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SPECULATING ABOUT MOUNTAINS

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Abstract. The definition of the weak slope of continuous functions introduced by Degiovanni and Marzocchi (cf. [8]) and its interrelation with the notion "steepness" of locally Lipschitz functions are discussed. A deformation lemma and a mountain pass theorem for usco mappings are proved. The relation between these results and the respective ones for lower semicontinuous functions (cf. [7]) is considered.

0. Introduction. The classical mountain pass theorem of A. Ambrosetti and P.H.Rabinowitz (cf. [1]) has been extended in various directions, one of them being the relaxation of the smoothness assumption on the considered functional. These extensions require notions corresponding to the norm of the derivative (and in particular, to a critical point) in the C^1 case. A well known generalization is for locally Lipschitz functionals (cf. [2, 16, 3]) where the most popular notion of the derivative is the Clarke subdifferential (cf. [4]). Recently in [8], [7], [12] and [13] the mountain pass result was extended to the continuous case and a version of this result for lower semicontinuous functionals was proved there. The principal aim of this paper is to propose an alternative approach to the lower semicontinuous case using multivalued usco mappings.

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The paper is organized as follows.

In section 1 we discuss some interrelations between the notions generalizing the norm of the derivative of a C^1 functional to the locally Lipschitz and continuous cases.

In section 2 we introduce the notion of steepness of upper semicontinuous compact-valued mappings. It is shown how it generates a corresponding notion for lower semicontinuous functionals. Two examples motivate our approach. In section 3 a natural multivalued version of the classical deformation lemma and in section 4 a mountain-pass theorem for usco mappings are proved.

1. The locally Lipschitz and the continuous cases. We next recall the definitions of the two notions which substitute the norm of the derivative of a C^1-functional in the locally Lipschitz and in the continuous case, respectively.

Definition 1.1. *Let X be a Banach space, $S \subseteq X$ be a neighbourhood of $x \in X$, $f\colon S \to \mathbb{R}$ be Lipschitz continuous and $f^\circ(x,h)$ be the Clarke derivative of f at x in direction $h \in X$. The number*

$$-\inf\{f^\circ(x,h)\colon h \in X,\ \|h\|_X = 1\}$$

is called steepness of f at x and is denoted by $st\, f(x)$.

The real number c is said to be a critical value of f if there exists $x \in X$ (called critical point of f) such that $c = f(x)$ and $0 \in \partial f(x)$ where $\partial f(x)$ is the Clarke subdifferential of f at x.

Remark 1.2. In [16] we have defined the steepness as

$$\inf\{f^\circ(x;h)\colon h \in X,\ \|h\|_X = 1\}.$$

In [2] the respective notion is

$$\lambda(x) \colon= \min\{\|x^*\|_{X^*}\colon x^* \in \partial f(x)\} \text{ (p. 113)}$$

In fact, if $st\, f(x)$ is defined as in Definition 1.1, we have $\lambda(x) = 0$ if $st\, f(x) \le 0$ and $\lambda(x) = st\, f(x)$ otherwise. In addition $st\, f(x) = \lambda(x) = \|f'(x)\|$ if $f(x)$ is C^1. That is why in this paper we prefer to stick to the Definition 1.1 of the steepness rather than to the one given in [16].

The notion of steepness can be introduced in a natural way for locally Lipschitz functionals defined on a Finsler manifold (cf. [15, 17]).

Definition 1.3 (cf. [8]). *Let X be a metric space endowed with the metric d, $f\colon X \to \mathbb{R}$ be a continuous function, $x \in X$ and let $B(x,\delta)$ be the closed ball with centre x and radius $\delta > 0$. The supremum of the numbers $\sigma \in [0,+\infty)$ such that there exist $\delta > 0$ and a continuous map $H\colon B(x,\delta)\times[0,\delta] \to X$, such that for each $y \in B(x,\delta)$ and for each $t \in [0,\delta]$ we have*

$$d(H(y,t),y) \le t \quad \textit{and} \quad f(H(y,t)) \le f(y) - \sigma t,$$

is called weak slope of f at x and is denoted by $|df|(x)$.

As in the C^1 and in the locally Lipschitz cases, $x \in X$ is called critical if $|df|(x) = 0$.

Remark 1.4. Independently Ioffe and Schwartzman introduced and studied a slightly different notion of steepness for nonsmooth functionals (called δ-regularity and denoted by $\delta(f, x)$) which is in the spirit of Definition 1.3 (cf. [12], [13]).

In the Definition 1.3 existence of local deformations is assumed, whereas it has to be proved on the basis of different assumptions in the previous versions of the so called "deformation lemma".

If f is C^1, we have

$$|df|(x) = st\, f(x) = \|f'(x)\|$$

(cf. [8, Proposition 2.10 and Theorem 2.11], and [4, §2.2]).

It was proved in [8] (cf. Theorem 2.17) that if f is locally Lipschitz then $|df|(x) \geq st\, f(x)$. In particular this means that the set of critical points defined by $|df|(x) = 0$, is smaller than the one, defined by $st\, f(x) \leq 0$.

Unfortunately, unlike the C^1 case, if f is locally Lipschitz, $|df|(x) = st\, f(x)$ is not necessarily true, as the following example shows.

Example 1.4. Let us divide the plane $\mathbb{R}^2$ into four regions:

$$\begin{aligned}
\Omega_1 &= \{(x, y)\colon\ 0 \leq x && \text{and} \quad -\tfrac{1}{2}x \leq y \leq x\} \\
\Omega_2 &= \{(x, y)\colon\ 0 \leq x \leq y && \text{or} \quad x \leq \min(0, 2y)\} \\
\Omega_3 &= \{(x, y)\colon\ 0 \leq x && \text{and} \quad -x \leq y \leq -\tfrac{1}{2}x\} \\
\Omega_4 &= \{(x, y)\colon\ 0 \leq x \leq -y && \text{or} \quad 2y \leq x \leq 0\}
\end{aligned}$$

The function $f\colon \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(x, y) = \begin{cases} -\dfrac{x}{2} + \dfrac{3y}{2} & \text{if } (x, y) \in \Omega_1 \\ \dfrac{3x}{2} - \dfrac{y}{2} & \text{if } (x, y) \in \Omega_2 \\ -\dfrac{3x}{2} - \dfrac{y}{2} & \text{if } (x, y) \in \Omega_3 \\ \dfrac{x}{2} + \dfrac{3y}{2} & \text{if } (x, y) \in \Omega_4 \end{cases}$$

is clearly locally Lipschitz.

Next we show that $st\, f(0) \leq 0$. Assuming the contrary, let $h = (\alpha, \beta)$ with $\alpha^2 + \beta^2 = 1$, be the direction along which we have

$$0 > f^0(0, h) = \limsup_{\substack{t \to 0, t > 0 \\ (x,y) \to (0,0)}} \frac{1}{t}[f(x + t\alpha, y + t\beta) - f(x, y)].$$

Since in every neighbourhood of the origin there are points from the interior of $\Omega_i, i = 1,2,3,4$, this inequality implies the following inconsistent system of linear inequalities:

$$\begin{aligned} -\alpha + 3\beta &< 0 \\ 3\alpha - \beta &< 0 \\ -3\alpha - \beta &< 0 \\ \alpha + 3\beta &< 0. \end{aligned}$$

Hence $st\, f(0) \leq 0$, i.e. the origin is a critical point according to Definition 1.1. But $|df|(0) \geq \dfrac{1}{\sqrt{2}}$ holds true. We show this by defining

$$H((x,y),t) = \begin{cases} (x + \frac{t}{\sqrt{2}}, y - \frac{t}{\sqrt{2}}) & \text{if } (x,y) \in \Omega_3 \cup \Omega_4 \\ (x - \frac{t}{\sqrt{2}}, y - \frac{t}{\sqrt{2}}) & \text{if } (x,y) \notin \Omega_3 \cup \Omega_4 \text{ and } t \leq t_{(x,y)} \\ H(H((x,y),t_{(x,y)}), t - t_{(x,y)}) & \text{if } (x,y) \notin \Omega_3 \cup \Omega_4 \text{ and } t > t_{(x,y)} \end{cases}$$

where $t_{(x,y)}$ is the greatest positive number t satisfying $H((x,y),t) \in \Omega_1 \cup \Omega_2$ for some $(x,y) \notin \Omega_3 \cup \Omega_4$.

This example shows that the set of critical points in the sense of Definition 1.1 of a given locally Lipschitz function may be strictly larger than the respective set in the sense of Definition 1.3.

2. The weak slope for multivalued usco mappings and for lower semi-continuous single valued functionals. Let $F: X \to Y$ be a multivalued mapping and A be a subset of Y. We denote

$$\begin{aligned} F^{-1}(A) &:= \{x \in X: F(x) \cap A \neq \emptyset\} \\ F_{-1}(A) &:= \{x \in X: F(x) \subset A\}. \end{aligned}$$

Definition 2.1. *A multivalued mapping $F: X \to Y$, where X and Y are topological spaces, is said to be upper semicontinuous at the point $x_0 \in X$, if for every open set V in Y which contains $F(x_0)$ there exists a neighbourhood W of x_0 such that $W \subset F_{-1}(V)$. F is said to be upper semicontinuous if it is upper semicontinuous at every point $x \in X$. The correspondence F is called usco if it is upper semicontinuous and $F(x)$ is non-empty and compact for every x in its domain.*

It is straightforward that the "big preimage" $F^{-1}(M)$ of a closed set M is closed and the "little preimage" $F_{-1}(U)$ of an open set U is open provided F is upper semicontinuous.

We introduce the notion of weak slope of an usco mapping F as a complete analogue to Definition 1.3 (applied to $\sup F$).

Definition 2.2. *Let (X,d) be a metric space and $F\colon X \to \mathbb{R}$ be an usco mapping. We fix a point $x \in X$. The supremum of the reals $\sigma \in [0,+\infty)$ such that there exist $\delta > 0$ and a continuous map $H\colon B(x,\delta) \times [0,\delta] \to X$ such that for each $y \in B(x,\delta)$ and for each $t \in [0,\delta]$ we have*

$$d(H(y,t),y) \leq t \text{ and } \sup\ F(H(y,t)) \leq \sup\ F(y) - \sigma\, t$$

is called weak slope of F at x and is denoted by $|dF|(x)$.

Note that for an usco F the supremum of $F(x)$ is finite and in fact it is a maximum. The function $\sup F\colon X \to \mathbb{R}$ is upper semicontinuous in the sense that $(\sup F)(x) \geq \limsup(\sup F)(x_n)$ for every sequence $\{x_n\}_{n=1}^{\infty}$ tending to x. It is more convenient to work with multivalued usco's than with upper semicontinuous real-valued functions (see Example 2.4 and Section 3). The mapping $|dF| : X \to [0,+\infty]$ is again a lower semicontinuous one as in the continuous case.

In [8, 7] a notion of weak slope for lower semicontinuous real functions is introduced and studied and some results about the existence of critical points (of mountain pass type) are proved.

Here we propose another approach to lower semicontinuous functions which follows the classical pattern with the deformation lemma.

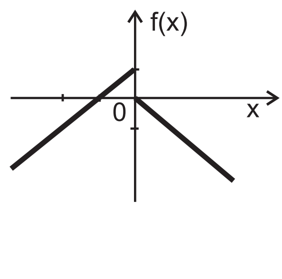
Let (X,d) be a metric space and $f\colon X \to \mathbb{R}$ be a lower semicontinuous real functional (i.e. $f(x_0) \leq \liminf\ f(x_n)$ whenever $x_n \to x_0$). The graph of f, $G = \{(x,t) :\ t = f(x)\}$ is a subset of $X \times \mathbb{R}$. Let $\overline{G}$ be its closure. We define a multivalued mapping $F\colon X \to \mathbb{R}$ corresponding to f by $F(x) = \{t \in \mathbb{R}\colon (x,t) \in \overline{G}\}$. The proof of the following proposition is straightforward.

Proposition 2.3. *Let X, f and F be as above. Then $f(x) \in F(x)$, moreover, $f(x) = \min F(x)$ for every $x \in X$. If, in addition, f is continuous at the point $x \in X$, then $F(x) = \{f(x)\}$. The correspondence F is usco provided f is locally bounded from above.*

Having in mind Proposition 2.3, it is natural instead of a lower semicontinuous functional f to consider its "extension" F. So we have a "weak slope" for f using Definition 2.2 for F.

Example 2.4. Let us consider $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ -x & \text{if } x \geq 0 \end{cases}$$



The function f is lower semicontinuous and it is natural from a geometrical point of view to consider the origin as a mountain pass point.

Indeed, the weak slope of the extension of f

$$F(x) = \begin{cases} x+1, & x<0 \\ -x, & x>0 \\ \{0,1\}, & x=0 \end{cases}$$

is zero at the origin i.e. $|dF|(0) = 0$.

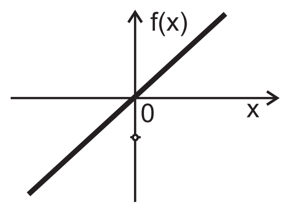
The weak slope of f as introduced in [8, 7] is 1 at 0 although it is obvious that there is no reasonable deformation in a neighbourhood of the origin.

This example shows that if one wants to prove a deformation lemma for a semicontinuous real function the assumption that the weak slope is "big" in a neighbourhood of $f^{-1}([1-\varepsilon, 1+\varepsilon])$ (or of $(\sup F)^{-1}([0-\varepsilon, 0+\varepsilon])$) is not sufficient. This motivates the use of the multivalued "extension" F and our formulation of the deformation lemma in section 3.

Example 2.5. The lower semicontinuous function in the previous example had a critical point in our sense which was not critical in the sense of [8, 7], being yet a reasonable mountain pass point.

Here we see that a point can be critical in the sense of [8, 7], not being critical in our sense which is due to the fact that we take into account only "essential values of the lsc function", i.e. values which are cluster points of the set of the values of f. Indeed, let

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ -1 & \text{if } x = 0 \end{cases}$$



Its extension is

$$F(x) = \begin{cases} x & \text{if } x \neq 0 \\ \{-1, 0\} & \text{if } x = 0 \end{cases}$$

and $|dF|(x) = 1$ for every $x \in \mathbb{R}$, although the origin is a local minimum.

Remark 2.6. The definition of critical point given by Ioffe and Schwartzman applies to any function and gives critical points in both examples (cf. [12],[13]).

3. Deformation lemma for usco mappings. Let X be a metric space with metric d. By $B(x, r)$ we denote the closed ball with centre x and radius r. For any subset S of X we set

$$S_r = \{x \in X : \operatorname{dist}(x, S) \leq r\}$$

where $\operatorname{dist}(x, S) = \inf\{d(x, y) : y \in S\}$.

Theorem 3.1 (deformation lemma). *Let X be a complete metric space with metric d and $F : X \to \mathbb{R}$ be an usco correspondence. Let ε, δ be two positive reals and S be a subset of X. If Q is an open neighbourhood of $F^{-1}([c-\varepsilon, c+\varepsilon]) \cap S_\delta$ such* *that for every* $y \in Q$ *the weak slope* $|dF|(y) > \dfrac{2\varepsilon}{\delta}$, *then there exists a continuous map* $\eta:\ X \times [0,+\infty) \to X$ *with the following properties:*

$$\begin{array}{llll} (i) & \eta(x,0) = x & \textit{for every} & x \in X; \\ (ii) & \eta(x,t) = x & \textit{for every} & x \in X \setminus Q \quad \textit{and} \quad t \in [0,+\infty); \\ (iii) & \textit{if} \quad x \in \overline{S} & \textit{and} & \sup F(x) \le c + \varepsilon, \quad \textit{then} \\ & & & \sup F(\eta(x,\delta)) \le c - \varepsilon; \\ (iv) & d(x,\eta(x,t)) \le t & \textit{whenever} & x \in X \quad \textit{and} \quad t \in [0,+\infty). \end{array}$$

P r o o f. Let $\sigma = \frac{2\varepsilon}{\delta}$ for short notation. Since $|dF|(x) > \sigma$ for every $x \in Q$ by definition 2.2 we obtain that for every $x \in Q$ there exist a positive real δ_x and $H_x \in C(B(x,\delta_x) \times [0,\delta_x], X)$, such that $B(x,\delta_x) \subset Q$ and for every $y \in B(x,\delta_x)$ and every $t \in [0,\delta_x]$ the following two inequalities hold true:

$$d(H_x(y,t), y) \le t \tag{1a}$$

$$\sup F(H_x(y,t)) \le \sup F(y) - \sigma\, t. \tag{1b}$$

Let us denote by U_x the open ball with centre x and radius $\dfrac{\delta_x}{2}$. Then $\{U_x\}_{x\in Q} \cup \{X \setminus (S_\delta \cap F^{-1}([c-\varepsilon, c+\varepsilon]))\}$ is an open covering of the metric space X. Let $\{U_\gamma\}_{\gamma\in\Gamma} \cup \{X \setminus (S_\delta \cap F^{-1}([c-\varepsilon, c+\varepsilon]))\}$ be a locally finite refinement of the above covering and $\{\alpha_\gamma\}_{\gamma\in\Gamma} \cup \{\alpha\}$ be a Lipschitz partition of unity subordinate to this refinement. Let $U_\gamma \subset U_{x_\gamma}$, $x_\gamma \in Q$ and for short $\delta_\gamma = \delta_{x_\gamma}$, $H_\gamma = H_{x_\gamma}$. Without loss of generality we can have

$$Q = \cup\{U_\gamma :\ \gamma \in \Gamma\} \supset S_\delta \cap F^{-1}([c-\varepsilon, c+\varepsilon]).$$

Let $\Gamma = [0,\gamma_0)$ be well ordered. We set $t_x = \frac{1}{2}\,\min\{\delta_\gamma :\ x \in \overline{U_\gamma}\}$ if $x \in Q$ and $t_x = 0$ if $x \in X \setminus Q$. We define inductively the mappings $\{\xi_\gamma(x,t)\}_{\gamma\in[0,\gamma_0]}$:

(a) $\xi_0(x,t) = x$ for every $x \in X$ and $0 \le t \le t_x$;

(b) if γ has a predecessor, then for every $x \in X$ and $t \in [0,t_x]$

$$\xi_\gamma(x,t) = \begin{cases} H_{\gamma-1}(\xi_{\gamma-1}(x,t),\ \alpha_{\gamma-1}(x).t) & \text{if } x \in U_{\gamma-1} \\ \xi_{\gamma-1}(x,t) & \text{if } x \notin U_{\gamma-1}; \end{cases} \tag{2}$$

(c) if γ is a limit ordinal, then

$$\xi_\gamma(x,t) = \lim_{\beta<\gamma} \xi_\beta(x,t) \text{ for each } x \in X,\ \ t \in [0,t_x].$$

Next we show that for each $\gamma \in [0, \gamma_0]$ and for $x \in X$, $t \in [0, t_x]$ the mapping $\xi_\gamma(x,t)$ is well defined and continuous and the following properties hold true:

$$d(\xi_\gamma(x,t), x) \le (\sum_{\beta<\gamma} \alpha_\beta(x)).t; \tag{3a}$$

$$\sup F(\xi_\gamma(x,t)) \le \sup F(x) - \sigma(\sum_{\beta<\gamma} \alpha_\beta(x)).t. \tag{3b}$$

We will proceed by induction on γ.

For $\gamma = 0$ the claim is clear.

Let the claim be true for every $\beta < \gamma$.

Case I: γ has a predecessor.

If $x \notin U_{\gamma-1}$, then $\xi_\gamma(x,t)$ is clearly well defined.

If $x \in U_{\gamma-1}$, then $\alpha_{\gamma-1}(x).t \le t_x \le \frac{1}{2}\delta_{\gamma-1}$. Using (3a) for $\gamma - 1$ we have $d(\xi_{\gamma-1}(x,t), x) \le t \le t_x \le \frac{1}{2}\delta_{\gamma-1}$ and therefore $\xi_{\gamma-1}(x,t) \in B(x_{\gamma-1}, \delta_{\gamma-1})$ so $\xi_\gamma(x,t) = H_{\gamma-1}(\xi_{\gamma-1}(x,t), \alpha_{\gamma-1}(x).t)$ is well defined.

Moreover, whenever $x \in X$ and $t \in [0, t_x]$ we have

$$\begin{aligned} d(\xi_\gamma(x,t), x) &\le d(\xi_\gamma(x,t), \xi_{\gamma-1}(x,t)) + d(\xi_{\gamma-1}(x,t), x) \le \\ &\le \alpha_{\gamma-1}(x).t + \left(\sum_{\beta<\gamma-1} \alpha_\beta(x)\right).t = \left(\sum_{\beta<\gamma} \alpha_\beta(x)\right).t \end{aligned}$$

according to (1a), (2) and inductive assumption and (3a) is proved. Now

$$\begin{aligned} \sup F(\xi_\gamma(x,t)) &= \sup F(\xi_\gamma(x,t)) - \sup F(\xi_{\gamma-1}(x,t)) + \sup F(\xi_{\gamma-1}(x,t)) \le \\ &\le \sup F(\xi_\gamma(x,t)) - \sup F(\xi_{\gamma-1}(x,t)) + \sup F(x) - \sigma(\sum_{\beta<\gamma-1} \alpha_\beta(x)).t. \end{aligned}$$

If $x \in U_{\gamma-1}$, then by (1b) and (2) we have

$$\sup F(\xi_\gamma(x,t)) - \sup F(\xi_{\gamma-1}(x,t)) =$$

$$= \sup F(H_{\gamma-1}(\xi_{\gamma-1}(x,t), \alpha_{\gamma-1}(x).t)) - \sup F(\xi_{\gamma-1}(x,t)) \le -\sigma.\alpha_{\gamma-1}(x).t.$$

If $x \notin U_{\gamma-1}$, then $\alpha_{\gamma-1}(x) = 0$ and $\sup F(\xi_\gamma(x,t)) - \sup F(\xi_{\gamma-1}(x,t)) = 0 = -\sigma.\alpha_{\gamma-1}(x).t$.

Hence

$$\begin{aligned} \sup F(\xi_\gamma(x,t)) &\le -\sigma.\alpha_{\gamma-1}(x).t + \sup F(x) - \sigma(\sum_{\beta<\gamma-1} \alpha_\beta(x).t = \\ &= \sup F(x) - \sigma(\sum_{\beta<\gamma} \alpha_\beta(x).t, \end{aligned}$$

thus proving (3b).

Next we establish the continuity of ξ_γ at (x_0, t_0), where $0 \le t_0 \le t_{x_0}$. Let $x_n \to x_0$ and $t_n \to t_0$, where $0 \le t_n \le t_{x_n}$. There are two possibilities: $x_0 \in U_{\gamma-1}$ or $x_0 \notin U_{\gamma-1}$.

If $x_0 \in U_{\gamma-1}$ then $x_n \in U_{\gamma-1}$ for n sufficiently large. As above $d(\xi_{\gamma-1}(x_n, t_n), x_n) \le t_n \le t_{x_n} \le \frac{1}{2}\delta_{\gamma-1}$ for every $n \ge n_0$ and for $n = 0$, so

$$\xi_{\gamma-1}(x_n, t_n) \in B(x_{\gamma-1}, \delta_{\gamma-1}),\ n \ge n_0,\ \xi_{\gamma-1}(x_0, t_0) \in B(x_{\gamma-1}, \delta_{\gamma-1}).$$

Now the continuity of ξ_γ at (x_0, t_0) follows from (2) and from the continuity of $\xi_{\gamma-1}, \alpha_{\gamma-1}$ and $H_{\gamma-1}$ (on the set $B(x_{\gamma-1}, \delta_{\gamma-1}) \times [0, \delta_{\gamma-1}]$).

If $x_0 \notin U_{\gamma-1}$, then the sequence $\{x_n\}_{n=1}^{\infty}$ consists of two subsequences:

$$\{x_{k_n}\}_{n=1}^{\infty} \subset X \setminus U_{\gamma-1} \text{ and } \{x_{l_n}\}_{n=1}^{\infty} \subset U_{\gamma-1}.$$

For the first subsequence we have $\xi_\gamma(x_{k_n}, t_{k_n}) = \xi_{\gamma-1}(x_{k_n}, t_{k_n})$ and the continuity of $\xi_{\gamma-1}$ implies

$$\lim_{n\to\infty} \xi_\gamma(x_{k_n}, t_{k_n}) = \xi_{\gamma-1}(x_0, t_0).$$

The second subsequence may be finite. If not, $x_0 \in \overline{U_{\gamma-1}}$ and so $t_0 \le \dfrac{1}{2}\,\delta_{\gamma-1}$, $\xi_{\gamma-1}(x_0, t_0) \in B(x_{\gamma-1}, \delta_{\gamma-1})$. Therefore

$$\begin{aligned}
\lim_{n\to\infty} \xi_\gamma(x_{l_n}, t_{l_n}) &= \lim_{n\to\infty} H_{\gamma-1}(\xi_{\gamma-1}(x_{l_n}, t_{l_n}), \alpha_{\gamma-1}(x_{l_n}).t_{l_n}) = \\
&= H_{\gamma-1}(\xi_{\gamma-1}(x_0, t_0), \alpha_{\gamma-1}(x_0).t_0) = H_{\gamma-1}(\xi_{\gamma-1}(x_0, t_0), 0) = \\
&= \xi_{\gamma-1}(x_0, t_0).
\end{aligned}$$

Thus the continuity of ξ_γ is proved because $\xi_\gamma(x_0, t_0) = \xi_{\gamma-1}(x_0, t_0)$ when $x_0 \notin U_{\gamma-1}$.

Case II: γ has no predecessor.

Let $x \in X$ and $B(x, r_x) \cap U_\beta = \emptyset$ for each $\beta \notin \{\gamma_1, \gamma_2, \ldots, \gamma_s\}$. Denote $\tilde{\gamma} = \max\{\gamma_i < \gamma \ :\ i = 1, 2, \ldots, s\} + 1$. Then $\xi_\gamma(y, t) = \xi_{\tilde{\gamma}}(y, t)$ for every $y \in B(x, r_x)$ and $t \in [0, t_y]$. Indeed, a simple induction on $\beta \in [\tilde{\gamma}, \gamma]$ shows $\xi_\beta(y, t) = \xi_{\tilde{\gamma}}(y, t)$ using (2). Case II is done.

Let us denote ξ_{γ_0} by ξ. This mapping has the properties:

(4*a*) $$d(\xi(x, t), x) \le t \text{ for each } x \in X,\quad t \in [0,\ t_x];$$

(4*b*) $$\sup F(\xi(x, t)) \le \sup F(x) \text{ for each } x \in X \text{ and } t \in [0,\ t_x];$$

(4*c*) $$\sup F(\xi(x, t)) \le \sup F(x) - \sigma.t \text{ for each } x \in S_\delta \cap F^{-1}([c - \varepsilon,\ c + \varepsilon])$$

and $t \in [0,\, t_x]$.

These properties follow from (3a), (3b), because $\sum_{\beta<\gamma_0} \alpha_\beta(x) = 1$ on $S_\delta \cap F^{-1}([c-\varepsilon,\, c+\varepsilon])$.

In the sequel we shall need the lower semicontinuity of the mapping $x \to t_x \in [0,+\infty)$ on X. If $x \notin Q$ we have $t_x = 0$ and the lower semicontinuity follows from $t_y \geq 0$ for every $y \in X$. Now let $x \in Q = \cup_{\gamma\in\Gamma} U_\gamma$. Since $\{U_\gamma\}_{\gamma\in\Gamma}$ is locally finite, there exists a ball $B(x,r_x)$, such that $B(x,r_x) \cap U_\gamma \neq \emptyset$ only for finitely many γ. Without loss of generality

$$B(x,r_x) \cap U_\gamma \neq \emptyset \text{ iff } \gamma \in \{\beta \in \Gamma :\, x \in \overline{U_\beta}\}.$$

If $y \in B(x,r_x)$, then $\{\beta \in \Gamma :\, y \in \overline{U_\beta}\} \subset \{\beta \in \Gamma :\, x \in \overline{U_\beta}\}$, i.e. $t_y \geq t_x$ and the lower semicontinuity of $x \to t_x \in [0,+\infty)$ is proved. It implies the existence of a continuous function $\tau :\, X \to [0,+\infty)$ such that $\tau(x) \leq t_x$ on X and $\tau(x) > 0$ iff $t_x > 0$, i.e. $\tau(x) > 0$ iff $x \in Q$.

Next we define inductively

$$\eta_k \in C(X \times [0,+\infty), X) \text{ and } \tau_k \in C(X) \text{ for } k = 0,1,2,\ldots$$

as follows:

$\tau_0(x) \equiv 0$ on X;
$\eta_0(x,t) = x$ for every $x \in X$ and $t \geq 0$;
$\tau_{k+1}(x) = \tau_k(x) + \tau(\eta_k(x,\tau_k(x)))$;

$$\eta_{k+1}(x,t) = \begin{cases} \eta_k(x,t) & \text{for every} \quad t \in [0,\tau_k(x)] \\ \xi(\eta_k(x,\tau_k(x)), t-\tau_k(x)) & \text{for every} \quad t \in [\tau_k(x),\tau_{k+1}(x)] \\ \xi(\eta_k(x,\tau_k(x)), \tau_{k+1}(x)-\tau_k(x)) & \text{for every} \quad t \geq \tau_{k+1}(x). \end{cases}$$

The following properties of η_k are corollaries of the properties (4a), (4b) and (4c) of ξ:

(5a) $$d(\eta_k(x,t),x) \leq t \text{ for every } x \in X \text{ and } t \geq 0.$$

(5b) $$\sup F(\eta_k(x,t)) \leq \sup F(x) \text{ for every } x \in X \text{ and } t \geq 0.$$

(5c) $$\sup F(\eta_k(x,t)) \leq \sup F(x) - \sigma.t$$

for every x satisfying $\eta_i(x,\tau_i(x)) \in S_\delta \cap F^{-1}([c-\varepsilon,\, c+\varepsilon])$ whenever $i \in \{0,1,\ldots,k-1\}$ and for every $t \in [0,\, \tau_k(x)]$.

We shall prove only (5c) since (5a) and (5b) are straightforward. Again, we proceed by induction on k. The first step is trivial. Next we estimate from above sup $F(\eta_{k+1}(x,t))$. If $t \in [0, \tau_k(x)]$ we have

$$\sup F(\eta_{k+1}(x,t)) = \sup F(\eta_k(x,t)) \le \sup F(x) - \sigma.t$$

by the inductive assumption. If $t \in [\tau_k(x), \tau_{k+1}(x)]$, then

$$\begin{aligned} \sup F(\eta_{k+1}(x,t)) &= \sup F(\xi(\eta_k(x,\tau_k(x)), t - \tau_k(x))) \le \\ &\le \sup F(\eta_k(x,\tau_k(x))) - \sigma.(t - \tau_k(x)) \le \\ &\le \sup F(x) - \sigma.\tau_k(x) - \sigma.t + \sigma.\tau_k(x) = \sup F(x) - \sigma.t. \end{aligned}$$

The first of the above inequalities is (4c) applied to

$$\eta_k(x,\tau_k(x)) \in S_\delta \cap F^{-1}([c-\varepsilon,\ c+\varepsilon]) \text{ and } t - \tau_k(x) \in [0, t_{\eta_k(x,\tau_k(x))}]$$

and the second one is the inductive assumption.

The following claim is the critical step in the proof.

Claim. Let us denote $A = cl(\{x \in X :\ \sup F(x) \in [c-\varepsilon,\ c+\varepsilon]\} \cap \overline{S})$. Then for every $x \in A$ there is a neighbourhood V of x and a positive integer s such that for every $y \in \overline{S} \cap V$ satisfying $\sup F(y) \le c + \varepsilon$ we have $\sup F(\eta_s(y, \tau_s(y))) < c - \varepsilon$.

P r o o f o f t h e c l a i m. Let x be fixed in A. There are three cases.

Case I. $\{\eta_k(x,\tau_k(x))\}_{k=1}^{\infty} \not\subset F^{-1}([c-\varepsilon,\ c+\varepsilon])$, i.e. there exists a positive integer s such that $\eta_s(x,\tau_s(x)) \notin F^{-1}([c-\varepsilon,\ c+\varepsilon])$. Since F is upper semicontinuous and $\eta_s(.,\tau_s(.))$ is continuous on X, there exists a neighbourhood V of x such that $\eta_s(y,\tau_s(y)) \notin F^{-1}([c-\varepsilon,\ c+\varepsilon])$ for every $y \in V$. Then if $y \in V$ and $\sup F(y) \le c+\varepsilon$, the property (5b) completes the proof in this case.

Case II. $\sup\{\tau_k(x) :\ k = 1, 2, \ldots\} > \delta$. Then there is s such that $\tau_s(x) > \delta$ and a neighbourhood V of x such that $\tau_s(y) > \delta$ for every $y \in V$. Let us fix $y \in V \cap \overline{S}$ with $\sup F(y) \le c+\varepsilon$. Aiming for contradiction we assume $\sup F(\eta_s(y,\tau_s(y))) \ge c-\varepsilon$. Let $s_y = \min\{k :\ \tau_k(y) > \delta\}$. Then $y \in \overline{S}$ and (5a) yield $\eta_i(y,\tau_i(y)) \in S_\delta$ for $i = 0, 1, \ldots, s_y - 1$. Moreover, by (4b), we have $c - \varepsilon \le \sup F(\eta_i(y,\tau_i(y))) \le c + \varepsilon$ for $i = 0, 1, \ldots, s_y - 1$, hence $\eta_i(y,\tau_i(y)) \in S_\delta \cap F^{-1}([c-\varepsilon,\ c+\varepsilon])$ for these i. Applying (4b) and (5c) we obtain

$$\begin{aligned} c - \varepsilon &\le \sup F(\eta_s(y,\tau_s(y))) \le \sup F(\eta_{s_y}(y,\tau_{s_y}(y))) \le \sup F(y) - \sigma.\tau_{s_y}(y) < \\ &< \sup F(y) - \frac{2\varepsilon}{\delta}.\delta \le c + \varepsilon - 2\varepsilon = c - \varepsilon, \end{aligned}$$

which is a contradiction.

Case III. $\{\eta_k(x,\tau_k(x))\}_{k=1}^{\infty} \subset F^{-1}([c-\varepsilon,\, c+\varepsilon])$ and $\sup\{\tau_k(x) : k = 1,2,\ldots\} \le \delta$. Because of (4a) we have

$$d(\eta_{k+1}(x,\tau_{k+1}(x)),\eta_k(x,\tau_k(x))) = d(\xi(\eta_k(x,\tau_k(x)),\tau_{k+1}(x)-\tau_k(x)),\,\eta_k(x,\tau_k(x))) \le$$

$\tau_{k+1}(x)-\tau_k(x)$. Since $\{\tau_k(x)\}_{k=1}^{\infty}$ is convergent, the sequence $\{\eta_k(x,\tau_k(x))\}_{k=1}^{\infty}$ is a Cauchy one. Since X is complete, there exists $z = \lim\limits_{k\to\infty} \eta_k(x,\tau_k(x))$. Moreover

$$\begin{aligned} d(x,z) &= \lim_{k\to\infty} d(x,\eta_k(x,\tau_k(x))) \le \lim_{k\to\infty} \sum_{i=1}^{k} d(\eta_{i-1}(x,\tau_{i-1}(x)),\eta_i(x,\tau_i(x))) \le \\ &\le \lim_{k\to\infty} \sum_{i=1}^{k} (\tau_i(x)-\tau_{i-1}(x)) = \lim_{k\to\infty} \tau_k(x) \le \delta, \end{aligned}$$

i.e. $z \in S_\delta$ because $x \in \overline{S}$. Further on, $z \in F^{-1}([c-\varepsilon,\, c+\varepsilon])$ because $F^{-1}([c-\varepsilon,\, c+\varepsilon])$ is closed. Hence $z \in S_\delta \cap F^{-1}([c-\varepsilon,\, c+\varepsilon]) \subset Q$. Therefore $\tau(z) > 0$ and the continuity of τ yields that $\tau(\eta_k(x,\tau_k(x))) > \frac{\tau(z)}{2}$ for all k sufficiently large. On the other hand

$$\tau(\eta_k(x,\tau_k(x))) = \tau_{k+1}(x) - \tau_k(x) \to_{k\to\infty} 0.$$

The contradiction ends the proof of the claim.

So, for every $x \in A$ the claim provides a neighbourhood $V_x \subset Q$ of x and a positive integer s_x such that for each $y \in V_x \cap \overline{S}$ satisfying $\sup F(y) \in [c-\varepsilon,\, c+\varepsilon]$ we have $\sup F(\eta_{s_x}(y,\tau_{s_x}(y))) < c-\varepsilon$. Now $\{V_x\}_{x\in A} \cup \{X \setminus A\}$ is an open covering of X. Let $\{V_\beta\}_{\beta\in B} \cup \{X\setminus A\}$ be a locally finite refinement of this covering and $\{\omega_\beta\}_{\beta\in B} \cup \{\omega\}$ be a Lipschitz partition of unity subordinate to the refinement. We denote

$$\tau^*(x) = \sum_{\beta\in B} \omega_\beta(x).\tau_{s_\beta}(x).$$

Since $\{V_\beta\}_{\beta\in B}$ is locally finite, $\tau^*(x) : X \to [0,+\infty)$ is continuous. Let $k(x) = \max\{s_\beta :\ x \in \overline{V_\beta}\}$. We define $\eta :\ X \times [0,+\infty) \to X$ as follows:

$$\eta(x,t) = \begin{cases} \eta_{k(x)}(x,t) & \text{if} \quad t \le \tau^*(x) \\ \eta_{k(x)}(x,\tau^*(x)) & \text{if} \quad t \ge \tau^*(x) \end{cases}$$

Let $x \in X$. Then there exists $r > 0$ such that $B(x,r) \cap V_\beta = \emptyset$ whenever $x \notin \overline{V_\beta}$. If $y \in B(x,r)$ we have

$$\tau^*(y) \le \max\{\tau_{s_\beta}(y) :\ y \in V_\beta\} \le \max\{\tau_{s_\beta}(y) :\ y \in \overline{V_\beta}\} =$$

$$= \tau_{\max\{s_\beta : y\in\overline{V_\beta}\}}(y) = \tau_{k(y)}(y) \le \tau_{k(x)}(y),$$

because $k(y) \le k(x)$. Therefore $\eta(y,t) = \eta_{k(y)}(y,t) = \eta_{k(x)+1}(y,t)$ for every $y \in B(x,r)$ and $t \in [0, \tau^*(y)]$. Now the continuity of $\eta_{k(x)+1}$ and τ^* yields the continuity of η.

It remains only to verify that the so defined mapping $\eta \in C(X \times [0,+\infty), X)$ has the properties (i)—(iv) from the statement of the deformation lemma. The property (i) follows from the definition of η_k, (ii) follows from the fact that $\tau^*(x) = 0$ if $x \notin Q$ and (iv) follows from (5a). Let $x \in \overline{S}$ and $\sup F(x) \le c + \varepsilon$. If $\sup F(x) < c - \varepsilon$, (iii) follows from (5b). If $\sup F(x) \ge c - \varepsilon$, then $x \in A$ and hence there exists $\beta \in B$ with $x \in V_\beta$. Let $s_{\beta^*} = \min\{s_\beta : x \in V_\beta\}$. Then $\tau^*(x) \ge \tau_{s_{\beta^*}}(x)$ and we have different possibilities depending on the order of the reals δ, $\tau^*(x)$, $\tau_{s_{\beta^*}}(x)$.

We will denote by l the last index for which $\tau_l(x) \le \min\{\tau^*(x), \delta\}$. If $\delta \ge \tau^*(x) \ge \tau_{s_{\beta^*}}(x)$, then using (4b) several times we obtain

$$\begin{aligned} \sup F(\eta(x,\delta)) &\le \sup F(\eta(x,\tau^*(x))) = \sup F(\xi(\eta_l(x,\tau_l(x)), \tau^*(x) - \tau_l(x))) \le \\ &\le \sup F(\eta_l(x,\tau_l(x))) = \sup F(\xi(\eta_{l-1}(x,\tau_{l-1}(x)), \tau_l(x) - \tau_{l-1}(x))) \le \\ &\le \sup F(\eta_{l-1}(x,\tau_{l-1}(x))) \le \ \ldots \ \le \sup F(\eta_{s_{\beta^*}}(x, \tau_{s_{\beta^*}}(x))). \end{aligned}$$

If $\tau^*(x) \ge \delta \ge \tau_{s_{\beta^*}}(x)$, then as above

$$\begin{aligned} \sup F(\eta(x,\delta)) &= \sup F(\eta_{l+1}(x,\delta)) = \sup F(\xi(\eta_l(x,\tau_l(x)), \delta - \tau_l(x))) \le \\ &\le \sup F(\xi(\eta_l(x,\tau_l(x))) \le \sup F(\eta_{s_{\beta^*}}(x, \tau_{s_{\beta^*}}(x))). \end{aligned}$$

In both cases $x \in \overline{S} \cap V_{\beta^*}$, $\sup F(x) \in [c-\varepsilon,\, c+\varepsilon]$ and the choice of V_{β^*} yields $\sup F(\eta_{s_{\beta^*}}(x, \tau_{s_{\beta^*}}(x))) < c - \varepsilon$, hence $\sup F(\eta(x,\delta)) < c - \varepsilon$. If $\tau^*(x) \ge \tau_{s_{\beta^*}}(x) > \delta$, we repeat the argument in the proof in Case II of the claim. If there exists $i \in \{0, 1, \ldots, l\}$ with $\sup F(\eta_i(x, \tau_i(x))) < c - \varepsilon$, then

$$\begin{aligned} \sup F(\eta(x,\delta)) &= \sup F(\eta_{l+1}(x,\delta)) = \sup F(\xi(\eta_l(x,\tau_l(x)), \delta - \tau_l(x))) \le \\ &\le \sup F(\eta_l(x,\tau_l(x))) \le \sup F(\eta_i(x,\tau_i(x))) < c - \varepsilon. \end{aligned}$$

If $\sup F(\eta_i(x,\tau_i(x))) \ge c - \varepsilon$ for every $i \in \{0, 1, \ldots, l\}$, then

$$\eta_i(x, \tau_i(x)) \in S_\delta \cap F^{-1}([c-\varepsilon,\, c+\varepsilon]),\ i = 0, 1, \ldots, l$$

and applying (5c) we have

$$\sup F(\eta(x,\delta)) = \sup F(\eta_{l+1}(x,\delta)) \le \sup F(x) - \sigma.\delta \le c + \varepsilon - \frac{2\varepsilon}{\delta}.\delta = c - \varepsilon$$

thus finishing the proof that η has the property (iii). □

Remark 3.2. A careful examination of the above proof shows that it is sufficient for Q to be an open set containing $S_\delta \cap cl\{x \in X : \sup F(x) \in [c-\varepsilon,\, c+\varepsilon]\}$.

4. A min–max principle. In this section we show how the deformation lemma can be used to obtain a min-max principle for usco mappings.

Here is the setting (see [9], [10] and [11]): Let X be a complete metric space with distance d and $F : X \to \mathbb{R}$ be an usco correspondence. Let $\mathcal{M}$ be a family of subsets of X and M be a subset of X. Throughout this section we shall denote

$$c(M, F, \mathcal{M}) = \inf\{\sup\ (\cup\{F(x) : x \in A \cap M\}) : A \in \mathcal{M}\}.$$

Definition 4.1 (cf. [9]). *Let $B \subset X$. We shall say that a class $\mathcal{M}$ of subsets of X is a homotopy stable family with boundary B if*
(a) every set in $\mathcal{M}$ contains B;
(b) for any set A in $\mathcal{M}$ and any $\eta \in C(X \times [0, 1], X)$ verifying $\eta(x, t) = x$ for all (x, t) in $(X \times \{0\}) \cup (B \times [0, 1])$ we have

$$\eta(A, 1) = \{x \in X : x = \eta(y, 1) \ \ \textit{for some} \ \ y \in A\} \in \mathcal{M}.$$

Theorem 4.2. *Let X be a complete metric space, $F : X \to \mathbb{R}$ be an usco correspondence, $\mathcal{M}$ be a homotopy stable family of subsets of X with boundary B and M be a subset of X verifying*

$$\text{(6}a\text{)} \qquad \operatorname{dist}(M, B) = \inf\ \{d(x, y) : x \in M, y \in B\} > 0,$$

$$\text{(6}b\text{)} \qquad M \cap A \neq \emptyset \ \ \textit{for all} \ \ A \in \mathcal{M},$$

and

$$\text{(6}c\text{)} \qquad \inf\{\sup\ F(x) : x \in M\} \geq c = c(X, F, \mathcal{M}).$$

Let $\varepsilon \in (0, \operatorname{dist}(M, B)/2)$. Then for every $A \in \mathcal{M}$ satisfying

$$\sup\ (\cup\{F(x) : x \in A \cap M_{\varepsilon/3}\}) < c + \frac{\varepsilon^2}{12}$$

there exists $x_\varepsilon \in X$ with the properties:

(i) $x_\varepsilon \in F^{-1}([c - \frac{\varepsilon^2}{3}, c + \frac{\varepsilon^2}{12}])$
(ii) $|dF|(x_\varepsilon) \leq \varepsilon$
(iii) $\operatorname{dist}(x_\varepsilon, M) \leq \varepsilon$
(iv) $\operatorname{dist}(x_\varepsilon, A) \leq \varepsilon$.

P r o o f. We first note that (6b) and (6c) imply $c(M, F, \mathcal{M}) = c(X, F, \mathcal{M}) = c$. Hence there is $A \in \mathcal{M}$ (appearing in the formulation of the theorem) satisfying

$$\sup\,(\cup\{F(x) : x \in A \cap M_{\varepsilon/3}\}) < c + \frac{\varepsilon^2}{12}$$

because

$$c = c(M, F, \mathcal{M}) \le c(M_{\varepsilon/3}, F, \mathcal{M}) \le c(X, F, \mathcal{M}) = c.$$

We set

$$\psi_\varepsilon(x) = \max\{0, \frac{\varepsilon^2}{4} - \frac{\varepsilon}{2} \cdot \text{dist}\,(x, M)\}$$
$$F_\varepsilon(x) = F(x) + \psi_\varepsilon(x)$$

for $x \in X$, and $c_\varepsilon = c + \dfrac{\varepsilon^2}{4}$. It is easy to check that

$$c_\varepsilon \le c(M, F_\varepsilon, \mathcal{M}) \le c(X, F_\varepsilon, \mathcal{M}) \le c_\varepsilon.$$

Since $0 \le \psi_\varepsilon(x) \le \dfrac{\varepsilon^2}{4}$ for each $x \in X$,

$$\sup\,(\cup\{F_\varepsilon(x) : x \in A \cap M_{\frac{\varepsilon}{3}}\}) < c_\varepsilon + \frac{\varepsilon^2}{12} \tag{7}$$

holds true.

Let us choose S to be the set $A \cap M_{\varepsilon/3}$. Then

$$S_{\varepsilon/3} = (M_{\varepsilon/3} \cap A)_{\varepsilon/3} \subset M_{2\varepsilon/3} \cap A_{\varepsilon/3}.$$

Let us assume that for every

$$x \in D_\varepsilon = cl(M_{2\varepsilon/3} \cap A_{\varepsilon/3} \cap \{x \in X : \sup\; F_\varepsilon(x) \in [c_\varepsilon - \varepsilon^2/12, c_\varepsilon + \varepsilon^2/12]\})$$

we have $|dF_\varepsilon|(x) > \varepsilon/2$. Let U_x be an open neighbourhood of x such that $U_x \subset M_\varepsilon$ and $|dF_\varepsilon|(y) > \varepsilon/2$ whenever $y \in U_x$. We set $Q_\varepsilon = \cup\{U_x : x \in D_\varepsilon\}$. This set is open, contains D_ε and $|dF_\varepsilon|(y) > \varepsilon/2$ whenever $y \in Q_\varepsilon$. We are ready to apply the deformation lemma (see Remark 3.2) for the following choice of $F, S, c, \varepsilon, \delta$ and Q respectively: F_ε, $A \cap M_{\varepsilon/3}$, c_ε, $\varepsilon^2/12$, $\varepsilon/3$ and Q_ε. We thus obtain $\eta \in C(X \times [0, \infty), X)$ satisfying :

(a) $\eta(x, 0) = x$ for each $x \in X$;

(b) $\eta(x, t) = x$ for each $(x, t) \in (X \setminus Q_\varepsilon) \times [0, \infty)$;

(c) if $x \in A \cap M_{\varepsilon/3}$ and $\sup\; F_\varepsilon(x) \le c_\varepsilon + \varepsilon^2/12$, then $\eta(x, \varepsilon/3) \in A_{\varepsilon/3} \cap M_{2\varepsilon/3}$ and $\sup\; F_\varepsilon(\eta(x, \varepsilon/3)) \le c_\varepsilon - \varepsilon^2/12$;

(d) $d(x, \eta(x, t)) \le t$ for each $x \in X$ and $t \ge 0$.

Let us consider $\hat{A} = \eta(A, \varepsilon/3)$. Since $Q_\varepsilon \subset M_\varepsilon$ we have $B \cap Q_\varepsilon = \emptyset$ and, hence, $\eta(x, t) = x$ for all $(x, t) \in (B \times [0, \infty))$. Since $\mathcal{M}$ is homotopy stable with boundary B, we have $\hat{A} \in \mathcal{M}$. It follows from (d) that $\hat{A} \cap M \subset \eta(A \cap M_{\varepsilon/3}, \varepsilon/3)$. On the other hand $A \cap M_{\varepsilon/3} \subset \{x \in X : \sup\; F_\varepsilon(x) \le c_\varepsilon + \varepsilon^2/12\}$ because of (7). Applying (c) we obtain $\sup\; F_\varepsilon(\eta(A \cap M_{\frac{\varepsilon}{3}}, \frac{\varepsilon}{3})) \le c_\varepsilon - \varepsilon^2/12$. Then

$$c_\varepsilon = c(M, F_\varepsilon, \mathcal{M}) \le \sup\,(\cup\{F_\varepsilon(x) : x \in \hat{A} \cap M\}) \le c_\varepsilon - \frac{\varepsilon^2}{12},$$

which is a contradiction. Hence there is a point $x_\varepsilon \in D_\varepsilon$ with $|dF_\varepsilon|(x) \leq \dfrac{\varepsilon}{2}$. Moreover, we know that

$$x_\varepsilon = \lim_{n\to\infty} x_n \text{ where } x_n \in A_{\varepsilon/3} \cap M_{2\varepsilon/3} \text{ and } \sup\ F_\varepsilon(x_n) \in [c_\varepsilon - \frac{\varepsilon^2}{12}, c_\varepsilon + \frac{\varepsilon^2}{12}].$$

By the compactness of $F_\varepsilon(x_n)$ we get $y_n \in F_\varepsilon(x_n)$ satisfying $y_n = \sup\ F_\varepsilon(x_n)$. The sequence $\{y_n\}_{n=1}^\infty$ of reals is bounded. Let $\{y_{n_k}\}_{k=1}^\infty$ be a convergent subsequence of it with a limit y_0. Then $y_0 \in F_\varepsilon(x_\varepsilon)$ and $y_0 \in [c_\varepsilon - \frac{\varepsilon^2}{12}, c_\varepsilon + \frac{\varepsilon^2}{12}]$ because F_ε is usco. Therefore $x_\varepsilon \in F_\varepsilon^{-1}([c_\varepsilon - \frac{\varepsilon^2}{12}, c_\varepsilon + \frac{\varepsilon^2}{12}])$. Since $F(x_\varepsilon) = F_\varepsilon(x_\varepsilon) - \psi_\varepsilon(x_\varepsilon)$ and $-\frac{\varepsilon^2}{4} \leq -\psi_\varepsilon(x_\varepsilon) \leq 0$, $x_\varepsilon \in F^{-1}([c - \frac{\varepsilon^2}{3}, c + \frac{\varepsilon^2}{12}])$. Thus we checked that x_ε satisfies (i), (iii) and (iv) from the statement of the theorem. It remains to prove (ii). Let us assume the contrary, i.e. $|dF|(x_\varepsilon) > \varepsilon$. Let ε' be a positive real with $|dF|(x_\varepsilon) > \varepsilon' > \varepsilon$. Then there exist a positive real δ and a deformation $\mathcal{H} \in C(B(x_\varepsilon, \delta) \times [0, \delta], X)$ such that for every $x \in B(x_\varepsilon, \delta)$ and for every $t \in [0, \delta]$ the following properties hold true:

$$d(x, \mathcal{H}(x,t)) \leq t$$

$$\sup\ F(\mathcal{H}(x,t)) \leq \sup\ F(x) - \varepsilon'.t.$$

Since ψ_ε is a globally Lipschitz function with Lipschitz constant $\frac{\varepsilon}{2}$, we have

$$\begin{aligned}
\sup\ F_\varepsilon(\mathcal{H}(x,t)) &= \sup\ F(\mathcal{H}(x,t)) + \psi_\varepsilon(\mathcal{H}(x,t)) \leq \\
&\leq \sup\ F(x) - \varepsilon'.t + \psi_\varepsilon(\mathcal{H}(x,t)) = \\
&= \sup\ F_\varepsilon(x) - \psi_\varepsilon(x) - \varepsilon'.t + \psi_\varepsilon(\mathcal{H}(x,t)) \leq \\
&\leq \sup\ F_\varepsilon(x) - \varepsilon'.t + (\varepsilon/2)d(x, \mathcal{H}(x,t)) \leq \\
&\leq \sup\ F_\varepsilon(x) - \varepsilon'.t + (\varepsilon/2).t = \\
&= \sup\ F_\varepsilon(x) - (\varepsilon' - (\varepsilon/2)).t.
\end{aligned}$$

Hence $|dF_\varepsilon|(x_\varepsilon) \geq (\varepsilon' - (\varepsilon/2)) > (\varepsilon/2)$. The obtained contradiction completes the proof. □

Definition 4.3. *Let X be a complete metric space, $c \in \mathbb{R}$ and $F : X \to \mathbb{R}$ be an usco correspondence. We say that c is a critical value of F if there exists a point $x_0 \in X$ such that $c \in F(x_0)$ and $|dF|(x_0) = 0$. We say that F satisfies the condition $(PS)_c$ if, whenever a sequence $\{x_n\}_{n=1}^\infty$ is such that $c = \lim_{n\to\infty} y_n$, where $y_n \in F(x_n)$ and $\liminf\ |dF|(x_n) = 0$, then c is a critical value of F.*

Next we introduce the final necessary notation: Let u, v be two distinct points of the connected metric space X. We denote

$$\Gamma = \{g \in C([0,1], X) : g(0) = u, g(1) = v\}$$

(the set of paths connecting u and v).

Corollary 4.4. *Let X be a complete metric space, $F : X \to \mathbb{R}$ be an usco correspondence, D be a closed subset of X and u, v be two points from X belonging to disjoint components of $X \setminus D$. Assume $c(X, F, \Gamma) = c(D, F, \Gamma) = c$. If F verifies $(PS)_c$ then c is a critical value of F.*

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