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QUADRATIC MEAN RADIUS OF A POLYNOMIAL IN $\mathbb{C}(Z)$

K. G. Ivanov, A. Sharma

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Dedicated to the memory of Prof. N. Obreshkoff

Abstract. A Schoenberg conjecture connecting quadratic mean radii of a polynomial and its derivative is verified for some kinds of polynomials, including fourth degree ones.

1. Introduction. Let $P_n(z) = z^n + a_2 z^{n-2} + \cdots + a_n$, $(n > 2)$ be a polynomial with real or complex coefficients and with $a_1 = 0$. If $P_n(z) = \prod_1^n (z - z_j)$, then $a_1 = 0$ implies that $z_1 + z_2 + \cdots + z_n = 0$. Following Schoenberg [1], we define the quadratic mean radius of P_n by

$$R(P_n) := \left(\frac{1}{n} \sum_1^n |z_j|^2 \right)^{1/2}. \tag{1.1}$$

Recently Schoenberg compared the quadratic mean radii of P_n and P_n' and stated the following

Conjecture. *The quadratic mean radii $R(P_n)$ and $R(P_n')$ satisfy the inequality*

$$R(P_n') \le \sqrt{\frac{n-2}{n-1}} R(P_n), \tag{1.2}$$

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with equality sign if and only if all the zeros z_j of $P_n(z)$ are on a straight line, as a partial case, all zeros z_j are real.

Schoenberg proved the conjecture when $n = 3$ and also for polynomials of the form

$$z^n + a_k z^{n-k} \tag{1.3a}$$

which he calls "binomial" polynomials.

Schoenberg's proof of the conjecture when $P_n(z)$ has three simple zeros is very elegant and instructive. The object of this note is two-fold: We first prove the conjecture when

$$P_n(z) = \prod_{j=1}^{3} (z - z_j)^{m_j}, \quad m_1 + m_2 + m_3 = n \tag{1.3}$$

and $\sum_1^3 m_j z_j = 0$.

The elegant method of Schoenberg's proof does not seem to extend to polynomials of degree > 3. So our second objective is to prove the conjecture for biquadratic polynomials and to verify it in some other cases.

2. Proof of Schoenberg's Conjecture when $P(z)$ is given by (1.3). If $P_n(z)$ is given by (1.3), and if $n = m_1 + m_2 + m_3$, let w_1, w_2 be the zeros of

$$\frac{P'(z)}{P(z)} = \sum_1^3 \frac{m_j}{z - z_j}. \tag{2.1}$$

It is easily seen that w_1, w_2 are the zeros of the quadratic polynomial

$$nw^2 - n(z_1 + z_2 + z_3)w + m_1 z_2 z_3 + m_2 z_1 z_3 + m_3 z_1 z_2 = 0. \tag{2.2}$$

From (2.2), we have

$$\begin{aligned} |w_1|^2 + |w_2|^2 &= \frac{1}{2}(|w_1 + w_2|^2 + |w_1 - w_2|^2) \\ &= \frac{1}{2}\{|z_1 + z_2 + z_3|^2 + |M(z_1, z_2, z_3)|\}, \end{aligned}$$

where

$$M(z_1, z_2, z_3) := (z_1 + z_2 + z_3)^2 - \frac{4}{n}(m_1 z_2 z_3 + m_2 z_1 z_3 + m_3 z_1 z_2).$$

Schoenberg's conjecture (1.2), in this case is equivalent to

$$F(z_1, z_2, z_3) \ge 0, \tag{2.3}$$

where

$$\begin{aligned} F(z_1, z_2, z_3) \ := \ & \left(1 - \frac{2m_1}{n}\right)|z_1|^2 + \left(1 - \frac{2m_2}{n}\right)|z_2|^2 + \left(1 - \frac{2m_3}{n}\right)|z_3|^2 \\ & - \frac{1}{2}\{|z_1 + z_2 + z_3|^2 + |M(z_1, z_2, z_3)|\}. \end{aligned} \tag{2.4}$$

Putting $z_3 = -\dfrac{m_1 z_1 + m_2 z_2}{m_3}$ and supposing without loss of generality that $m_3 \ge \max(m_1, m_2)$, then after some elementary simplification, we see that

$$\begin{aligned} F(z_1, z_2, z_3) = \ & \sum_{j=1}^{2}\left(1 - \frac{2m_j}{n}\right)|z_j|^2 + \left(1 - \frac{2m_3}{n}\right)\cdot\left|\frac{m_1 z_1}{m_3} + \frac{m_2 z_2}{m_2}\right|^2 \\ & - \frac{1}{2}\left|\sum_{j=1}^{2}\left(1 - \frac{m_j}{m_3}\right) z_j\right|^2 - \frac{1}{2}|A(z_1, z_2)|, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} A(z_1, z_2) \ := \ & \left\{\left(1 - \frac{m_1}{m_3}\right)^2 + \frac{4m_1 m_2}{n m_3}\right\} z_1^2 + \left\{\left(1 - \frac{m_2}{m_3}\right)^2 + \frac{4m_1 m_2}{n m_3}\right\} z_2^2 \\ & - 2\left\{1 - \frac{m_1 + m_2}{m_3} - \frac{m_1 m_2}{m_3^2} + \frac{4m_1 m_2}{n m_3}\right\} z_1 z_2. \end{aligned}$$

We shall need the following lemma which may be of independent interest.

Lemma 1. *If* $c_1, c_2 \in \mathbb{C}$, $\alpha \in \mathbb{R}$, $|\alpha| \le 1$, *then*

$$|c_1^2 + c_2^2 + 2\alpha c_1 c_2| \le |c_1|^2 + |c_2|^2 + 2\alpha \operatorname{Re} c_1 \overline{c}_2. \tag{2.6}$$

An equality in (2.6) holds iff $\arg c_1^2 = \arg c_2^2$.

Proof. Since $c_1^2 + c_2^2 + 2\alpha c_1 c_2 = (1 - |\alpha|)(c_1^2 + c_2^2) + |\alpha|(c_1 + c_2 \operatorname{sgn} \alpha)^2$, we have

$$\begin{aligned} |c_1^2 + c_2^2 + 2\alpha c_1 c_2| \ & \le \ (1 - |\alpha|)(|c_1|^2 + |c_2|^2) + |\alpha||c_1 + c_2 \operatorname{sgn} \alpha|^2 \\ & = \ |c_1|^2 + |c_2|^2 + 2\alpha \operatorname{Re} c_1 \overline{c}_2, \end{aligned}$$

which completes the proof of the lemma. □

We now set

$$(2.7)\quad \begin{gathered} c_1=\left\{\left(1-\frac{m_1}{m_3}\right)^2+\frac{4m_1m_2}{nm_3}\right\}^{1/2}z_1,\quad c_2=\left\{\left(1-\frac{m_2}{m_3}\right)^2+\frac{4m_1m_2}{nm_3}\right\}^{1/2}z_2 \\ \alpha=\frac{\left\{1-\frac{m_1+m_2}{m_3}-\frac{m_1m_2}{m_3^2}+\frac{4m_1m_2}{nm_3}\right\}z_1z_2}{c_1c_2}. \end{gathered}$$

Then $A(z_1,z_2)=c_1^2+c_2^2-2\alpha c_1c_2$. If we show that $|\alpha|\le 1$ (which we show later), then by the above lemma, we have

$$|A(z_1,z_2)|=|c_1|^2+|c_2|^2-2\alpha\ \mathrm{Re}\, c_1\overline{c}_2.$$

Then from (2.5), we see that

$$\begin{aligned} F(z_1,z_2,z_3)\ &\ge\ \left(1-\frac{2m_1}{n}\right)|z_1|^2+\left(1-\frac{2m_2}{n}\right)|z_2|^2 \\ &\quad+\left(1-\frac{2m_3}{n}\right)\left\{\frac{m_1^2}{m_3^2}|z_1|^2+\frac{m_2^2}{m_3^2}|z_2|^2+\frac{2m_1m_2}{m_3^2}\ \mathrm{Re}\, z_1\overline{z}_2\right\} \\ &\quad-\frac{1}{2}\left\{\left(1-\frac{m_1}{m_3}\right)^2|z_1|^2+\left(1-\frac{m_2}{m_3}\right)^2|z_2|^2\right. \\ &\quad\left.+2\left(1-\frac{m_1}{m_3}\right)\left(1-\frac{m_2}{m_3}\right)\mathrm{Re}\, z_1\overline{z}_3\right\} \\ &\quad-\frac{1}{2}\left[|c_1|^2+|c_2|^2-2\left\{1-\frac{m_1+m_2}{m_3}-\frac{m_1m_2}{m_3^2}+\frac{4m_1m_2}{nm_3}\right\}\mathrm{Re}\, z_1\overline{z}_2\right] \\ &=\ 0, \end{aligned}$$

since on using (2.7) for $|c_1|^2$ and $|c_2|^2$ in the above, we see that the coefficients of $|z_1|^2$, $|z_2|^2$ and $\mathrm{Re}\, z_1\overline{z}_2$ vanish, as is easy to verify.

Thus $F(z_1,z_2,z_3)\ge 0$, which proves the conjecture. The case of equality holds iff $\arg z_1^2=\arg z_2^2$, i.e. all three points z_1, z_2, z_3 lie on the line $\{z\in\mathbb{C}:\arg z^2=\arg z_1^2\}$.

It only remains to show that $|\alpha|\le 1$, where α is given by (2.7). In order to prove this, we set

$$a:=1-\frac{m_1}{m_3},\quad b:=1-\frac{m_2}{m_3},\quad c:=\frac{4m_1m_2}{nm_3},\quad d:=\frac{2m_1m_2}{m_3^2},$$

so that

$$\alpha=\frac{ab+c-d}{(a^2+c)^{1/2}(b^2+c)^{1/2}}.$$

From our supposition that $m_3 \geq \max(m_1, m_2)$, we have $a, b \geq 0$. Also

$$2c = \frac{8m_1m_2}{nm_3} > \frac{2m_1m_2}{m_3^2} = d,$$

if $3m_3 > m_1 + m_2$ which follows from the fact that $m_3 \geq \max(m_1, m_2)$. It follows that

$$(a^2 + c)(b^2 + c) - (ab + c - d)^2 = c(a - b)^2 + 2abd + d(2c - d) \geq 0.$$

This shows that $|\alpha| \leq 1$. $\square$

Remark 1. A simpler proof of the above can also be given on the lines of Schoenberg's proof using a theorem of Van der Berg [2].

Remark 2. If we suppose that

$$\frac{1}{n}\sum_{1}^{n} z_j = b \neq 0,$$

for a polynomial $P_n(z) = \prod_{1}^{n}(z - z_j)$, then Schoenberg's conjecture is equivalent to the inequality

$$F(z_1, \ldots, z_n) \geq 0, \tag{2.8}$$

where

$$F(z_1, \ldots, z_n) = \frac{n-2}{n}\sum_{1}^{n}|z_j|^2 + |b|^2 - \sum_{1}^{n-1}|w_k|^2, \tag{2.9}$$

where w_k's are the zeros of $P_n'(z)$ and an equality in (2.8) holds iff all points $z_1, \ldots, z_n$ lie on a straight line (through b).

Remark 3. Schoenberg's example (1.3a) can be easily extended to polynomials $P_n(z)$ of the form $z^{n-\ell k}(z^k - 1)^\ell$, ℓ, k positive integers, $n \geq \ell k$. If $k \geq 2$, $\sum_{1}^{n} z_j = 0$, but if $k = 1$, $\frac{1}{n}\sum_{1}^{n} z_j = \frac{\ell}{n}$. Indeed, we have $R(P_n) = \left(\frac{k\ell}{n}\right)^{1/2}$ and

$$R(P_n') = \left[\frac{1}{n-1}\left\{k\left(\frac{n-k\ell}{n}\right)^{2/k} + k(\ell - 1)\right\}\right]^{1/2},$$

since $P_n'=(z^k-1)^{\ell-1}z^{n-\ell k-1}[nz^k-(n-k\ell)]$. It is easy to see that $R(P_n') \le \sqrt{\frac{n-2}{n-1}}R(P_n)$ is equivalent to

$$\left(1-\frac{k\ell}{n}\right)^2 \le \left(1-\frac{2\ell}{n}\right)^k,$$

which is true if $k \ge 2$.

For $k=1$ we have to show from (2.9) that

$$\frac{n-2}{n}\sum_1^n |z_j|^2 + \frac{\ell^2}{n^2} - \sum_1^{n-1}|w_j|^2 \ge 0.$$

But in this case we have

$$\frac{n-2}{n}\ell + \frac{\ell^2}{n^2} - \left(\ell - 1 + \left(\frac{n-\ell}{n}\right)^2\right) = 0.$$

3. Case when $P_n(z) = (1+z)^n - a^n z^n$, $a \in \mathbb{C}$, $a^n \neq 1$. If $P_n(z) = (1+z)^n - a^n z^n$ and if $a = \rho e^{i\alpha}$, then

$$z_k = \left(\rho e^{i\left(\alpha+\frac{2k\pi}{n}\right)} - 1\right)^{-1}, \quad (k = 0, 1, \dots, n-1) \tag{3.1}$$

and

$$w_k = \left(a^{\frac{n}{n-1}} e^{\frac{2\pi i k}{n-1}} - 1\right)^{-1}, \quad (k = 0, 1, \dots, n-2). \tag{3.2}$$

Also $\frac{1}{n}\sum_0^{n-1} z_k = \frac{1}{a^n - 1}$.

From (2.9), we see that

$$\begin{aligned} F(z_1,\dots,z_n) \;\; = \;\; & \frac{n-2}{n}\sum_{k=0}^{n-1}\left|\rho e^{i\left(\alpha+\frac{2k\pi}{n}\right)} - 1\right|^{-2} + |\rho^n e^{in\alpha} - 1|^{-2} \\ & - \sum_{k=0}^{n-2}\left|\rho^{\frac{n}{n-1}} e^{i\left(\frac{n\alpha}{n-1}+\frac{2k\pi}{n-1}\right)} - 1\right|^{-2}. \end{aligned}$$

On simplifying the above, we have

$$
\begin{aligned}
F(z_1,\ldots,z_n) &= \frac{n-2}{n}\sum_{k=0}^{n-1}\left(\rho^2+1-2\rho\cos\left(\alpha+\frac{2\pi k}{n}\right)\right)^{-1} \\
&\quad +(\rho^{2n}+1-2\rho^n\cos n\alpha)^{-1} \\
&\quad -\sum_{k=0}^{n-2}\left(\rho^{\frac{2n}{n-1}}+1-2\rho^{\frac{n}{n-1}}\cos\left(\frac{n\alpha}{n-1}+\frac{2k\pi}{n-1}\right)\right)^{-1} \\
&= \frac{n-2}{n}S_1+S_2+S_3.
\end{aligned}
\tag{3.3}
$$

Since

$$
\begin{aligned}
S_1 &= \frac{1}{2\rho}\sum_{k=0}^{n-1}\frac{1}{t-\cos\left(\alpha+\frac{2k\pi}{n}\right)},\quad t=\frac{\rho^2+1}{2\rho}, \\
&= \frac{1}{2\rho}\cdot\frac{Q'(t)}{Q(t)},
\end{aligned}
$$

where

$$
Q(t):=\prod_{k=0}^{n-1}\left(t-\cos\left(\alpha+\frac{2k\pi}{n}\right)\right),
$$

and since

$$
\begin{aligned}
Q(t) &= \frac{1}{(2\rho)^n}\prod_{k=0}^{n-1}\left(\rho^2+1-2\rho\cos\left(\alpha+\frac{2k\pi}{n}\right)\right) \\
&= \frac{1}{(2\rho)^n}(\rho^{2n}+1-2\rho^n\cos n\alpha),
\end{aligned}
$$

we have

$$
\begin{aligned}
S_1 &= \frac{1}{2\rho}\left\{\frac{d\rho}{dt}\cdot\frac{d}{d\rho}\left[\frac{\rho^{2n}+1-2\rho^n\cos n\alpha}{(2\rho)^n}\right]\right\}/Q(t) \\
&= \frac{n(\rho^{2n}-1)}{(\rho^2-1)(\rho^{2n}+1-2\rho^n\cos n\alpha)}.
\end{aligned}
\tag{3.4}
$$

Similarly, we can prove that

$$
S_3=-\sum_{k=0}^{n-2}|w_k|^2=\frac{-(n-1)(\rho^{2n}-1)}{\left(\rho^{\frac{2n}{n-1}}-1\right)(\rho^{2n}+1-2\rho^n\cos n\alpha)}.
\tag{3.5}
$$

From (3.3), (3.4) and (3.5), we see that

$$F(z_1, z_2, \dots, z_n)(\rho^{2n} + 1 - 2\rho^n \cos n\alpha)$$

$$= (n-2)\frac{(\rho^{2n}-1)}{\rho^2 - 1} + 1 - \frac{(n-1)(\rho^{2n}-1)}{\rho^{\frac{2n}{n-1}} - 1}$$

$$= (n-2)f(n) + f(1) - (n-1)f(n-1),$$

where $f(t) = (\rho^{2n} - 1)/(\rho^{2n/t} - 1)$ and f is strictly convex for $t > 0$. Thus

$$F(z_1, \dots, z_n) > 0$$

for $n \geq 3$. This completes the verification of Schoenberg's conjecture for the polynomials $(1+z)^n - a^n z^n$, $(a^n \neq 1)$.

4. Biquadratic polynomials. It is easy to see that any biquadratic polynomial $P(z)$ with zeros $\{z_j\}_1^4$ such that $\sum_1^4 z_j = 0$, can be written as the product of two quadratic polynomials. Indeed we have

$$P(z) = (z^2 - 2\alpha z + \beta)(z^2 + 2\alpha z + \gamma), \quad \alpha, \beta, \gamma \in \mathbb{C}, \tag{4.1}$$

so that

$$\frac{1}{4}P'(z) = z^3 - \left(2\alpha^2 - \frac{\beta+\gamma}{2}\right) z + \frac{\alpha(\beta - \gamma)}{2}. \tag{4.2}$$

It follows from (4.1) that

$$\frac{1}{2}\sum_{j=1}^{4} |z_j|^2 = 2|\alpha^2| + |\alpha^2 - \beta| + |\alpha^2 - \gamma|. \tag{4.3}$$

If w_j $(j = 1, 2, 3)$ denote the zeros of $P'(z)$, and if we set

$$w_j = u\omega^j + v\omega^{2j}, \quad j = 1, 2, 3 \quad \text{with} \quad \omega^3 = 1, \ \omega \neq 1$$

then $\sum_{j=1}^{3} |w_j|^2 = 3(|u|^2 + |v|^2)$. Since w_j are the zeros of $P'(z)$, we see from (4.2) that

$$\begin{cases} u^3 + v^3 = -\dfrac{\alpha}{2}(\beta - \gamma) \\ uv = \dfrac{1}{3}\left(2\alpha^2 - \dfrac{\beta + \gamma}{2}\right). \end{cases} \tag{4.4}$$

Schoenberg's conjecture in this case reduces to

$$\frac{1}{2}\sum_{j=1}^{4}|z_j|^2 - \sum_{k=1}^{3}|w_k|^2 \geq 0. \tag{4.5}$$

If $\alpha = 0$ then (4.5) reduces to the triangle inequality $|\beta| + |\gamma| - |\beta+\gamma| \geq 0$. The case $\alpha \neq 0$ is equivalent to $\alpha = 1$. Then the left side of (4.5) becomes $F(u,v)$ on using (4.4) and (4.3), where

$$F(u,v) \;:=\; 2 + |u^3+v^3-1+3uv| + |u^3+v^3+1-3uv| - 3|u|^2 - 3|v|^2.$$

Since $u^3+v^3+1-3uv = (u+v+1)(u^2+v^2+1-uv-u-v)$, we may put $u+v=\zeta$, $u-v=W$ so that

$$4F(u,v) = 8 + |\zeta+1|.|(\zeta-2)^2+3W^2| + |\zeta-1|.|(\zeta+2)^2+3W^2| - 6|\zeta|^2 - 6|W|^2.$$

Putting $3W^2 = w$, we have

$$G(\zeta,w) := 4F(u,v) = 8+|\zeta-1|.|(\zeta+2)^2+w|+|\zeta+1|.|(\zeta-2)^2+w|-6|\zeta|^2-2|w|. \tag{4.6}$$

If $\zeta, w \in \mathbb{C}$, let $\zeta = \xi + i\eta$, $\xi,\eta \in \mathbb{R}$, $w = re^{i\varphi}$, $r \geq 0$ and $0 \leq \varphi \leq 2\pi$. Then we set

$$p := |\zeta-1| = \sqrt{(\xi-1)^2+\eta^2}, \quad q := |\zeta+1| = \sqrt{(\xi+1)^2+\eta^2}.$$

If we set $(\zeta+2)^2 := a+ib$, $(\zeta-2)^2 := c+id$, then

$$\begin{aligned} a &= (\xi+2)^2-\eta^2, \quad b = 2\eta(\xi+2), \\ c &= (\xi-2)^2-\eta^2, \quad d = 2\eta(\xi-2). \end{aligned} \tag{4.6a}$$

We can now see that if we set

$$A := |(\zeta+2)^2+w|, \quad B := |(\zeta-2)^2+w|,$$

then

$$\begin{aligned} A &= |a + r\cos\varphi + i(b + r\sin\varphi)| \\ B &= |c + r\cos\varphi + i(d + r\sin\varphi)| \end{aligned}$$

and $A^2 = a_1^2 + a_2^2$, $B^2 = b_1^2 + b_2^2$, where

$$\begin{aligned} a_1 &:= -a\sin\varphi + b\cos\varphi, \qquad a_2 = r + a\cos\varphi + b\sin\varphi, \\ b_1 &:= -c\sin\varphi + d\cos\varphi, \qquad b_2 = r + c\cos\varphi + d\sin\varphi. \end{aligned}$$

With the above substitution, we obtain

$$G(\zeta, w) = 8 + pA + qB - 6(\xi^2 + \eta^2) - 2r. \tag{4.7}$$

Now Schoenberg's conjecture is $G(\zeta, w) \geq 0$. In order to prove it we first consider the case of real ζ, i.e. $\eta = 0$. It will turn out that this is the only possibility for $G(\zeta, w) = 0$. Using

$$(1+\zeta)(2-\zeta)^2 + (1-\zeta)(2+\zeta)^2 = 8 - 6\zeta^2 \quad \text{and} \quad (1+\zeta) + (1-\zeta) = 2$$

we get from (4.6)

$$\begin{aligned} G(\zeta, w) &= \operatorname{sgn}(1+\zeta)|1+\zeta|(2-\zeta)^2 + \operatorname{sgn}(1-\zeta)|1-\zeta|(2+\zeta)^2 \\ &\quad + |1+\zeta|.|(2-\zeta)^2 + w| + |1-\zeta|.|(2+\zeta)^2 + w| \\ &\quad - \operatorname{sgn}(1+\zeta)|1+\zeta|.|w| - \operatorname{sgn}(1-\zeta)|1-\zeta|.|w| \\ &= |1+\zeta|\{|(2-\zeta)^2 + w| + \operatorname{sgn}(1+\zeta)[(2-\zeta)^2 - |w|]\} \\ &\quad + |1-\zeta|\{|(2+\zeta)^2 + w| + \operatorname{sgn}(1-\zeta)[(2+\zeta)^2 - |w|]\} \geq 0. \end{aligned}$$

The only cases of equality $G(\zeta, w) = 0$ are

$$\begin{array}{lll} |\zeta| < 1 \quad \text{and} \quad w \leq -(2+|\zeta|)^2 & & \text{or} \\ |\zeta| = 1 \quad \text{and} \quad w \leq -1 & & \text{or} \\ |\zeta| > 1 \quad \text{and} \quad -(2+|\zeta|)^2 \leq w \leq -(2-|\zeta|)^2. & & \end{array}$$

The zeros of P are $1 \pm \sqrt{1-\beta}$ and $-1 \pm \sqrt{1-\gamma}$ with $1-\beta = -(1-\zeta)((2+\zeta)^2 + w)/4$, $1-\gamma = -(1+\zeta)((2-\zeta)^2 + w)/4$. Therefore the equality $G(\zeta, w) = 0$ implies only real zeros for P.

In the general case we fix $\zeta \in \mathbb{C}$ and we want to find $\inf_{w \in \mathbb{C}} G(\zeta, w)$. The infimum can occur only at points w for which:

i) G is not differentiable with respect to w (i.e. $r = 0$ or $A = 0$ or $B = 0$);

ii) $r = \infty$,

iii) $\dfrac{\partial G}{\partial \varphi} = \dfrac{\partial G}{\partial r} = 0$.

i) If $r = 0$, then from the identity

$$(\zeta - 1)(\zeta + 2)^2 - (\zeta + 1)(\zeta - 2)^2 = 6\zeta^2 - 8,$$

we see that

$$\begin{aligned} G(\zeta, 0) &= 8 - 6|\zeta|^2 + |(\zeta - 1)(\zeta + 2)^2| + |(\zeta + 1)(\zeta - 2)^2| \\ &\geq 8 - 6|\zeta|^2 + |6\zeta^2 - 8| \geq 0. \end{aligned}$$

Moreover $G(\zeta,0)=0$ iff $\zeta=\pm 2$, which implies real zeros for P. If $A=0$ or $B=0$ then $w=-(\zeta\pm 2)^2$ and

$$\begin{aligned} G(\zeta,-(\zeta\pm 2)^2) &= 8+8|\zeta(\zeta\pm 1)|-6|\zeta|^2-2|\zeta\pm 2|^2 \\ &= 4+8|\zeta(\zeta\pm 1)|-4|\zeta|^2-4|\zeta\pm 1|^2 \\ &= 4-4(|\zeta|-|\zeta\pm 1|)^2\geq 0. \end{aligned}$$

Moreover $G(\zeta,-(\zeta\pm 2)^2)=0$ only on subsets of the real line which implies only real zeros for P.

ii) If $\eta\neq 0$, then $|\zeta-1|+|\zeta+1|>2$ and so

$$\lim_{|w|\to\infty} G(\zeta,w)=+\infty.$$

iii) In the sequel, we assume $\eta\neq 0$ and $r,A,B>0$ and solve the system of equations

$$\frac{\partial G}{\partial\varphi}=0 \quad\text{and}\quad \frac{\partial G}{\partial r}=0. \tag{4.8}$$

From (4.7), we see that (4.8) is equivalent to

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial G}{\partial\varphi}=\left\{\dfrac{p}{A}a_1+\dfrac{q}{B}b_1\right\}=0 \\ \dfrac{\partial G}{\partial r}=p\dfrac{a_2}{A}+q\dfrac{b_2}{B}-2=0. \end{cases} \tag{4.9}$$

We shall prove

Lemma 2. *If $A^2=a_1^2+a_2^2$, $B^2=b_1^2+b_2^2$ and*

$$p=\sqrt{(\xi-1)^2+\eta^2},\quad q=\sqrt{(\xi+1)^2+\eta^2},\quad A,B>0,\quad \eta\neq 0,$$

then the system of equations (4.9) *is equivalent to* (4.10), (4.11), (4.12) *and* (4.13) *(where $\varepsilon=\pm 1$):*

$$\eta a_2=\varepsilon(1-\xi)a_1, \tag{4.10}$$

$$\eta b_2=-\varepsilon(1+\xi)b_1, \tag{4.11}$$

$$\operatorname{sgn} a_2 \ \ \operatorname{sgn}(1-\xi)\geq 0, \tag{4.12}$$

$$\operatorname{sgn} b_2 \ \ \operatorname{sgn}(1+\xi)\geq 0. \tag{4.13}$$

Proof. We shall first show that (4.10) – (4.13) imply (4.9). Indeed from (4.10) and (4.11), we get

$$(4.10)' \qquad \eta^2 a_2^2 = (1-\xi)^2 a_1^2$$

$$(4.11)' \qquad \eta^2 b_2^2 = (1+\xi)^2 b_1^2$$

so that on adding $(1-\xi)^2 a_2^2$ (or $(1+\xi)^2 b_2^2$) to $(4.10)'$ (or to $(4.11)'$), we obtain

$$p^2 a_2^2 = (1-\xi)^2 A^2 \quad \text{and} \quad q^2 b_2^2 = (1+\xi)^2 B^2,$$

and in view of (4.12) and (4.13), we have

$$(4.14) \qquad pa_2 = (1-\xi)A \quad \text{and} \quad qb_2 = (1+\xi)B.$$

From (4.14) we easily get the second condition in (4.9).

From (4.14), in view of (4.10) and (4.11), we also have

$$\begin{aligned} pa_1 A^{-1} &= a_1 a_2^{-1}(1-\xi) = \eta/\varepsilon, \\ qb_1 B^{-1} &= b_1 b_2^{-1}(1+\xi) = -\eta/\varepsilon \end{aligned}$$

which on adding yield the first equation in (4.9).

We shall now show that (4.9) imply (4.10) – (4.13). Indeed from (4.9), we have

$$pa_1 A^{-1} = -qb_1 B^{-1}, \quad pa_2 A^{-1} = 2 - qb_2 B^{-1}$$

$$(\text{or } qb_2 B^{-1} = 2 - pa_2 A^{-1}).$$

Adding the square of the first equation to the squares of the second and third, we obtain

$$p^2 = 4 - 4qb_2 B^{-1} + q^2, \quad q^2 = 4 - 4pa_2 A^{-1} + p^2$$

so that

$$(4.15) \qquad \begin{cases} qb_2 B^{-1} = 1 + \dfrac{q^2 - p^2}{4} = 1 + \xi \\[2ex] pa_2 A^{-1} = 1 - \dfrac{q^2 - p^2}{4} = 1 - \xi \end{cases}$$

where $\xi = (q^2 - p^2)/4$ by assumptions. These equations imply (4.12), (4.13).

Squaring (4.15) yields

$$\begin{aligned} q^2 b_2^2 &= ((1+\xi)^2+\eta^2)b_2^2 = (1+\xi)^2(b_1^2+b_2^2),\\ p^2 a_2^2 &= ((1-\xi)^2+\eta^2)a_2^2 = (1-\xi)^2(a_1^2+a_2^2), \end{aligned}$$

whence we have

$$\eta^2 b_2^2 = (1+\xi)^2 b_1^2, \quad \eta^2 a_2^2 = (1-\xi)^2 a_1^2.$$

Equivalently, we get

$$\tag{4.16} \begin{cases} \eta b_2 = (1+\xi) b_1 \varepsilon_1, \\ \eta a_2 = (1-\xi) a_1 \varepsilon_2 \end{cases}$$

where $\varepsilon_j = \pm 1$, $(j = 1, 2)$. In order to find the relation between ε_1 and ε_2, we put a_1 and b_1 from (4.16) in the first equation in (4.9). Then we have on using (4.15),

$$\begin{aligned} p a_1 A^{-1} + q b_1 B^{-1} &= p A^{-1} \eta a_2 (1-\xi)^{-1} \varepsilon_2 + q B^{-1} \eta b_2 (1+\xi)^{-1} \varepsilon_1 \\ &= \eta \varepsilon_2 + \eta \varepsilon_1 = 0 \end{aligned}$$

so that $\varepsilon_1 = -\varepsilon$, $\varepsilon_2 = \varepsilon$ and we get (4.12) and (4.13) from (4.9).

This completes the proof of the lemma. □

5. Proof of the conjecture for biquadratics. We shall show that

$$\tag{5.0} \inf_{w \in \mathbb{C}} G(\zeta, w) \geq 0,$$

where $G(\zeta, w)$ is given by (4.7). The conditions $\dfrac{\partial G}{\partial \varphi} = 0$, $r \neq 0$ and $\dfrac{\partial G}{\partial r} = 0$, after using Lemma 2, yield (with $\varepsilon = \pm 1$):

$$\tag{5.1} \eta(r + a\cos\varphi + b\sin\varphi) = \varepsilon(1-\xi)(-a\sin\varphi + b\cos\varphi),$$

$$\tag{5.2} \eta(r + c\cos\varphi + d\sin\varphi) = -\varepsilon(1+\xi)(-c\sin\varphi + d\cos\varphi),$$

$$\tag{5.3} \begin{cases} \operatorname{sgn}(r + a\cos\varphi + b\sin\varphi) \cdot \operatorname{sgn}(1-\xi) \geq 0, \\ \operatorname{sgn}(r + c\cos\varphi + d\sin\varphi) \cdot \operatorname{sgn}(1+\xi) \geq 0. \end{cases}$$

From the definitions of a, b, c, d in (4.6a), we have

$$\begin{cases} a - c = 8\xi, & a + c = 2(\xi^2 - \eta^2 + 4), \\ b - d = 8\eta, & b + d = 4\xi\eta. \end{cases} \tag{5.4}$$

Subtracting (5.2) from (5.1) and simplifying on using (5.4), we obtain after elementary calculations

$$[3\xi^2\varepsilon + (\varepsilon - 4)\eta^2 - 4\varepsilon]\sin\varphi = (4 + 2\varepsilon)\xi\eta\cos\varphi. \tag{5.5}$$

We now consider two cases: (i) when $\varepsilon = 1$ and (ii) when $\varepsilon = -1$.

(i) *Case when* $\varepsilon = 1$.
In this case (5.5) becomes

$$(3\xi^2 - 3\eta^2 - 4)\sin\varphi = 6\xi\eta\cos\varphi. \tag{5.6}$$

If we set $\Phi^2 := (3\xi^2 - 3\eta^2 - 4)^2 + (6\xi\eta)^2$ and $\sigma = \pm 1$, then $\Phi > 0$ because $\eta \neq 0$. From (5.6), we have

$$\sin\varphi = \frac{\sigma}{\Phi} 6\xi\eta, \quad \cos\varphi = \frac{\sigma(3\xi^2 - 3\eta^2 - 4)}{\Phi}. \tag{5.7}$$

From (5.1) with $\varepsilon = 1$, using the values of a, b, $\sin\varphi$, $\cos\varphi$ from (4.6a), (5.7) we get

$$r = \frac{\sigma}{\Phi}[-3(\xi^2 + \eta^2)^2 - 4\eta^2 + 12\xi^2]. \tag{5.8}$$

Since $\operatorname{sgn} r > 0$, σ is determined from (5.8). Indeed, we have

$$\sigma = \begin{cases} 1 & \text{if } \eta^2 < \dfrac{3}{4} \text{ and } 2 - \eta^2 - 2\sqrt{1 - \dfrac{4}{3}\eta^2} < \xi^2 < 2 - \eta^2 + 2\sqrt{1 - \dfrac{4}{3}\eta^2} \\ -1 & \text{otherwise.} \end{cases} \tag{5.9}$$

(Note that $r > 0$ excludes the case $-3(\xi^2 + \eta^2)^2 - 4\eta^2 + 12\xi^2 = 0$.) The values of r, $\sin\varphi$, $\cos\varphi$ given by (5.7) and (5.8) must satisfy (5.2) and (5.3).

Using (5.1), (5.7) and (5.8) some calculation yields

$$r + a\cos\varphi + b\sin\varphi = 4(1 - \xi)(-3\xi^2 - 3\eta^2 - 8\xi - 4)\frac{\sigma}{\Phi}, \tag{5.10}$$

which together with (5.3) implies that

$$\sigma(-3\xi^2 - 3\eta^2 - 8\xi - 4) \geq 0. \tag{5.11}$$

Similarly from (5.2), (5.7) and (5.8), we have

$$r + c\cos\varphi + d\sin\varphi = 4(1+\xi)(-3\xi^2 - 3\eta^2 + 8\xi - 4)\frac{\sigma}{\Phi} \tag{5.12}$$

and together with (5.3), we obtain

$$\sigma(-3\xi^2 - 3\eta^2 + 8\xi - 4) \geq 0. \tag{5.13}$$

Now (5.11) and (5.13) are satisfied simultaneously if and only if

$$\sigma = -1 \quad \text{and} \quad 3\xi^2 + 3\eta^2 + 4 \geq 8|\xi|.$$

Equivalently, (5.11) and (5.13) are valid at the same time if and only if

$$\sigma = -1 \quad \text{and} \quad \begin{cases} \text{either} \ \ \eta^2 \geq \dfrac{4}{9} \\ \text{or} \ \ \eta^2 < \dfrac{4}{9} \quad \text{and} \quad \left||\xi| - \dfrac{4}{3}\right| \geq \dfrac{1}{3}\sqrt{4 - 9\eta^2}. \end{cases}$$

From this and from (5.7), (5.8) and (5.9), we see that the solution to (5.1) – (5.3) is

$$\begin{cases} r = \dfrac{1}{\Phi}\{3(\xi^2+\eta^2)^2 + 4\eta^2 - 12\xi^2\} \\ \sin\varphi = -\dfrac{6\xi\eta}{\Phi}, \quad \cos\varphi = \dfrac{1}{\Phi}(-3\xi^2 + 3\eta^2 + 4) \end{cases} \tag{5.14}$$

for the following two cases:
(a) $\eta^2 > \frac{3}{4}$.
(b) $0 < \eta^2 \leq \frac{3}{4}$ and $\xi^2 \in \left[0, 2 - \eta^2 - 2\sqrt{1 - \frac{4}{3}\eta^2}\right) \bigcup \left(2 - \eta^2 + 2\sqrt{1 - \frac{4}{3}\eta^2}, \infty\right)$.

For these cases, on using (5.1) and (5.10), we obtain

$$\begin{aligned} A^2 &= (r + a\cos\varphi + b\sin\varphi)^2\left(1 + \frac{\eta^2}{(1-\xi)^2}\right) \\ &= 16(3\xi^2 + 3\eta^3 + 8\xi + 4)^2\{(1-\xi)^2 + \eta^2\}/\Phi^2, \end{aligned}$$

and similarly from (5.2) and (5.12) we get

$$B^2 = 16(3\xi^2 + 3\eta^2 - 8\xi + 4)^2\{(1+\xi)^2 + \eta^2\}/\Phi^2.$$

Hence we see on using (4.7), (5.11) and (5.13) that

$$\begin{aligned}\inf_{w\in\mathbb{C}} G(\zeta,w) &= 8 + 4\{(1-\xi)^2+\eta^2\}\frac{(3\xi^2+3\eta^2+8\xi+4)}{\Phi} \\ &+ 4\{(1+\xi)^2+\eta^2\}\frac{(3\xi^2+3\eta^2-8\xi+4)}{\Phi} - 6(\xi^2+\eta^2) \\ &- 2\{3(\xi^2+\eta^2)^2+4\eta^2-12\xi^2\}/\Phi.\end{aligned}$$

Elementary calculation yields

$$\inf_{w\in\mathbb{C}} G(\zeta,w) = 2\left\{4-3\xi^2-3\eta^2+\sqrt{(3\xi^2+3\eta^2-4)^2+48\eta^2}\right\} > 0$$

because $\eta \neq 0$.

(ii) *Case when* $\varepsilon = -1$.

In this case (5.5) becomes

$$(3\xi^2+5\eta^2-4)\sin\varphi = -2\xi\eta\cos\varphi.$$

If we set $\Psi^2 = (3\xi^2+5\eta^2-4)^2+(2\xi\eta)^2$, then $\Psi \geq 0$ and $\Psi = 0$ if and only if

$$\xi = 0 \quad \text{and} \quad \eta^2 = \frac{4}{5}.$$

We shall first consider the case when $\Psi > 0$. Then we have (for $\sigma = \pm 1$)

$$\sin\varphi = -\frac{\sigma}{\Psi}2\xi\eta, \quad \cos\varphi = \frac{\sigma}{\Psi}(-4+3\xi^2+5\eta^2). \tag{5.15}$$

From (5.1), (4.6a) and (5.15), we have

$$r = \frac{\sigma}{\Psi}\{5(\xi^2+\eta^2)^2-28\xi^2-44\eta^2+32\}. \tag{5.16}$$

As in (i), we see that

$$\sigma = \begin{cases} -1 & \text{if } 0<\eta^2<\frac{4}{5} \text{ and } \left|\xi^2+\eta^2-\frac{14}{5}\right| < \frac{2}{5}\sqrt{9+20\eta^2} \\ & \text{or } \frac{4}{5}\leq\eta^2<8 \text{ and } 0\leq\xi^2<\frac{14}{5}-\eta^2+\frac{2}{5}\sqrt{9+20\eta^2}; \\ 1 & \text{otherwise.}\end{cases} \tag{5.17}$$

(Note that $r > 0$ excludes the case $5(\xi^2+\eta^2)^2-28\xi^2-44\eta^2+32 = 0$.) Using (5.15) and (5.16), we get from (5.1) with $\varepsilon = -1$

$$r + a\cos\varphi + b\sin\varphi = -(1-\xi)\frac{\sigma}{\Psi}\{8(\xi^2+\eta^2)\xi+20(\xi^2+\eta^2)-16\} \tag{5.18}$$

which together with (5.3) yields

$$\sigma\{2\xi(\xi^2+\eta^2)+5(\xi^2+\eta^2)-4\} \le 0. \tag{5.19}$$

Similarly from (5.2), (5.15) and (5.16), we obtain

$$r + c\cos\varphi + d\sin\varphi = -(1+\xi)\frac{\sigma}{\Psi}\{-8\xi(\xi^2+\eta^2)+20(\xi^2+\eta^2)-16\} \tag{5.20}$$

which in view of (5.3), yields

$$\sigma\{-2\xi(\xi^2+\eta^2)+5(\xi^2+\eta^2)-4\} \le 0. \tag{5.21}$$

Now (5.19) and (5.21) are simultaneously satisfied if and only if

$$|5(\xi^2+\eta^2)-4| \ge 2|\xi|(\xi^2+\eta^2) \quad \text{and} \quad \sigma(4-5(\xi^2+\eta^2)) \ge 0. \tag{5.22}$$

From (5.1) and (5.18), we have

$$\begin{aligned} A^2 &= (r+a\cos\varphi+b\sin\varphi)^2+(-a\sin\varphi+b\cos\varphi)^2 \\ &= 16\{(2\xi+5)(\xi^2+\eta^2)-4\}^2\{(1-\xi)^2+\eta^2\}\Psi^{-2} \end{aligned}$$

and similarly from (5.2) and (5.20) we get

$$\begin{aligned} B^2 &= (r+c\cos\varphi+d\sin\varphi)^2+(-c\sin\varphi+d\cos\varphi)^2 \\ &= 16\{(-2\xi+5)(\xi^2+\eta^2)-4\}^2\{(1+\xi)^2+\eta^2\}\Psi^{-2}. \end{aligned}$$

Hence on using (4.7), (5.16), (5.19) and (5.21) we get

$$\begin{aligned} \inf_{w\in\mathbb{C}} G(\zeta,w) &= 8+4\{(1-\xi)^2+\eta^2\}\{4-(2\xi+5)(\xi^2+\eta^2)\}\sigma\Psi^{-1} \\ &\quad + 4\{(1+\xi)^2+\eta^2\}\{4-(-2\xi+5)(\xi^2+\eta^2)\}\sigma\Psi^{-1} \\ &\quad - 6(\xi^2+\eta^2)-2\{5(\xi^2+\eta^2)^2-28\xi^2-44\eta^2+32\}\sigma\Psi^{-1} \\ &= 2\{4-3\xi^2-3\eta^2-\sigma\Psi\}. \end{aligned}$$

Since

$$\Psi^2-(3\xi^2+3\eta^2-4)^2 = 16\eta^2(\xi^2+\eta^2-1), \tag{5.23}$$

we see that if $\xi^2+\eta^2>1$, then (5.22) implies $\sigma=-1$ and $\Psi>|3\xi^2+3\eta^2-4|$, so that $\inf G(\zeta,w)>0$. If $\xi^2+\eta^2<1$, then from (5.23) $\Psi<4-3\xi^2-3\eta^2$ and again we get

$\inf G(\zeta, w) > 0$. Finally if $\xi^2 + \eta^2 = 1$ then (5.22) implies $\sigma = -1$, (5.23) gives $\Psi = 1$ and hence $\inf G(\zeta, w) = 4$.

In the case $\Psi = 0$ we get by continuity from the case $\Psi > 0$ that $\inf G(\zeta, w) = \frac{16}{5}$. Thus, in the case $\varepsilon = -1$ we always have $\inf G(\zeta, w) > 0$. $\square$

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K. G. Ivanov
Bulgarian Academy of Science
Institute of Mathematics and Informatics
Acad. G. Bonchev str., bl. 8
1113 Sofia
Bulgaria

A. Sharma
Department of Mathematical Sciences
University of Alberta
Edmonton, Alberta
Canada T6G 2G1

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