, q < \infty$

$$\begin{aligned} &\frac{1}{q} < \frac{1}{p} \quad , \quad \frac{1}{q} + \frac{1}{p} \leq 1, \\ &\frac{n-3}{2} < \frac{n}{q} - \frac{1}{p} \end{aligned} \tag{5.20}$$

for $\varepsilon > 0$ sufficiently small. These conditions coincide with the assumption (1.8) of Theorem 1.

From (5.18), (5.4)–(5.5) and (5.9) for $\varepsilon > 0$ sufficiently small we obtain

$$\begin{aligned} \alpha &< \frac{n-1}{2} - \frac{n}{q}, \\ \beta &= \gamma - \frac{n+1}{2} + \frac{n}{p} - \frac{1}{q}. \end{aligned} \tag{5.21}$$

Further, the condition (5.6) can be rewritten as follows

$$\begin{aligned} &\frac{n+1}{2}\left(1 - \frac{1}{p} - \frac{1}{q}\right) + \frac{1}{q} < \gamma, \\ &\gamma < n\left(1 - \frac{1}{p} - \frac{1}{q}\right) + \frac{1}{q} \end{aligned} \tag{5.22}$$

or equivalently as

$$\frac{n-1}{2p} - \frac{n+1}{2q} < \beta < \frac{n-1}{2} - \frac{n}{q}. \tag{5.23}$$

Finally, (5.7) takes the form

$$\delta > 1 - \frac{1}{p}. \tag{5.24}$$

Thus, we conclude that the assumptions of Theorem 1 mean that the estimate (5.15) is fulfilled.

This completes the proof of Theorem 1.

We shall need also an estimate of the solution of the Cauchy problem for the homogeneous wave equation

$$\begin{aligned} &(\partial_t^2 - \Delta)u = 0, \\ &u(0,x) = \varepsilon f(x) \quad , \quad \partial_t u(0,x) = \varepsilon g(x). \end{aligned} \tag{5.25}$$

We shall assume that $f, g \in C_0^\infty(\mathbb{R}^n)$.

For the purpose we shall use the following weighted Sobolev inequality due to S. Klainerman [17]

$$|u(t,x)| \le C(1+t+|x|)^{-(n-1)/2}(1+|t-|x||)^{-1/2} \sum_{|\alpha|\le m} \|\Gamma^\alpha u(t,.)\|_{L^2}. \tag{5.26}$$

Here $\Gamma^\alpha = \Gamma_0^{\alpha_0}\Gamma_1^{\alpha_1}, \cdots, \Gamma_N^{\alpha_N}$, where $\Gamma_0, \Gamma_1, \cdots, \Gamma_N$ are the generators of a conformal group in $\mathbb{R}^{n+1}$. To be more precise, $\Gamma_1, \cdots, \Gamma_N$ are the vector fields

$$\begin{aligned} &\partial_t, \partial_{x_1}, \cdots, \partial_{x_n}, \\ &t\partial_{x_j} + x_j\partial_t, \quad x_j\partial_{x_k} - x_k\partial_{x_j} \end{aligned} \tag{5.27}$$

and Γ_0 is the scaling, i.e. $\Gamma_0 = t\partial_t + \sum_{j=1}^n x_j \partial_{x_j}$. The vector fields $\Gamma_0, \Gamma_1, \cdots, \Gamma_N$ generate a Lie algebra. Each of the generators in (5.27) commutes with $(\partial_t^2 - \Delta)$. Moreover, we have $[(\partial_t^2 - \Delta), \Gamma_0] = 2(\partial_t^2 - \Delta)$.

Applying the conformal energy estimate (see [18]) for the wave equation, we dominate the right side of (5.26) from above by

$$C\varepsilon(1+t+|x|)^{-(n-1)/2}(1+|t-|x||)^{-1/2}.$$

Therefore, we have the estimate

$$|u(t,x)| \leq C\varepsilon(1+t+|x|)^{-(n-1)/2}(1+|t-|x||)^{-1/2}.$$

From this estimate and the Hölder inequality we obtain the following.

Proposition 5.1. *The solution of* (5.25) *satisfies the estimate*

$$\|\tau_+^\alpha \tau_-^\beta u\|_{L^q(\mathbb{R}_+^{n+1})} \leq C\varepsilon \tag{5.28}$$

provided $q \geq 2n/(n-1)$ *and the nonnegative parameters* α, β *satisfy*

$$\alpha < \frac{n-1}{2} - \frac{n}{q} \quad , \quad \beta \leq \frac{1}{2} - \frac{1}{q}. \tag{5.29}$$

6. Application to the semilinear wave equation. In this section we shall prove Theorem 2.

It is evident that for $\lambda > 1$ the requirements (1.15) mean that

$$(n-1)\lambda^2 - (n+1)\lambda - 2 > 0 \tag{6.1}$$

and $\lambda < (n+3)/(n-1)$.

We shall look for the solution of (1.12) in the form

$$u = v + u_0, \tag{6.2}$$

where u_0 is the solution of the linear Cauchy problem

$$\begin{aligned} &(\partial_t^2 - \Delta)u_0 = 0, \\ &u_0(0,x) = \varepsilon f, \qquad \partial_t u_0(0,x) = \varepsilon g. \end{aligned} \tag{6.3}$$

Then (1.12) is reduced to the following nonlinear problem for v

$$\begin{aligned} (\partial_t^2 - \Delta)v &= F(v+u_0), \\ v(0,x) &= \partial_t v(0,x) = 0. \end{aligned} \tag{6.4}$$

To prove the existence of a solution v, we construct the recurrent sequence $\{v_j\}_{j=1}^{\infty}$ as follows: $v_0 = 0$

$$
\begin{aligned}
(\partial_t^2 - \Delta)v_{j+1} &= F(v_j + u_0), \\
v_{j+1}(0,x) &= \partial_t v_{j+1}(0,x) = 0.
\end{aligned} \tag{6.5}
$$

Our goal is to show that v_j is a Cauchy sequence in $L^q_{\alpha,\beta}$.

The plan to prove the existence of a global solution consists then of two steps, namely we have to show that

$$
\|v_j\|_{L^q_{\alpha,\beta}} \leq C\varepsilon \tag{6.6}
$$

for suitable q, α, β. Further , we shall prove the estimate

$$
\|v_{j+1} - v_j\|_{L^q_{\alpha,\beta}} \leq C\varepsilon^{\lambda-1}\|v_j - v_{j-1}\|_{L^q_{\alpha,\beta}}. \tag{6.7}
$$

Taking $\varepsilon > 0$ sufficiently small, we obtain via the contraction mapping theorem the existence and uniqueness of the solution in the Banach space $L^q_{\alpha,\beta}$. To establish (6.6) we apply Theorem 1 and Proposition 5.1 and obtain

$$
\|\tau_+^{\alpha}\tau_-^{\beta} v_{j+1}\|_{L^q} \leq C\varepsilon + \|\tau_+^{\gamma}\tau_-^{\delta}|v_j + u_0|^{\lambda}\|_{L^p}, \tag{6.8}
$$

where $\alpha, \beta, \gamma, \delta, p, q$ satisfy the assumptions of Theorem 1.

Taking

$$
\frac{1}{p} = \frac{\lambda}{q}, \tag{6.9}
$$

and

$$
\begin{aligned}
\gamma &= \alpha\lambda, \\
\delta &= \beta\lambda,
\end{aligned} \tag{6.10}
$$

we get

$$
\|\tau_+^{\alpha}\tau_-^{\beta} v_{j+1}\|_{L^q} \leq C\varepsilon + C\|\tau_+^{\alpha}\tau_-^{\beta}(v_j + u_0)\|_{L^q}^{\lambda} \tag{6.11}
$$

Proposition 5.1 yields

$$
\|\tau_+^{\alpha}\tau_-^{\beta} u_0\|_{L^q} \leq C\varepsilon. \tag{6.12}
$$

Now the estimate

$$
\|\tau_+^{\alpha}\tau_-^{\beta} v_{j+1}\|_{L^q} \leq C\varepsilon + C\|\tau_+^{\alpha}\tau_-^{\beta} v_j\|_{L^q}^{\lambda}
$$

leads to the desired estimate (6.6). The other estimate (6.7) can be derived in a similar manner if we take into account the fact that the difference $w_{j+1} = v_{j+1} - v_j$ satisfies the equation

$$\begin{aligned}(\partial_t^2 - \Delta)w_{j+1} &= F(v_j + u_0) - F(v_{j-1} + u_0).\\ w_{j+1}(0,x) &= \partial_t w_{j+1}(0,x) = 0.\end{aligned}$$

Then we can apply the second assumption in (1.13) and we arrive at (6.7).

Therefore, it remains to find the parameters $\alpha, \beta, \gamma, \delta, p, q$ so that the assumptions of Theorem 1 as well as the conditions (6.9) and (6.10) are fulfilled.

To simplify the computations we take (as in the Strichartz inequality)

$$\frac{1}{q} = \frac{n-1}{2(n+1)}. \tag{6.13}$$

Then the couple $1/q, 1/p = \lambda/q$ satisfies the assumptions (1.7) of Theorem 1 for $1 < \lambda < (n+3)/(n-1)$.

From (6.10) and the assumption (1.10) we see that β, γ, δ can be expressed as functions of α, λ, q namely we have

$$\begin{aligned}\gamma &= \lambda\alpha,\\ \beta &= \lambda\left(\alpha + \frac{n}{q}\right) - \frac{n+1}{2} - \frac{1}{q} - \theta,\\ \delta &= \lambda^2\left(\alpha + \frac{n}{q}\right) - \lambda\frac{n+1}{2} - \frac{\lambda}{q} - \lambda\theta.\end{aligned} \tag{6.14}$$

The assumptions (1.10) serve for determination of the admissible domain for the parameters α, θ and they can be written in the form

$$\alpha < \frac{n-1}{2} - \frac{n}{q}, \tag{6.15}$$

$$\lambda\left(\alpha + \frac{n}{q}\right) - \frac{n+1}{2} - \frac{1}{q} - \theta < \frac{n-1}{2} - \frac{n}{q}, \tag{6.16}$$

$$\lambda\frac{n-1}{2q} - \frac{n+1}{2q} < \lambda\left(\alpha + \frac{n}{q}\right) - \frac{n+1}{2} - \frac{1}{q} - \theta, \tag{6.17}$$

$$\lambda^2\left(\alpha + \frac{n}{q}\right) - \lambda\frac{n+1}{2} - \frac{\lambda}{q} + \theta - \lambda\theta > 1 - \frac{\lambda}{q}. \tag{6.18}$$

The first requirement (6.15) suggests us to take

$$\alpha = \frac{n-1}{2} - \frac{n}{q} - \sigma \tag{6.19}$$

for suitable small $\sigma > 0$.

The verification of (6.16) , (6.17) and (6.18) is done in the following.

Lemma 6.1. *Suppose $n \geq 2$, the parameters $\alpha = \alpha(\lambda, \sigma)$ and q are given by* (6.13), (6.19) *and*

$$\lambda_0(n) < \lambda < \frac{n+3}{n-1}. \tag{6.20}$$

Here $\lambda_0(n)$ is the positive root of the equation $\phi_0(\lambda) = 0$ and

$$\phi_0(\lambda) \equiv (n-1)\lambda^2 - (n+1)\lambda - 2. \tag{6.21}$$

Then there exist $\theta = \theta(n, \lambda)$ and a sufficiently small $\sigma_0 = \sigma_0(n, \lambda)$, such that for $0 < \sigma \leq \sigma_0$ the inequalities (6.16), (6.17) *and* (6.18) *are fulfilled.*

Proof. It is sufficient to prove the assertion of the Lemma for $\sigma = 0$.

The estimate (6.16) is equivalent in this case to

$$\theta > \lambda\frac{n-1}{2} - n\left(1 - \frac{1}{q}\right) - \frac{1}{q} \equiv \theta_1(\lambda), \tag{6.22}$$

while (6.17) is equivalent to

$$\theta < \lambda\frac{n-1}{2}\left(1 - \frac{1}{q}\right) - \frac{n+1}{2}\left(1 - \frac{1}{q}\right) - \frac{1}{q} \equiv \theta_2(\lambda). \tag{6.23}$$

Finally, (6.18) means that

$$\theta < \frac{\phi_0(\lambda)}{2(\lambda - 1)} \equiv \theta_3(\lambda). \tag{6.24}$$

The assertion of the Lemma follows from

$$\theta_1(\lambda) < \min(\theta_2(\lambda), \theta_3(\lambda)) \tag{6.25}$$

and the fact that the right side in the above estimate is positive. In fact, $\theta_1(\lambda) < \theta_2(\lambda)$ means that $\lambda < (n+3)/(n-1)$. The inequality $\theta_1(\lambda) < \theta_3(\lambda)$ is equivalent to

$$\lambda > 1 + \frac{4(n+1)}{(n-1)(n+3)}. \tag{6.26}$$

The number $\theta_3(\lambda)$ is positive in view of (6.20). The number $\theta_2(\lambda)$ is positive if and only if (6.26) holds. It is clear that (6.26) is equivalent to

$$\phi_0\left(1+\frac{4(n+1)}{(n-1)(n+3)}\right)<0. \tag{6.27}$$

A direct computation shows that this estimate is equivalent to $n>1$. Therefore, (6.27) is true and this completes the proof of the Lemma. □

A. Appendix. Here we shall recall for completeness the sollowing Sobolev estimate

Proposition A.1. *Suppose* $0\le a<n$. *Then we have*

$$\||.|^{-a}\hat{f}\|_{L^2(\mathbb{R}^n)}\le C\|f\|_{L^p(\mathbb{R}^n)}, \tag{A.1}$$

where $1/p-1/2=a/n$. *If* $\gamma>a$, *then we have*

$$\||.|^{-a}\hat{f}\|_{L^2(\mathbb{R}^n)}\le C\|(1+|.|)^\gamma f\|_{L^2(\mathbb{R}^n)}. \tag{A.2}$$

If $\operatorname{supp} f\subset\{|x|\le T\}$ *and* $0\le\gamma<a$, *then*

$$\||.|^{-a}\hat{f}\|_{L^2(\mathbb{R}^n)}\le CT^{a-\gamma}\|(1+|.|)^\gamma f\|_{L^2(\mathbb{R}^n)}. \tag{A.3}$$

Proof. Consider the operator

$$I_a(f)(x)=\int e^{ix\xi}|\xi|^{-a}\hat{f}(\xi)d\xi.$$

Since

$$I_a(f)(x)=c\int|x-y|^{a-n}dy,$$

we see that the classical Sobolev inequality (see [25])

$$\|I_a(f)\|_{L^2}\le C\|f\|_{L^p},\ 1/p-1/2=a/n,$$

implies (A.1).

The inequality (A.2) follows from

$$\|f\|_{L^p}\le C\|(1+|.|)^\gamma f\|_{L^2}$$

for $\gamma>n/p-n/2=a$. Finally, the inequality (A.3) follow from

$$\|f\|_{L^p}\le C\|(1+|.|)^{-\gamma}\chi(\text{supp}F)\|_{L^r}\|(1+|.|)^\gamma f\|_{L^2} \tag{A.4}$$

for $1/p = 1/r + 1/2$. Combining (A.4) with the assumption supp$f \subset \{|x| \leq T\}$, we arrive at (A.3).

This completes the proof. □

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