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## NONTRIVIAL SOLUTIONS OF QUASILINEAR EQUATIONS IN $BV$

Marco Marzocchi

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ABSTRACT. The existence of a nontrivial critical point is proved for a functional containing an area-type term. Techniques of nonsmooth critical point theory are applied.

**1. Introduction.** Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$  ( $n \geq 3$ ) and  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  a Carathéodory function with  $g(x, 0) = 0$ . A classical result of Ambrosetti and Rabinowitz [1, 12, 13] says that the semilinear problem

$$\begin{cases} -\Delta u = g(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits a nontrivial solution  $u$ , provided that the following conditions are satisfied:

(C1) there exist  $a \in L^{\frac{2n}{n+2}}(\Omega)$ ,  $b \in \mathbb{R}$  and  $p \in ]2, \frac{2n}{n-2}[$  such that

$$|g(x, s)| \leq a(x) + b|s|^{p-1};$$

(C2) there exist  $q > 2$  and  $R > 0$  such that

$$|s| \geq R \implies 0 < qG(x, s) \leq sg(x, s),$$

where  $G(x, s) = \int_0^s g(x, t) dt;$

(C3) it is

$$\lim_{s \rightarrow 0} \frac{g(x, s)}{s} = 0$$

uniformly with respect to  $x$ .

Such a nontrivial solution  $u$  is found as a mountain pass point of the functional  $f : H_0^1(\Omega) \rightarrow \mathbb{R}$  defined by

$$f(u) = \frac{1}{2} \int_{\Omega} |Du|^2 dx - \int_{\Omega} G(x, u) dx.$$

Our aim is to get a similar result for a class of functionals which contains, as a model example, the functional

$$f(u) = \int_{\Omega} |Du| dx - \int_{\Omega} G(x, u) dx.$$

The correct expression of  $f$ , which requires a relaxation procedure, will be given in section 4. Here we want to observe that the natural adaptation of condition (C1) would be

$$|g(x, s)| \leq a(x) + b|s|^{p-1}$$

with  $a \in L^n(\Omega)$  and  $p \in ]1, \frac{n}{n-1}[$ . On the other hand, the natural domain of  $f$  is now the space  $BV(\Omega)$ . In such a space also nonsmooth versions of critical point theory cannot be directly applied, as the Palais-Smale condition fails (see [11]). To overcome this difficulty, it is possible to consider the functional  $f$  on  $L^p(\Omega)$  (with value  $+\infty$  outside its natural domain). If we add the stronger condition that  $a \in L^{p'}(\Omega)$ , then  $f$  is the sum of a convex term and a functional of class  $C^1$ , and the expected result can be obtained. Such a strategy has been applied in [11], to treat the case where  $f$  is even. However, this further condition on  $a$  seems to be merely technical. Our aim is to show that the assumption  $a \in L^n(\Omega)$  is in fact sufficient. As in [11], we apply the nonsmooth critical point theory developed in [4, 6], which provides general results for continuous functionals defined on metric spaces. Among lower semicontinuous functionals (as  $f$  on

$L^p(\Omega)$ ), some particular classes can be treated. The main part of this paper, namely section 3, is devoted to the study of a class of lower semicontinuous functionals, which contains  $f$  and for which the theory of [4, 6] can be applied. Then, in the last section, we prove the existence of a mountain pass point for  $f$ .

**2. Some notions of nonsmooth critical point theory.** Let us recall some notions of nonsmooth critical point theory from [4, 6]. A similar approach to nonregular functionals can be found also in [10, 9]. In the following of this section,  $X$  will denote a metric space endowed with the metric  $d$ .

**Definition 2.1.** *Let  $f : X \rightarrow \mathbb{R}$  be a continuous function and let  $u \in X$ . We denote by  $|df|(u)$  the supremum of the  $\sigma$ 's in  $[0, +\infty[$  such that there exist  $\delta > 0$  and a continuous map  $\mathcal{H} : B(u, \delta) \times [0, \delta] \rightarrow X$  such that*

$$\begin{aligned} \forall v \in B(u, \delta), \forall t \in [0, \delta] : \quad & d(\mathcal{H}(v, t), v) \leq t, \\ \forall v \in B(u, \delta), \forall t \in [0, \delta] : \quad & f(\mathcal{H}(v, t)) \leq f(v) - \sigma t. \end{aligned}$$

The extended real number  $|df|(u)$  is called the weak slope of  $f$  at  $u$ .

The above notion can be extended also to lower semicontinuous functions, by means of a tool introduced for the first time in [5].

**Definition 2.2.** *Let  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be a lower semicontinuous function and  $b \in \mathbb{R}$ . We set*

$$\begin{aligned} \mathcal{D}(f) &= \{u \in X : f(u) < +\infty\}, \\ f^b &= \{u \in X : f(u) \leq b\}, \\ \text{epi}(f) &= \{(u, \xi) \in X \times \mathbb{R} : f(u) \leq \xi\}. \end{aligned}$$

We define the function  $\mathcal{G}_f : \text{epi}(f) \rightarrow \mathbb{R}$  putting  $\mathcal{G}_f(u, \xi) = \xi$ .

In the following  $\text{epi}(f)$  will be endowed with the metric

$$d((u, \xi), (v, \mu)) = (d(u, v)^2 + (\xi - \mu)^2)^{\frac{1}{2}},$$

so that  $\mathcal{G}_f$  is Lipschitz continuous of constant 1. Therefore Definition 2.1 can be applied to  $\mathcal{G}_f$  and  $|d\mathcal{G}_f|(u, \xi) \leq 1$  for every  $(u, \xi) \in \text{epi}(f)$ .

**Definition 2.3.** Let  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be a lower semicontinuous function and let  $u \in \mathcal{D}(f)$ . We set

$$|df|(u) = \begin{cases} \frac{|d\mathcal{G}_f|(u, f(u))}{\sqrt{1 - (|d\mathcal{G}_f|(u, f(u)))^2}} & \text{if } |d\mathcal{G}_f|(u, f(u)) < 1, \\ +\infty & \text{if } |d\mathcal{G}_f|(u, f(u)) = 1. \end{cases}$$

It is shown in [6, Proposition 2.3] that the above definition is consistent with Definition 2.1.

**Definition 2.4.** Let  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be a lower semicontinuous function. We say that  $u \in X$  is a (lower) critical point for  $f$ , if  $|df|(u) = 0$ . A real number  $c$  is said to be a (lower) critical value, if there exists  $u \in \mathcal{D}(f)$  such that  $|df|(u) = 0$  and  $f(u) = c$ .

**Definition 2.5.** Let  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$  be a lower semicontinuous function and  $c \in \mathbb{R}$ . We say that  $f$  satisfies the Palais-Smale condition at level  $c$  ( $(PS)_c$  for short), if from every sequence  $(u_h)$  in  $\mathcal{D}(f)$  with  $|df|(u_h) \rightarrow 0$  and  $f(u_h) \rightarrow c$  it is possible to extract a subsequence  $(u_{h_k})$  converging in  $X$ .

**3. Some abstract results.** As pointed out in [6], the essential difficulty when dealing with lower semicontinuous functions is that we do not know in general the behaviour of  $|d\mathcal{G}_f|(u, \xi)$  at the points with  $\xi > f(u)$ .

Therefore, the main result of this section is a theorem in the spirit of [6, Theorem 3.13] and [4, Theorem 4.4].

**Theorem 3.1.** Let  $X$  be a linear space,  $\|\cdot\|, \|\cdot\|_0$  two norms on  $X$  and  $c > 0$  such that  $\|\cdot\|_0 \leq c\|\cdot\|$ . Let  $X_0$  (resp.  $X_1$ ) be the space  $X$  endowed with the norm  $\|\cdot\|_0$  (resp.  $\|\cdot\|$ ).

Let  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ ,  $f = f_0 + f_1$ , such that:

- (a)  $f_0 : X_0 \rightarrow \mathbb{R} \cup \{+\infty\}$  is convex, lower semicontinuous and for every  $u_0 \in X$  there exists  $r > 0$  such that

$$\lim_{\substack{\|u\| \rightarrow \infty \\ \|u - u_0\|_0 \leq r}} f_0(u) = +\infty;$$

- (b)  $f_1 : X_1 \rightarrow \mathbb{R}$  is of class  $C^1$ ;

(c) for every  $\varepsilon > 0$  there exist  $\varphi : X_1 \rightarrow \mathbb{R}$  Lipschitz of constant  $\varepsilon$  and  $\psi : X_0 \rightarrow \mathbb{R}$  of class  $C^1$  such that  $f_1 = \varphi + \psi$ .

Then the following facts hold:

(i)  $f : X_0 \rightarrow \mathbb{R} \cup \{+\infty\}$  is lower semicontinuous and for every  $u_0 \in X$  there exists  $r > 0$  such that

$$(3.1) \quad \liminf_{\substack{\|u\| \rightarrow \infty \\ \|u - u_0\|_0 \leq r}} \frac{f(u)}{\|u\|} > 0;$$

(ii)  $f_1 : X_0 \rightarrow \mathbb{R}$  is continuous on  $X_1$ -bounded subsets;

(iii)  $\xi > f(u) \implies |d_0 \mathcal{G}_f|(u, \xi) = 1$ ;

(iv) if  $u \in \mathcal{D}(f)$ , then  $|df|(u) \leq c |d_0 f|(u)$ , where  $|d_0 f|$  (resp.  $|df|$ ) denotes the weak slope of  $f$  in  $X_0$  (resp. in  $X_1$ ).

Proof. (i) Let  $u_0 \in X$ ,  $r > 0$  according to (a). Without loss of generality, we can suppose there exists  $v_0 \in \mathcal{D}(f_0)$  such that  $\|v_0 - u_0\|_0 \leq r$ . Let  $r_1 > 0$  be such that

$$\forall u \in X : \|u - u_0\|_0 \leq r, \|u - v_0\| \geq r_1 \implies f_0(u) \geq f_0(v_0) + 1.$$

If  $\|u - u_0\|_0 \leq r$  and  $\|u - v_0\| \geq r_1$ , taking into account the convexity of  $f_0$ , we deduce that

$$f_0(v_0) + 1 \leq f_0\left(v_0 + \frac{r_1}{\|u - v_0\|}(u - v_0)\right) \leq f_0(v_0) + \frac{r_1}{\|u - v_0\|}(f_0(u) - f_0(v_0)),$$

hence

$$f_0(u) \geq f_0(v_0) + \frac{1}{r_1} \|u - v_0\|.$$

Thus, we have shown that, for every  $u_0 \in X$  with corresponding  $r$  according to (a), it is

$$(3.2) \quad m_{u_0, r} := \liminf_{\substack{\|u\| \rightarrow \infty \\ \|u - u_0\|_0 \leq r}} \frac{f_0(u)}{\|u\|} > 0.$$

Let  $u_0 \in X$  and  $r > 0$  according to (a). Let  $\varepsilon \in ]0, m_{u_0, r}[$  and  $f_1 = \varphi + \psi$  according to hypothesis (c). Then, for every  $u \in X$  it is

$$f(u) = f_0(u) + \varphi(u) + \psi(u) \geq f_0(u) + \psi(u) + \varphi(0) - \varepsilon \|u\|.$$

Unless reducing  $r$ , we can suppose that  $\psi$  is bounded on  $B_0(u_0, r)$ . Therefore, from (3.2) it follows that for every  $u_0 \in X$  and  $r > 0$  according to (a) it is

$$\liminf_{\substack{\|u\| \rightarrow \infty \\ \|u-u_0\|_0 \leq r}} \frac{f(u)}{\|u\|} \geq m_{u_0,r} - \varepsilon > 0.$$

Now, if  $(u_h)$  is a sequence convergent to  $u$  in  $X_0$  with  $f(u_h) \leq c$ , it follows that  $(u_h)$  is bounded also in  $X_1$ . From assumptions (a) and (b) we deduce that

$$f(u) \leq \liminf_h f(u_h),$$

namely that  $f : X_0 \rightarrow \mathbb{R} \cup \{+\infty\}$  is lower semicontinuous.

(ii) Let  $\varepsilon = 1/h$  and let  $\varphi_h, \psi_h$  as in (c). Then we have

$$\|u\| \leq K \implies |\varphi_h(u) - \varphi_h(0)| \leq \frac{1}{h} \|u\| \leq \frac{1}{h} K.$$

It follows that  $(\psi_h(\cdot) + \varphi_h(0))$  is uniformly convergent to  $f_1$  on  $X_1$ -bounded subsets, whence the assertion.

(iii) Let  $(u, \xi) \in \text{epi}(f)$  with  $\xi > f(u)$ . Without loss of generality, we can assume that  $f_1(u) = 0$ . By (i), there exists  $r \in ]0, 1[$  and  $K > 0$  such that

$$(3.3) \quad \forall v \in X : \begin{cases} f(v) \leq \xi + 1 \\ \|v - u\|_0 \leq r \end{cases} \implies \|v - u\| \leq K.$$

Let  $\delta \in ]0, \min\{\frac{\xi - f_0(u)}{2}, r\}[$  and  $\varepsilon > 0$  such that  $\varepsilon K < \delta/4$ . Choose  $\varphi$  and  $\psi$  according to (c) with  $\varphi(u) = \psi(u) = 0$  and set, for every  $v \in X$ ,

$$\begin{aligned} \tilde{f}_0(v) &= f_0(v) + \langle d\psi(u), v - u \rangle_0, \\ \tilde{\psi}(v) &= \psi(v) - \langle d\psi(u), v - u \rangle_0, \end{aligned}$$

so that  $f = \tilde{f}_0 + \varphi + \tilde{\psi}$  and  $d\tilde{\psi}(u) = 0$  in  $X'_0$ .

Let  $\mathcal{H} : (B((u, \xi), \delta) \cap \text{epi}(\tilde{f}_0)) \times [0, \delta] \rightarrow \text{epi}(\tilde{f}_0)$  be defined by

$$\mathcal{H}((v, \mu), t) = \left( v + \frac{t(u - v)}{\sqrt{\|v - u\|_0^2 + |\mu - \tilde{f}_0(u)|^2}}, \mu - (\mu - \tilde{f}_0(u)) \frac{t}{\sqrt{\|v - u\|_0^2 + |\mu - \tilde{f}_0(u)|^2}} \right).$$

As in the proof of [6, Theorem 3.13], it follows that for every  $(v, \mu) \in B_0((u, \xi), \delta) \cap \text{epi}(\tilde{f}_0)$  and every  $t \in [0, \delta]$ , it is

$$d(\mathcal{H}((v, \mu), t), (v, \mu)) \leq t,$$

$$\mathcal{G}_{\tilde{f}_0}(\mathcal{H}((v, \mu), t)) \leq \mathcal{G}_{\tilde{f}_0}(v, \mu) - \frac{\xi - \delta - \tilde{f}_0(u)}{\sqrt{\delta^2 + (\xi + \delta - \tilde{f}_0(u))^2}} t.$$

Let  $\delta' \in ]0, \delta/2[$  such that  $|\tilde{\psi}(v)| < \delta/2$  if  $v \in B_0(u, \delta')$ . Then, if  $(v, \mu) \in B_0((u, \xi), \delta') \cap \text{epi}(\tilde{f}_0 + \varphi)$  it is, taking into account (3.3),

$$|\mu - \varphi(v) - \xi| \leq |\mu - \xi| + |\varphi(v)| \leq \frac{\delta}{2} + \varepsilon \|v - u\| \leq \frac{\delta}{2} + \varepsilon K \leq \frac{\delta}{2} + \frac{\delta}{4} = \frac{3}{4}\delta$$

so that it is easy to check that  $(v, \mu - \varphi(v)) \in B_0((u, \xi), \delta) \cap \text{epi}(\tilde{f}_0)$ .

If  $\rho > 0$ , by the definition of  $\mathcal{H}$  we can deduce that, for every  $(v, \mu) \in B_0((u, \xi), \delta') \cap \text{epi}(\tilde{f}_0 + \varphi)$

$$\|\mathcal{H}_1((v, \mu - \varphi(v)), \rho t) - v\| = \frac{\|v - u\|}{\sqrt{\|v - u\|_0^2 + |\mu - \varphi(v) - \tilde{f}_0(u)|^2}} \rho t \leq \frac{\rho K}{\delta} t$$

since  $|\mu - \varphi(v) - \tilde{f}_0(u)| \geq |\xi - \tilde{f}_0(u)| - |\mu - \xi| - |\varphi(v)| > 2\delta - \frac{\delta}{2} - \frac{\delta}{4} > \delta$ .

Let now  $\rho = \left(1 + \frac{\varepsilon K}{\delta}\right)^{-1}$ , and define  $\tilde{\mathcal{H}} : (B_0((u, \xi), \delta') \cap \text{epi}(\tilde{f}_0 + \varphi)) \times [0, \delta'] \rightarrow \text{epi}(\tilde{f}_0 + \varphi)$  setting

$$\tilde{\mathcal{H}}((v, \mu), t) = \left(\mathcal{H}_1((v, \mu - \varphi(v)), \rho t), \mathcal{H}_2((v, \mu - \varphi(v)), \rho t) + \varphi(\mathcal{H}_1((v, \mu - \varphi(v)), \rho t))\right).$$

It is readily seen that  $\tilde{\mathcal{H}}$  actually takes his values in  $\text{epi}(\tilde{f}_0 + \varphi)$ .

Furthermore, since  $\varphi$  is continuous in  $X_0$  on  $X_1$ -bounded sets, (3.3) implies that  $\tilde{\mathcal{H}}$  is continuous.

It is

$$\begin{aligned} & \left\| \tilde{\mathcal{H}}((v, \mu), t) - (v, \mu) \right\|_{X_0 \times \mathbb{R}}^2 = \\ & = \|\mathcal{H}_1((v, \mu - \varphi(v)), \rho t) - v\|_0^2 + \left(\mathcal{H}_2((v, \mu - \varphi(v)), \rho t) + \varphi(\mathcal{H}_1((v, \mu - \varphi(v)), \rho t)) - \mu\right)^2 = \\ & = \|\mathcal{H}_1((v, \mu - \varphi(v)), \rho t) - v\|_0^2 + \left(\mathcal{H}_2((v, \mu - \varphi(v)), \rho t) - (\mu - \varphi(v))\right)^2 + \\ & + 2\left(\mathcal{H}_2((v, \mu - \varphi(v)), \rho t) - (\mu - \varphi(v))\right) \left(\varphi(\mathcal{H}_1((v, \mu - \varphi(v)), \rho t)) - \varphi(v)\right) + \end{aligned}$$



$$\begin{aligned}
& + \left( \varphi(\mathcal{H}_1((v, \mu - \varphi(v)), \rho t)) - \varphi(v) \right)^2 \leq \\
\leq & \rho^2 t^2 + 2\rho t \varepsilon \|\mathcal{H}_1((v, \mu - \varphi(v)), \rho t) - v\| + \varepsilon^2 \|\mathcal{H}_1((v, \mu - \varphi(v)), \rho t) - v\|^2 \leq \\
& \leq \rho^2 t^2 + 2\rho^2 \frac{\varepsilon K}{\delta} t^2 + \rho^2 \frac{\varepsilon^2 K^2}{\delta^2} t^2 = \rho^2 t^2 \left( 1 + \frac{\varepsilon K}{\delta} \right)^2 = t^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mathcal{G}_{\tilde{f}_0 + \varphi}(\tilde{\mathcal{H}}((v, \mu), t)) & = \mathcal{H}_2((v, \mu - \varphi(v)), \rho t) + \varphi(\mathcal{H}_1((v, \mu - \varphi(v)), \rho t)) = \\
& = \mathcal{G}_{\tilde{f}_0}(\mathcal{H}((v, \mu - \varphi(u)), \rho t)) + \varphi(\mathcal{H}_1((v, \mu - \varphi(v)), \rho t)) \leq \\
& \leq \mu - \varphi(v) - \frac{\xi - \delta - \tilde{f}_0(u)}{\sqrt{\delta^2 + (\xi + \delta - \tilde{f}_0(u))^2}} \rho t + \varphi(\mathcal{H}_1((v, \mu - \varphi(v)), \rho t)) \leq \\
& \leq \mu - \frac{\xi - \delta - \tilde{f}_0(u)}{\sqrt{\delta^2 + (\xi + \delta - \tilde{f}_0(u))^2}} \rho t + \varepsilon \frac{\rho K}{\delta} t = \\
& = \mu - \left( \frac{\xi - \delta - \tilde{f}_0(u)}{\sqrt{\delta^2 + (\xi + \delta - \tilde{f}_0(u))^2}} - \varepsilon \frac{K}{\delta} \right) \frac{t}{\left( 1 + \frac{\varepsilon K}{\delta} \right)};
\end{aligned}$$

therefore we have

$$\left| d_0 \mathcal{G}_{\tilde{f}_0 + \varphi} \right| (u, \xi) \geq \left( \frac{\xi - \delta - \tilde{f}_0(u)}{\sqrt{\delta^2 + (\xi + \delta - \tilde{f}_0(u))^2}} - \varepsilon \frac{K}{\delta} \right) \frac{1}{\left( 1 + \frac{\varepsilon K}{\delta} \right)}.$$

But by [6, Proposition 2.7] it is

$$\left| d_0 \mathcal{G}_f \right| (u, \xi) = \left| d_0 \mathcal{G}_{\tilde{f}_0 + \varphi + \tilde{\psi}} \right| (u, \xi) = \left| d_0 \mathcal{G}_{\tilde{f}_0 + \varphi} \right| (u, \xi),$$

hence by the arbitrariness of  $\varepsilon \in ]0, \frac{\delta}{4K}[$  it is

$$\left| d_0 \mathcal{G}_f \right| (u, \xi) \geq \frac{\xi - \delta - f_0(u)}{\sqrt{\delta^2 + (\xi + \delta - f_0(u))^2}},$$

and finally by the arbitrariness of  $\delta \in ]0, \min\{\frac{\xi - f_0(u)}{2}, r\}[$  we obtain

$$\left| d_0 \mathcal{G}_f \right| (u, \xi) \geq 1.$$

(iv) Let  $u \in \mathcal{D}(f)$  and  $r > 0$  corresponding to  $u$  as in (a). For every  $v \in X$ , let

$$\begin{aligned} \tilde{f}_0(v) &= f_0(v) + f_1(u) + \langle df_1(u), v - u \rangle_1, \\ \tilde{f}_1(v) &= f_1(v) - f_1(u) - \langle df_1(u), v - u \rangle_1. \end{aligned}$$

Then  $\tilde{f}_0$  is convex; furthermore, we can choose  $\varepsilon \in ]0, m_{u,r}[$  and obtain the decomposition

$$\tilde{f}_0(v) = f_0(v) + f_1(u) + \langle d\varphi(u), v - u \rangle_1 + \langle d\psi(u), v - u \rangle_0$$

with  $\|d\varphi(u)\|_{X'_1} \leq \varepsilon$ . It follows that

$$\liminf_{\substack{\|v\| \rightarrow \infty \\ \|v-u\|_0 \leq r}} \frac{\tilde{f}_0(v)}{\|v\|} > 0.$$

As in the proof of (i), we deduce that  $\tilde{f}_0 : X_0 \rightarrow \mathbb{R} \cup \{+\infty\}$  is lower semicontinuous, hence  $\tilde{f}_0$  satisfies assumption (a). Of course,  $\tilde{f}_1 : X_1 \rightarrow \mathbb{R}$  is of class  $C^1$  with  $\tilde{f}_1(u) = 0$  and  $d\tilde{f}_1(u) = 0$ . It is easy to see that  $\tilde{f}_1$  satisfies also assumption (c) with  $\tilde{\varphi}(v) = \varphi(v) - \varphi(u) - \langle d\varphi(u), v - u \rangle_1$ ,  $\tilde{\psi}(v) = \psi(v) - \psi(u) - \langle d\psi(u), v - u \rangle_0$  and  $d\tilde{\psi}(u) = 0$ . Therefore, it suffices to consider the case  $f_1(u) = 0, df_1(u) = 0$ . Moreover, when we apply (c), we can ask that  $d\psi(u) = 0$ .

Since the case  $|df|(u) = 0$  is obvious, let  $0 < \sigma < |df|(u)$ . From [6, Proposition 2.7] we deduce that  $|df_0|(u) = |df|(u)$ . As in the proof of [6, Theorem 2.11], we can find  $w \in X$  such that

$$f_0(w) < f_0(u) - \sigma \|w - u\| \leq f_0(u) - \frac{\sigma}{c} \|w - u\|_0.$$

Let  $\delta > 0$  be such that  $2\delta < \|w - u\|_0$  and

$$\forall v \in X : \|v - u\|_0 < \delta \implies f_0(w) < f_0(v) - \frac{\sigma}{c} \|w - v\|_0.$$

As in the proof of [6, Theorem 2.11], we can define a continuous map  $\mathcal{H} : B_0(u, \delta) \times [0, \delta] \rightarrow X_0$  by

$$\mathcal{H}(v, t) = v + \frac{t}{\|w - v\|_0} (w - v)$$

and we have

$$\|\mathcal{H}(v, t) - v\|_0 \leq t,$$

$$f_0(\mathcal{H}(v, t)) \leq f_0(v) - \frac{\sigma}{c}t.$$

By (i), there exists  $K > 0$  such that

$$\forall v \in X : \|v - u\|_0 < \delta, f(v) < f(u) + 1 \implies \|v\| \leq K.$$

Now let  $\varepsilon > 0$  and let  $f_1 = \varphi + \psi$  according to (c) with  $d\psi(u) = 0$ . If  $\|v - u\|_0 < \delta$  and  $f(v) < f(u) + 1$ , we have

$$|\varphi(\mathcal{H}(v, t)) - \varphi(v)| \leq \varepsilon \|\mathcal{H}(v, t) - v\| = \varepsilon \frac{\|w - v\|}{\|w - v\|_0} t \leq \varepsilon \frac{\|w\| + K}{\delta} t.$$

It follows

$$(f_0 + \varphi)(\mathcal{H}(v, t)) \leq (f_0 + \varphi)(v) - \left( \frac{\sigma}{c} - \varepsilon \frac{\|w\| + K}{\delta} \right) t,$$

hence  $|d_0(f_0 + \varphi)|(u) \geq \left( \frac{\sigma}{c} - \varepsilon \frac{\|w\| + K}{\delta} \right)$ . Since  $d\psi(u) = 0$ , from [6, Proposition 2.7] we deduce that

$$|d_0(f_0 + \varphi)|(u) \geq \frac{\sigma}{c} - \varepsilon \frac{\|w\| + K}{\delta}.$$

By the arbitrariness of  $\varepsilon$ , we have  $|d_0f|(u) \geq \sigma/c$ , and the assertion follows by the arbitrariness of  $\sigma$ .  $\square$

Now we prove a theorem of saddle-point type for our class of functionals.

**Theorem 3.2.** *Let  $X$  and  $f$  be as in Theorem 3.1.*

*Assume that*

(a) *there exist  $r > 0$  and  $\alpha > 0$  such that*

$$\forall u \in X : \|u\|_0 = r \implies f(u) \geq f(0) + \alpha;$$

(b) *there exists  $u_1 \in X$  with  $\|u_1\|_0 > r$  and  $f(u_1) \leq f(0)$ ;*

(c) *for every  $b \in \mathbb{R}$ ,  $f^b$  is complete with respect to  $\|\cdot\|_0$  and  $f : X_0 \rightarrow \mathbb{R} \cup \{+\infty\}$  satisfies  $(PS)_c$ , where*

$$c := \inf_{\gamma \in \Gamma} \sup_{0 \leq t \leq 1} f(\gamma(t)),$$

$$\Gamma = \left\{ \gamma \in C([0, 1]; X_0) : \gamma(0) = 0, \gamma(1) = u_1 \right\}.$$

Then there exists  $u \in X$  such that  $f(u) = c$  and

$$\forall v \in X : f_0(v) \geq f_0(u) - \langle df_1(u), v - u \rangle_1.$$

**Proof.** We apply Theorem 4.5 of [4]. Even if  $X_0$  is not complete, it is easy to see that  $\text{epi}(f)$  is complete, and this is enough.

By (i) and (iii) of Theorem 3.1,  $f : X_0 \rightarrow \mathbb{R} \cup \{+\infty\}$  is lower semicontinuous and satisfies condition (4.1) of [4]. Since  $f_0$  is convex, we have that  $\gamma(t) = tu_1$  belongs to  $\Gamma$  and that

$$c \leq \sup_{0 \leq t \leq 1} f(tu_1) < +\infty.$$

Then, by Theorem (4.5) of [4] there exist  $u \in X$  with  $f(u) = c$  and  $|d_0f|(u) = 0$ ; but for (iv) of Theorem 3.1 it is also  $|df|(u) = 0$ , hence the assertion follows from [6, Theorem 2.11].  $\square$

**4. An application.** In this section we apply our abstract framework to obtain an existence result for a class of functionals containing an area-type term.

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$  with Lipschitz boundary,  $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  two functions satisfying the following conditions:

( $\Psi$ ) the function  $\Psi$  is convex and there exist  $c, d > 0$  such that

$$\forall \xi \in \mathbb{R}^n : d|\xi| - c \leq \Psi(\xi) \leq c(|\xi| + 1);$$

( $g_1$ ) the function  $g$  satisfies the Carathéodory conditions and there exist  $a \in L^n(\Omega)$ ,  $b \in \mathbb{R}$  and  $p \in ]1, \frac{n}{n-1}[$  such that

$$|g(x, s)| \leq a(x) + b|s|^{p-1}$$

for a.e.  $x \in \Omega$  and every  $s \in \mathbb{R}$ .

Furthermore, let  $G(x, s) = \int_0^s g(x, t) dt$ .

According to [3, 8], we define  $f : BV(\Omega) \rightarrow \mathbb{R}$  setting  $f = f_0 + f_1$ , where

$$f_0(u) = \int_{\Omega} \Psi(\nabla u^a) dx + \int_{\Omega} \Psi^{\infty} \left( \frac{\nabla u^s}{|\nabla u^s|} \right) d|\nabla u^s|(x) + \int_{\partial\Omega} \Psi^{\infty}(u\nu) d\mathcal{H}^{n-1}(x),$$

$$f_1(u) = - \int_{\Omega} G(x, u) dx,$$

$\nabla u = \nabla u^a + \nabla u^s$  is the Lebesgue decomposition of  $\nabla u$ ,  $|\nabla u^s|$  is the total variation of  $\nabla u^s$ ,  $\nabla u^s/|\nabla u^s|$  is the Radon-Nikodym derivative of  $\nabla u^s$  with respect to  $|\nabla u^s|$ ,  $\Psi^\infty$  is the recession functional associated with  $\Psi$ , and  $\nu$  is the outer normal to  $\Omega$ .

As a norm in  $BV(\Omega)$ , we shall consider

$$\|u\|_{BV} = \int_{\Omega} |\nabla u^a| dx + \int_{\Omega} d|\nabla u^s|(x) + \int_{\partial\Omega} |u| d\mathcal{H}^{n-1}(x).$$

**Lemma 4.1.** *Let  $\gamma : [0, +\infty[ \rightarrow \mathbb{R}$  be a convex function such that*

$$\forall \xi \in \mathbb{R}^n : \Psi(\xi) \geq \gamma(|\xi|).$$

*Then we have*

$$\forall u \in BV(\Omega) : f_0(u) \geq \mathcal{L}^n(\Omega) \gamma\left(\frac{\|u\|_{BV}}{\mathcal{L}^n(\Omega)}\right).$$

*Proof.* First of all, for any  $x, y, z \in [0, +\infty[$  and  $\lambda > 0$  we have

$$(4.1) \quad \lambda\gamma\left(\frac{x+y+z}{\lambda}\right) \leq \lambda\gamma\left(\frac{x}{\lambda}\right) + \gamma^\infty(y) + \gamma^\infty(z).$$

In fact, for any  $\varepsilon \in ]0, 1/2]$  we have

$$\begin{aligned} \lambda\gamma\left(\frac{x+y+z}{\lambda}\right) &= \lambda\gamma\left((1-2\varepsilon)\frac{x}{\lambda} + \varepsilon\left(\frac{x}{\lambda} + \frac{y}{\varepsilon\lambda}\right) + \varepsilon\left(\frac{x}{\lambda} + \frac{z}{\varepsilon\lambda}\right)\right) \leq \\ &\leq \lambda(1-2\varepsilon)\gamma\left(\frac{x}{\lambda}\right) + \lambda\varepsilon\gamma\left(\frac{x}{\lambda} + \frac{y}{\varepsilon\lambda}\right) + \lambda\varepsilon\gamma\left(\frac{x}{\lambda} + \frac{z}{\varepsilon\lambda}\right) \leq \\ &\leq \lambda(1-2\varepsilon)\gamma\left(\frac{x}{\lambda}\right) + 2\lambda\varepsilon\gamma(0) + \lambda\varepsilon\gamma^\infty\left(\frac{x}{\lambda} + \frac{y}{\varepsilon\lambda}\right) + \lambda\varepsilon\gamma^\infty\left(\frac{x}{\lambda} + \frac{z}{\varepsilon\lambda}\right) = \\ &= \lambda(1-2\varepsilon)\gamma\left(\frac{x}{\lambda}\right) + 2\lambda\varepsilon\gamma(0) + \gamma^\infty(\varepsilon x + y) + \gamma^\infty(\varepsilon x + z). \end{aligned}$$

Going to the limit as  $\varepsilon \rightarrow 0$ , we get (4.1).

Now let  $u \in BV(\Omega)$ . Since  $\gamma^\infty(|\xi|) \leq \Psi^\infty(\xi)$ , from Jensen's inequality and (4.1) we deduce that

$$f_0(u) \geq \int_{\Omega} \gamma(|\nabla u^a|) dx + \int_{\Omega} \gamma^\infty(1) d|\nabla u^s|(x) + \int_{\partial\Omega} |u| \gamma^\infty(1) d\mathcal{H}^{n-1}(x) \geq$$

$$\begin{aligned} &\geq \mathcal{L}^n(\Omega) \gamma \left( \mathcal{L}^n(\Omega)^{-1} \int_{\Omega} |\nabla u^a| \, dx \right) + \gamma^\infty \left( \int_{\Omega} d|\nabla u^s|(x) \right) + \gamma^\infty \left( \int_{\partial\Omega} |u| \, d\mathcal{H}^{n-1}(x) \right) \geq \\ &\geq \mathcal{L}^n(\Omega) \gamma \left( \frac{\|u\|_{BV}}{\mathcal{L}^n(\Omega)} \right), \end{aligned}$$

whence the assertion.  $\square$

In particular, combining the previous lemma with assumption  $(\Psi)$ , we deduce that there exist  $\tilde{c}, \tilde{d} > 0$  such that

$$(4.2) \quad \forall u \in BV(\Omega) : \tilde{d} \|u\|_{BV} - \tilde{c} \leq f_0(u) \leq \tilde{c}(\|u\|_{BV} + 1).$$

**Lemma 4.2.** *For every  $\varepsilon > 0$  there exist  $\varphi : BV(\Omega) \rightarrow \mathbb{R}$  Lipschitz of constant  $\varepsilon$ ,  $\psi : L^p(\Omega) \rightarrow \mathbb{R}$  of class  $C^1$  such that  $f_1 = \varphi + \psi$ .*

*Proof.* Let  $\varepsilon > 0$  and

$$\begin{aligned} g_1(x, s) &= \min\{\max\{g(x, s), -a(x)\}, a(x)\}, \\ g_2(x, s) &= g(x, s) - g_1(x, s), \end{aligned}$$

so that  $|g_1(x, s)| \leq a(x)$  and  $|g_2(x, s)| \leq b|s|^{p-1}$ .

Let  $\bar{c} > 0$  be such that  $\|\cdot\|_{\frac{n}{n-1}} \leq \bar{c} \|\cdot\|_{BV}$ ; if  $k \in \mathbb{R}$ , let  $\bar{a}(x) = a(x)\chi_{\{a(x) \geq k\}}(x)$ , and choose  $k$  such that  $\bar{c} \|\bar{a}\|_n \leq \varepsilon$ . Now let

$$\begin{aligned} \bar{g}_1(x, s) &= g_1(x, s)\chi_{\{a(x) \geq k\}}(x), \\ \bar{\bar{g}}_1(x, s) &= g_1(x, s)\chi_{\{a(x) < k\}}(x), \end{aligned}$$

so that  $|\bar{g}_1(x, s)| \leq \bar{a}(x)$  and  $|\bar{\bar{g}}_1(x, s)| \leq k$ .

Finally, let

$$\begin{aligned} G_1(x, s) &= \int_0^s \bar{g}_1(x, t) \, dt, \\ G_2(x, s) &= \int_0^s [\bar{\bar{g}}_1(x, t) + g_2(x, t)] \, dt, \\ \varphi(u) &= \int_{\Omega} G_1(x, u) \, dx, \\ \psi(u) &= \int_{\Omega} G_2(x, u) \, dx. \end{aligned}$$

Since

$$|\bar{g}_1(x, s) + g_2(x, s)| \leq k + b |s|^{p-1},$$

it is well known that  $\psi : L^p(\Omega) \rightarrow \mathbb{R}$  is of class  $C^1$ .

Furthermore, it is

$$\begin{aligned} |\varphi(v) - \varphi(u)| &\leq \int_{\Omega} \bar{a}(x) |v - u| \, dx \leq \|\bar{a}\|_n \|v - u\|_{\frac{n}{n-1}} \leq \\ &\leq \bar{c} \|\bar{a}\|_n \|v - u\|_{BV} \leq \varepsilon \|v - u\|_{BV}, \end{aligned}$$

namely  $\varphi : BV(\Omega) \rightarrow \mathbb{R}$  is Lipschitz continuous of constant  $\varepsilon$ .  $\square$

Now we consider the space  $X = BV(\Omega)$ , we denote by  $\|\cdot\|$  the norm of  $BV(\Omega)$  and by  $\|\cdot\|_0$  the norm of  $L^p(\Omega)$ .

**Theorem 4.3.** *The following facts hold:*

- (i) for every  $b \in \mathbb{R}$ ,  $f^b$  is complete with respect to  $\|\cdot\|_0$ ;
- (ii)  $f_1 : X_0 \rightarrow \mathbb{R}$  is continuous on  $X_1$ -bounded subsets;
- (iii)  $\xi > f(u) \implies |d_0 \mathcal{G}_f|(u, \xi) = 1$ ;
- (iv) if  $u \in \mathcal{D}(f)$ , then  $|df|(u) \leq c |d_0 f|(u)$ .

**Proof.** For  $(\Psi)$ ,  $(g_1)$  and the previous lemma, hypotheses of Theorem 3.1 are satisfied; therefore (ii), (iii) and (iv) follow by Theorem 3.1

(i) Let  $(u_h)$  be a sequence in  $BV(\Omega)$  convergent to  $u$  in  $L^p(\Omega)$  with  $f(u_h) \leq b$ . Let  $\varepsilon = \tilde{d}/2$  and let  $\varphi, \psi$  be as in the previous lemma. Since  $(\psi(u_h))$  is bounded and

$$f_0(u_h) + \varphi(u_h) - \varphi(0) \geq \frac{\tilde{d}}{2} \|u_h\| - \bar{c},$$

we have that  $(u_h)$  is bounded in  $BV(\Omega)$ , so that  $u \in BV(\Omega)$ . From (i) of Theorem 3.1 assertion follows.  $\square$

Let us now assume the following superlinearity condition on  $G$ :

- ( $g_2$ ) there exist  $q > 1$  and  $R > 0$  such that for a.e.  $x \in \Omega$  and every  $s \in \mathbb{R}$  with  $|s| \geq R$  we have

$$0 < qG(x, s) \leq sg(x, s).$$

From  $(g_1)$  and  $(g_2)$  it follows (see e.g. [13, Theorem 6.2]) that there exists  $a_0 \in L^1(\Omega)$  such that for a.e.  $x \in \Omega$  and every  $s \in \mathbb{R}$  we have

$$(4.3) \quad G(x, s) \geq (R^{-q} \min\{G(x, R), G(x, -R)\}) |s|^q - a_0(x),$$

$$(4.4) \quad qG(x, s) \leq sg(x, s) + a_0(x).$$

Let us also recall that hypothesis  $(\Psi)$  implies that  $\Psi$  is Lipschitz continuous of constant  $c$ , and therefore there exists  $M \in \mathbb{R}$  such that

$$(4.5) \quad (q + 1)\Psi(\xi) - \Psi(2\xi) \geq \frac{q - 1}{2}\Psi(\xi) - M,$$

$$(4.6) \quad (q + 1)\Psi^\infty(\xi) - \Psi^\infty(2\xi) \geq \frac{q - 1}{2}\Psi^\infty(\xi).$$

**Theorem 4.4.** *The following facts hold:*

(a) *for every  $u \in BV(\Omega) \setminus \{0\}$  we have*

$$\lim_{t \rightarrow +\infty} f(tu) = -\infty;$$

(b) *for every  $c \in \mathbb{R}$ ,  $f : X_0 \rightarrow \mathbb{R} \cup \{+\infty\}$  satisfies  $(PS)_c$ .*

**Proof.** (a) Let  $u \in BV(\Omega) \setminus \{0\}$ ; from (4.2) and (4.3) it follows that

$$f(tu) \leq \tilde{c} |t| \|u\| - R^{-q} |t|^q \int_{\Omega} \min\{G(x, R), G(x, -R)\} |u|^q dx + \int_{\Omega} a_0 dx,$$

hence

$$\lim_{t \rightarrow +\infty} f(tu) = -\infty.$$

(b) Let  $c \in \mathbb{R}$  and let  $(u_h)$  be a sequence in  $BV(\Omega)$  such that  $|d_0 f|(u_h) \rightarrow 0$  and  $f(u_h) \rightarrow c$ .

From (iv) of Theorem 4.3 it follows that  $|df|(u_h) \rightarrow 0$ . From [6, Theorem 2.11], there exist  $w_h \in (BV(\Omega))'$  such that  $\|w_h\|_{(BV(\Omega))'} \rightarrow 0$  and

$$\forall v \in X : f_0(v) \geq f_0(u_h) + \int_{\Omega} g(x, u_h)(v - u_h) dx + \langle w_h, v - u_h \rangle.$$



Choosing  $v_h = 2u_h$  we have, by (4.4),

$$f_0(2u_h) \geq f_0(u_h) + \int_{\Omega} u_h g(x, u_h) dx + \langle w_h, u_h \rangle - \int_{\Omega} a_0 dx.$$

By the definition of  $f$ , we obtain

$$qf(u_h) - \langle w_h, u_h \rangle + \int_{\Omega} a_0 dx \geq (q + 1)f_0(u_h) - f_0(2u_h).$$

Therefore, by (4.2), (4.5) and (4.6), it follows that

$$\begin{aligned} qf(u_h) + \|w_h\|_{(BV(\Omega))'} \|u_h\| + \int_{\Omega} a_0 dx &\geq \frac{(q - 1)}{2} f_0(u_h) - M\mathcal{L}^n(\Omega) \geq \\ &\geq \frac{q - 1}{2} \tilde{d} \|u_h\| - \frac{q - 1}{2} \tilde{c} - M\mathcal{L}^n(\Omega). \end{aligned}$$

Hence  $(u_h)$  is bounded in  $BV(\Omega)$  and the assertion follows from the compact embedding of  $BV(\Omega)$  in  $L^p(\Omega)$  (see [7]).  $\square$

Now we state the last hypothesis needed for the geometrical conditions of the mountain pass theorem:

( $\alpha$ ) there exist  $\alpha \in [1, \frac{n}{n-1}[$  and  $a_1 \in L^{\frac{n}{n+\alpha-n\alpha}}(\Omega)$  such that

$$\liminf_{\xi \rightarrow 0} \frac{\Psi(\xi) - \Psi(0)}{|\xi|^\alpha} > 0,$$

$$\lim_{s \rightarrow 0} \frac{G(x, s)}{|s|^\alpha} = 0,$$

$$|G(x, s)| \leq a_1(x) |s|^\alpha + b |s|^{\frac{n}{n-1}}$$

for a.e.  $x \in \Omega$  and every  $s \in \mathbb{R}$ .

**Lemma 4.5.** *Let  $(u_h) \subseteq BV(\Omega)$  with  $\|u_h\| = 1$  and  $\rho_h > 0$  with  $\rho_h \rightarrow 0$ .*

*Then it is*

$$\liminf_{h \rightarrow \infty} \frac{f_0(\rho_h u_h) - f_0(0)}{\rho_h^\alpha} > 0.$$

**Proof.** Without loss of generality, we can assume  $\Psi(0) = 0$ . Let  $\delta > 0$  be such that

$$|\xi| \leq \delta \Rightarrow \Psi(\xi) \geq \delta |\xi|^\alpha.$$

Since  $\Psi$  is convex, for every  $\xi \in \mathbb{R}^n$  with  $|\xi| > \delta$  we have

$$\Psi\left(\frac{\delta}{|\xi|}\xi\right) \leq \frac{\delta}{|\xi|}\Psi(\xi),$$

hence

$$\Psi(\xi) \geq \frac{|\xi|}{\delta}\Psi\left(\frac{\delta}{|\xi|}\xi\right) \geq \delta^\alpha |\xi|.$$

If we define  $\gamma : [0, +\infty[ \rightarrow \mathbb{R}$  by

$$\gamma(s) = \begin{cases} \frac{\delta}{\alpha}s^\alpha & \text{if } 0 \leq s \leq \delta \\ \delta^\alpha s - \frac{\alpha-1}{\alpha}\delta^{\alpha+1} & \text{if } s \geq \delta \end{cases}$$

we have that  $\gamma$  is convex and satisfies

$$\forall \xi \in \mathbb{R}^n : \Psi(\xi) \geq \gamma(|\xi|).$$

Taking into account Lemma 4.1, we deduce that

$$f_0(\rho_h u_h) \geq \mathcal{L}^n(\Omega) \gamma\left(\frac{\rho_h}{\mathcal{L}^n(\Omega)}\right)$$

and the assertion follows.  $\square$

**Lemma 4.6.** *Let  $(u_h) \subseteq BV(\Omega)$  with  $\|u_h\| = 1$  and  $\rho_h > 0$  with  $\rho_h \rightarrow 0$ .*

*Then it is*

$$\lim_{h \rightarrow \infty} \frac{G(x, \rho_h u_h)}{\rho_h^\alpha} = 0$$

*in  $L^1(\Omega)$ .*

**Proof.** Up to a subsequence, we can assume that  $u_h(x) \rightarrow u(x)$  for a.e.  $x \in \Omega$ .

Moreover,  $(u_h)$  is bounded in  $L^{\frac{n}{n-1}}(\Omega)$ .

From  $(\alpha)$  we deduce that

$$\lim_{h \rightarrow \infty} \frac{G(x, \rho_h u_h)}{\rho_h^\alpha} = 0$$

for a.e.  $x \in \Omega$  and that

$$\frac{|G(x, \rho_h u_h)|}{\rho_h^\alpha} \leq a_1(x) |u_h|^\alpha + b \rho_h^{\frac{n}{n-1} - \alpha} |u_h|^{\frac{n}{n-1}}.$$

Since the right hand side of the last inequality is strongly convergent in  $L^1(\Omega)$ , the assertion follows from the Lebesgue theorem.  $\square$

**Theorem 4.7.** *There exists  $r > 0$  such that*

$$\forall u \in BV(\Omega) : \|u\|_0 = r \implies f(u) \geq f(0) + r^{\alpha+1}.$$

*Proof.* By contradiction, let  $(u_h)$  be a sequence in  $BV(\Omega)$  such that  $\|u_h\|_0 = 1/h$  and

$$f(u_h) < f(0) + \frac{1}{h^{\alpha+1}}.$$

From (i) of Theorem 3.1, it follows that  $\|u_h\|$  is bounded; hence, from (ii) of Theorem 3.1 we deduce that

$$\lim_{h \rightarrow \infty} f_1(u_h) = 0.$$

Define  $\gamma : [0, +\infty[ \rightarrow \mathbb{R}$  as in Lemma 4.5. Since

$$\limsup_{h \rightarrow \infty} \mathcal{L}^n(\Omega) \gamma \left( \frac{\|u_h\|}{\mathcal{L}^n(\Omega)} \right) \leq \limsup_{h \rightarrow \infty} (f_0(u_h) - f_0(0)) \leq 0,$$

we have  $\|u_h\| \rightarrow 0$ .

Let  $\rho_h = \|u_h\|$ ,  $w_h = u_h / \|u_h\|$ ; applying Lemmas 4.5 and 4.6 to  $\rho_h$  and  $w_h$  we deduce that

$$\begin{aligned} \liminf_{h \rightarrow \infty} \frac{f_0(u_h) - f_0(0)}{\rho_h^\alpha} &> 0, \\ \lim_{h \rightarrow \infty} \frac{f_1(u_h)}{\rho_h^\alpha} &= - \lim_{h \rightarrow \infty} \frac{\int_\Omega G(x, u_h) dx}{\rho_h^\alpha} = 0. \end{aligned}$$

It follows  $\liminf_{h \rightarrow \infty} \frac{f(u_h) - f(0)}{\rho_h^\alpha} > 0$ , whence a contradiction.  $\square$

Since  $f : BV(\Omega) \rightarrow \mathbb{R}$  is the sum of a convex term and a term of class  $C^1$  (when  $BV(\Omega)$  is endowed with its natural norm), it is natural to say that  $u \in BV(\Omega)$  is a (generalized) critical point for  $f$  if

$$\forall v \in BV(\Omega) : f_0(v) \geq f_0(u) - \langle df_1(u), v - u \rangle.$$

**Remark 4.8.** Under mild assumptions of  $\Psi$ , it is shown in [2] that the above relation implies that  $u$  satisfies a suitable Euler equation.

We may now prove the main result of this section.

**Theorem 4.9.** *Assume that  $(\Psi)$ ,  $(g_1)$ ,  $(g_2)$  and  $(\alpha)$  hold. Then there exists  $u \in BV(\Omega) \setminus \{0\}$  such that  $u$  is a critical point for  $f$ .*

Proof. For Theorem 4.4 and Theorem 4.7 all the hypotheses of Theorem 3.2 are satisfied. Hence the assertion follows from Theorem 3.2.  $\square$

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*Marco Marzocchi*  
*Dipartimento di Matematica*  
*Università Cattolica*  
*Via Trieste, 17*  
*I 25121 Brescia*  
*Italy*  
e-mail: `mrz@bs.unicatt.it`

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