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# PROBLEMS AND THEOREMS IN THE THEORY OF MULTIPLIER SEQUENCES

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Abstract. The purpose of this paper is (1) to highlight some recent and heretofore unpublished results in the theory of multiplier sequences and (2) to survey some open problems in this area of research. For the sake of clarity of exposition, we have grouped the problems in three subsections, although several of the problems are interrelated. For the reader's convenience, we have included the pertinent definitions, cited references and related results, and in several instances, elucidated the problems by examples.

**1. Introduction.** In this century, the Bulgarian mathematicians have played a prominent role in several areas of mathematics and, in particular, in the theory of distribution of zeros of polynomials and entire functions. The theory of multiplier sequences commenced with the work of Laguerre [13] and was solidified in the seminal work of Pólya and Schur [20]. Subsequently, this theory gained prominence at the hands of such renowned mathematicians as Iliev, Schoenberg and Obreshkov, just to mention a few names.

In the theory of distribution of zeros of polynomials, the following open problem is of central interest. Let $D$ be a subset of the complex plane. Characterize the linear transformations $T$, taking polynomials into polynomials such that if $p$ is a polynomial

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(either arbitrary or restricted to a certain class of polynomials), then the polynomial $T[p]$ has *at least* as many zeros in $D$ as $p$ has zeros in $D$. There is an analogous problem for transcendental entire functions. (For related questions and results see, for example, [4], [7], [8, Ch. 2, Ch. 4], [9], [10, Ch. 7], [15, Ch. 3–5] and [17, Ch. 1–2].) In the classical setting ($D = \mathbb{R}$) the problem (solved by Pólya and Schur [20]) is to characterize all real sequences $T = \{\gamma_k\}_{k=0}^{\infty}, \gamma_k \in \mathbb{R}$, such that if a polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ has only real zeros, then the polynomial

$$T[p(x)] = T\left[\sum_{k=0}^{n} a_k x^k\right] := \sum_{k=0}^{n} \gamma_k a_k x^k, \tag{1.1}$$

also has only real zeros (see (2.2) and (2.3) below). In this paper we shall consider the following more general problem. Characterize all real sequences $T = \{\gamma_k\}_{k=0}^{\infty}, \gamma_k \in \mathbb{R}$, such that if $p(x)$ is *any real polynomial*, then

$$Z_c(T[p(x)]) \leq Z_c(p(x)), \tag{1.2}$$

where $Z_c(p(x))$ denotes the number of nonreal zeros of $p(x)$, counting multiplicities.

**2. Multiplier sequences and totally positive sequences.** We commence here with some pertinent definitions involving the Laguerre–Pólya class, multiplier sequences and totally positive sequences.

**Definition 2.1.** *A real entire function* $\phi(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ *is said to be in the* Laguerre-Pólya class, *written* $\phi(x) \in \mathcal{L}\text{-}\mathcal{P}$, *if* $\phi(x)$ *can be expressed in the form*

$$\phi(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{x_k}\right) e^{-\frac{x}{x_k}}, \tag{2.1}$$

*where* $c, \beta, x_k \in \mathbb{R}$, $c \neq 0$, $\alpha \geq 0$, *$n$ is a nonnegative integer and* $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$. *If* $-\infty \leq a < b \leq \infty$ *and if* $\phi(x) \in \mathcal{L}\text{-}\mathcal{P}$ *has all its zeros in* $(a, b)$ (*or* $[a, b]$), *then we will use the notation* $\phi \in \mathcal{L}\text{-}\mathcal{P}(a, b)$ (*or* $\phi \in \mathcal{L}\text{-}\mathcal{P}[a, b]$). *If* $\gamma_k \geq 0$ (*or* $(-1)^k \gamma_k \geq 0$ *or* $-\gamma_k \geq 0$) *for all* $k = 0, 1, 2 \ldots$, *then* $\phi \in \mathcal{L}\text{-}\mathcal{P}$ *is said to be of* type I in the Laguerre-Pólya class, *and we will write* $\phi \in \mathcal{L}\text{-}\mathcal{P}I$. *We will also write* $\phi \in \mathcal{L}\text{-}\mathcal{P}I^+$, *if* $\phi \in \mathcal{L}\text{-}\mathcal{P}I$ *and if* $\gamma_k \geq 0$ *for all* $k = 0, 1, 2 \ldots$.

In order to clarify the terminology above, we remark that if $\phi \in \mathcal{L}\text{-}\mathcal{P}I$, then $\phi \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0]$ or $\phi \in \mathcal{L}\text{-}\mathcal{P}[0, \infty)$, but that an entire function in $\mathcal{L}\text{-}\mathcal{P}(-\infty, 0]$ need not belong to $\mathcal{L}\text{-}\mathcal{P}I$. Indeed, if $\phi(x) = \frac{1}{\Gamma(x)}$, where $\Gamma(x)$ denotes the gamma function, then $\phi(x) \in \mathcal{L}\text{-}\mathcal{P}(-\infty, 0]$, but $\phi(x) \notin \mathcal{L}\text{-}\mathcal{P}I$. This can be seen, for example, by noting that functions in $\mathcal{L}$-$\mathcal{P}I$ are of exponential type, whereas $\dfrac{1}{\Gamma(x)}$ is an entire function of order one of maximal type (see [3, p. 8] or [14, Chapter 2] for the definition of the "type" of an entire function). Also, it is easy to see that $\mathcal{L}$-$\mathcal{P}I^+ = \mathcal{L}$-$\mathcal{P}I(-\infty, 0]$.

**Definition 2.2.** *A sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ of real numbers is called a multiplier sequence if, whenever the real polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ has only real zeros, the polynomial $T[p(x)] = \sum_{k=0}^{n} \gamma_k a_k x^k$ also has only real zeros.*

The following are well-known characterizations of multiplier sequences (cf. [20], [19, pp. 100–124] or [17, pp. 29–47]). A sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence if and only if

$$\phi(x) = T[e^x] := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L}\text{-}\mathcal{P}I. \tag{2.2}$$

Moreover, the algebraic characterization of multiplier sequences asserts that a sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence if and only if

$$g_n(x) := \sum_{j=0}^{n} \binom{n}{j} \gamma_j x^j \in \mathcal{L}\text{-}\mathcal{P}I \quad \text{for all } n = 1, 2, 3 \ldots . \tag{2.3}$$

**Definition 2.3.** *A real sequence $\{\alpha_k\}_{k=0}^{\infty}$ is a totally positive sequence (TP-sequence) if $\alpha_0 = 1$, $\sum\limits_{k=0}^{\infty} \alpha_k x^k$ is an entire function and all the minors of all orders (i.e. determinants of all square submatrices) of the lower triangular matrix*

$$A := (\alpha_{i-j}) := \begin{pmatrix} \alpha_0 & 0 & 0 & 0 & 0 & \ldots \\ \alpha_1 & \alpha_0 & 0 & 0 & 0 & \ldots \\ \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 & \ldots \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \ldots \\ & & \ldots & & & \end{pmatrix} \tag{2.4}$$

*are nonnegative.*

The intimate connection between multiplier sequences and TP-sequences can be inferred from the following theorem.

**Theorem 2.4** ([1, p. 306])**.** *Let $\phi(x) = \sum\limits_{k=0}^{\infty} \alpha_k x^k$, $\alpha_0 = 1$ $\alpha_k \in \mathbb{R}$, be an entire function. Then $\{\alpha_k\}$ is a TP-sequence if and only if $\phi(x) \in \mathcal{L}$-$\mathcal{P}I^+$.*

Preliminaries aside, we are now in a position to state our first problem.

*Problem 1.* Let $\{\gamma_k\}_{k=0}^{\infty}$, $\gamma_0 = 1$, $\gamma_k > 0$, be a multiplier sequence. Then is it true that

$$(2.5)\qquad (-1)^{\frac{n(n-1)}{2}} \begin{vmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_n \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{n+1} \\ & & \cdots & \\ \gamma_n & \gamma_{n+1} & \cdots & \gamma_{2n} \end{vmatrix} \geq 0 \qquad (n = 1, 2, 3, \ldots)?$$

This problem is a special case of a more general problem that was raised by S. Karlin [10, p. 389] (see also [5, p. 258]). If $\lim_{n\to\infty} \sqrt[n]{\gamma_n} = 0$, so that $\phi(x) = \sum_{n=0}^{\infty} \gamma_n x^n$ is an entire function, then by Theorem 2.4, $\{\gamma_k\}_{k=0}^{\infty}$ is a TP-sequence and hence, by rearranging the columns of the matrix $(\gamma_{i-j})$ $(i, j = 1, 2, 3, \ldots)$, we see that Problem 1 has an affirmative answer. Additional evidence for an affirmative answer to Problem 1 is contained in the following theorem.

**Theorem 2.5** ([5, Theorem 2.13]). *If $\{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k > 0$, is a multiplier sequence, then*

$$(2.6)\qquad \begin{vmatrix} \gamma_k & \gamma_{k+1} & \gamma_{k+2} \\ \gamma_{k+1} & \gamma_{k+2} & \gamma_{k+3} \\ \gamma_{k+2} & \gamma_{k+3} & \gamma_{k+4} \end{vmatrix} \leq 0 \qquad \textit{for}\ \ k = 0, 1, 2, \ldots$$

We next turn to the general problem cited in the introduction. In [7] the authors established the following generalization of the Gauss-Lucas theorem.

**Theorem 2.6** ([7, Corollary 3.1]). *Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence with $0 \leq \gamma_0 \leq \gamma_1 \leq \ldots$. Let $K$ be a closed, unbounded convex set which contains the origin and all the zeros of the entire function $f(z) = \sum_{k=0}^{\infty} a_k z^k$, where $f(z)$ is of genus zero. Then the zeros of the entire function $\Gamma[f(z)] = \sum_{k=0}^{\infty} \gamma_k a_k z^k$ also lie in $K$.*

*Problem 2.* Does Theorem 2.6 remain valid if we omit the restrictive assumption that $f(z)$ is of genus zero?

For related questions and problems, we refer the reader to the fundamental paper of Korevaar [11] and the references cited therein.

In particular, one should note that the multiplier sequences in Theorem 2.6 are assumed to be increasing. Such sequences possess a very strong convexity property as the following proposition shows.

**Proposition 2.7** ([7, Proposition 4.2]). *Let $\phi(x) = \sum_{k=0}^{\infty} \gamma_k x^k/k!$, $0 \leq \gamma_0 \leq \gamma_1 \leq \ldots$, be an entire function in $\mathcal{L}$-$\mathcal{P}^+$. Then*

$$(2.7)\qquad \Delta^n \gamma_p \geq 0 \qquad n, p = 0, 1, 2, \ldots,$$

*where* $\Delta^0\gamma_p = \gamma_p$ *and* $\Delta^n\gamma_p = \sum_{j=0}^{n} \binom{n}{j}(-1)^{n-j}\gamma_{p+j}$, *for* $n, p = 0, 1, 2, \ldots$.

*Problem 3.* Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a sequence for which $\Delta^n\gamma_p \geq 0$ for $n, p = 0, 1, 2, \ldots$. Under what additional assumptions on this sequence will it follow that $\Gamma$ is a multiplier sequence? For a discussion of this problem see [7, p. 428].

**3. Complex zero decreasing sequences and $\boldsymbol{\lambda}$-sequences.** Once again, we first recall some relevant definitions and terminology.

**Definition 3.1.** *We say that a sequence* $\{\gamma_k\}_{k=0}^{\infty}$ *is a complex zero decreasing sequence (CZDS), if*

$$Z_c\left(\sum_{k=0}^{n} \gamma_k a_k x^k\right) \leq Z_c\left(\sum_{k=0}^{n} a_k x^k\right), \tag{3.1}$$

*for any real polynomial* $\sum_{k=0}^{n} a_k x^k$. *(The acronym CZDS will also be used in the plural.)*

Now it follows from (3.1) that any complex zero decreasing sequence is also a multiplier sequence. If $T = \{\gamma_k\}_{k=0}^{\infty}$ is a sequence of *nonzero* real numbers, then inequality (3.1) is equivalent to the statement that for any polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$, $T[p]$ has at least as many real zeros as $p$ has. There are, however, CZDS which have zero terms (cf. [6, Section 3]) and consequently it may happen that $\deg T[p] < \deg p$. When counting the real zeros of $p$, the number generally increases with the application of $T$, but may in fact decrease due to a decrease in the degree of the polynomial. For this reason, we count nonreal zeros rather than real ones. The existence of a *nontrivial* CZDS is a consequence of a theorem of Laguerre (see [17, Satz 3.2] or [6, Theorem 1.4]) and extended by Pólya (see Pólya [18] or [19, pp. 314-321]).

**Definition 3.2.** *A sequence of nonzero real numbers,* $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$, *is called a* $\lambda$*-sequence, if*

$$\Lambda[p(x)] = \Lambda\left[\sum_{k=0}^{n} a_k x^k\right] := \sum_{k=0}^{n} \lambda_k a_k x^k > 0 \text{ for all } x \in \mathbb{R}, \tag{3.2}$$

*whenever* $p(x) = \sum_{k=0}^{n} a_k x^k > 0$ *for all* $x \in \mathbb{R}$.

**Remarks.** (1) We remark that if $\Lambda$ is a sequence of nonzero real numbers and if $\Lambda[e^{-x}]$ is an entire function, then a *necessary* condition for $\Lambda$ to be a $\lambda$-sequence, is that $\Lambda[e^{-x}] \geq 0$ for all real $x$. (Indeed, if $\Lambda[e^{-x}] < 0$ for $x = x_0$, then continuity

considerations show that there is a positive integer $n$ such that $\Lambda\left[(1-\frac{x}{2n})^{2n}+\frac{1}{n}\right]<0$ for $x=x_0$.) This is a useful condition for showing that certain sequences are not $\lambda$-sequences (cf. [6, §2]).
(2) In [8, Ch. 4] (see also [12]) it was pointed out by Iliev that $\lambda$-sequences are precisely the positive definite sequences. (There are several known characterizations of positive definite sequences (see, for example, [16, Ch. 8] and [22, Ch. 3]).

The importance of $\lambda$-sequences in the investigation of CZDS stems from the fact that a *necessary condition* for a sequence $T=\{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k>0$, to be a CZDS is that the sequence of reciprocals $\Lambda=\{1/\gamma_k\}_{k=0}^{\infty}$ be a $\lambda$-sequence. Thus, for example, if $\gamma_k=\phi(k)$, $k=0,1,2,\ldots$, for $\phi\in\mathcal{L}\text{-}\mathcal{P}(-\infty,0)$, then $\{1/\gamma_k\}_{k=0}^{\infty}$ is a $\lambda$-sequence. On the other hand, there are multiplier sequences whose reciprocals are not $\lambda$-sequences. For example, in [6, Example 1.8] we have demonstrated that the sequence $T:=\{1+k+k^2\}_{k=0}^{\infty}$ is a multiplier sequence, but that the sequence of reciprocals, $\left\{\frac{1}{1+k+k^2}\right\}_{k=0}^{\infty}$ is not a $\lambda$-sequence.

In light of the foregoing discussion, the following natural problem arises.

*Problem 4.* Characterize those multiplier sequences $\Gamma=\{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k>0$, for which the sequences of reciprocals $\{1/\gamma_k\}_{k=0}^{\infty}$ are $\lambda$-sequences.

A similar problem can also be formulated in terms of the TP-sequences discussed in Section 2. Indeed, we note that if $\{\gamma_k\}_{k=0}^{\infty}$, $\gamma_0=1$, $\gamma_k>0$, is a multiplier sequence, then the sequence $\{\gamma_k/k!\}_{k=0}^{\infty}$ is a TP-sequence. For TP-sequences we have the following result.

**Theorem 3.3.** *The reciprocal of a TP-sequence need not be a $\lambda$-sequence. In particular, the multiplier sequence $\{(k-\frac{1}{2})^2\}_{k=0}^{\infty}$, arising from $(x+\frac{1}{2})^2e^x=\sum_{k=0}^{\infty}\frac{(k-\frac{1}{2})^2}{k!}x^k$, when divided by $k!$, fails to yield a CZDS.*

Proof. It is well known (see, for example, Iliev [8, Chap. 4]) that if a sequence $\{\mu_k\}_{k=0}^{\infty}$ is a $\lambda$-sequence, then the Hankel determinants $\det(\mu_{i-j})\geq 0$, $i,j=1,2,\ldots,n$, $n\in\mathbb{Z}^+$, are all nonnegative. Now set $\mu_k=\dfrac{k!}{(k-\frac{1}{2})^2}$. With $n=7$, one obtains a negative determinant, approximantly $-1.1357\times 10^{14}$. (The smaller order determinants are all positive.) $\square$

The next problem (related to (1.2)) was raised by Laguerre [13] in 1898 (cf. S. Karlin [10, p. 382]).

*Problem 5.* Characterize the multiplier sequences which are CZDS.

Recently Bakan and Golub [2] and the authors [6] have obtained some results which solve Problem 5 in certain special cases. As a sample result we quote here the following

**Theorem 3.4** ([6, Theorem 2.13]). *Let $h(x)$ be a real polynomial. The sequence $T = \{h(k)\}_{k=0}^{\infty}$ is a complex zero decreasing sequence (CZDS) if and only if either*

(1) *$h(0) \neq 0$ and all the zeros of $h$ are real and negative, or*

(2) *$h(0) = 0$ and the polynomial $h(x)$ has the form*

$$h(x) = x(x-1)(x-2)\cdots(x-m+1)\prod_{i=1}^{p}(x-b_i), \tag{3.3}$$

*where $m$ is a positive integer and $b_i < m$ for each $i = 1, \ldots, p$.*

It was shown in [6] that a quadratic polynomial gives rise to a CZDS $\{k^2+ak+b\}_{k=0}^{\infty}$ if and only if its zeros are real and negative. Indeed, this was extended to all polynomials via an induction argument. The essence of the argument for a quadratic polynomial was that if it induces a multiplier sequence and the zeros are not real and negative, then the reciprocal sequence is not a $\lambda$-sequence. This raised the question of whether a multiplier sequence whose reciprocal is a $\lambda$-sequence must always be a CZDS. In view of the techniques used in [6], it is surprising that this question is settled (in the negative) by cubic polynomials.

**Proposition 3.5.** *Let $p(x) = (x+c)((x+a)^2+b^2)$ with $a, b, c > 0$. Then*

(1) *$\{p(k)\}_{k=0}^{\infty}$ is not a CZDS.*

(2) *$\{p(k)\}_{k=0}^{\infty}$ is a multiplier sequence if and only if*

$$\begin{aligned}\Delta = &-4b^2c^4 + 4ac^4 + c^4 + 16ab^2c^3 - 16b^2c^3 - 16a^2c^3 + 12ac^3 + 4c^3 - 8b^4c^2\\ &-24a^2b^2c^2 + 24ab^2c^2 - 34b^2c^2 + 24a^3c^2 - 6a^2c^2 + 36ac^2 + 10c^2\\ &+16ab^4c + 48b^4c + 16a^3b^2c - 76ab^2c - 48b^2c - 16a^4c - 28a^3c + 24a^2c + 52ac\\ &+12c - 4b^6 - 8a^2b^4 - 12ab^4 - 3b^4 - 4a^4b^2 - 8a^3b^2 + 2a^2b^2 + 12ab^2\\ &+6b^2 + 4a^5 + 21a^4 + 44a^3 + 46a^2 + 24a + 5\end{aligned}$$

*is positive.*

(3) *If we write $a = c + \epsilon$, then*

$$\begin{aligned}\Delta &= 4\epsilon^5 + (4c+21)\epsilon^4 + (56c+44)\epsilon^3 + (36c^2+156c+46)\epsilon^2\\ &\quad +(216c^2+144c+24)\epsilon + 108c^3 + 108c^2 + 36c + 5 + O(b^2)\end{aligned}$$

*is positive for small $b$.*

(4) *The sequence* $\left\{\dfrac{1}{p(k)}\right\}_{k=0}^{\infty}$ *is a* $\lambda$*-sequence if and only if* $a \geq c$*. In this case,* $\left\{\dfrac{1}{p(k+1)}\right\}_{k=0}^{\infty}$ *is also a* $\lambda$*-sequence.*

Due to limitations of space, we omit the proof of Proposition 3.5 which will appear elsewhere.

In [6, Lemma 5.3] the authors have shown that if $\gamma_k = 1/k!$, $k = 0, 1, 2, \ldots$, then for each fixed $t > 0$, the sequence $\{g_k(t)\}_{k=0}^{\infty}$ is a CZDS, where $g_k(t) = \sum_{j=0}^{k} \binom{k}{j}\gamma_j t^j$.

*Problem 6.* Find a characterization for the multiplier sequences $\{\gamma_k\}_{k=0}^{\infty}$ for which the sequence $\{g_k(t)\}_{k=0}^{\infty}$ is a CZDS for each fixed $t > 0$.

**4. Interpolation and rapidly decreasing sequences.** Our next problem is intimately connected with Problem 5.

*Problem 7.* Characterize the multiplier sequences $\{\gamma_k\}_{k=0}^{\infty}$ which can be interpolated by functions in $\mathcal{L}$-$\mathcal{P}(-\infty, 0)$; that is, those sequences for which there exists a $\phi \in \mathcal{L}$-$\mathcal{P}(-\infty, 0)$ such that $\phi(k) = \gamma_k$ for each $k = 0, 1, 2, \ldots$.

In an attempt to attack this problem, the authors used Schoenberg's celebrated theorem [21, p. 354] on the representation of the reciprocal of a function in the Laguerre-Pólya class, to prove the following theorem.

**Theorem 4.1.** *Suppose that* $\phi(x) \in \mathcal{L}$-$\mathcal{P}I$*,* $\phi(x) > 0$ *if* $x > 0$*. Let* $p(x)$ *be a real polynomial with no nonreal zeros in the left half-plane* $\Re z < 0$*. Suppose that* $p(0)\phi(0) = 1$ *and set* $h(x) = p(x)\phi(x)$*. Then* $T = \{h(k)\}_{k=0}^{\infty}$ *is a CZDS if and only if* $p(x)$ *has only real negative zeros.*

The corresponding question when the nonreal zeros of $p(x)$ are not restricted to the right half-plane appears to be open and we state this as

*Problem 8.* Let $h(x) = p(x)\phi(x)$, where $\phi \in \mathcal{L}$-$\mathcal{P}I^+$ and $p(x) \in \mathbb{R}[x]$. Is it true that $T = \{h(k)\}_{k=0}^{\infty}$ is a CZDS if and only if $p(x)$ has only real negative zeros?

The following more general problem is also of interest.

*Problem 9.* Let $\phi(x)$ be a real entire function of exponential type. Suppose that all the zeros of $\phi(x)$ lie in the strip $|\Im z| \leq M$ for some positive bound $M$. If $\{\phi(k)\}_{k=0}^{\infty}$ is a CZDS, when does it follow that $\phi(x) \in \mathcal{L}$-$\mathcal{P}$?

In [6, Section 4] the authors investigated a special class of multiplier sequences which are rapidly decreasing in the following sense.

**Definition 4.2.** *A sequence* $\{\gamma_k\}_{k=0}^{\infty}$ *with* $\gamma_k \geq 0$ *and* $\gamma_0 = 1$ *is* rapidly decreasing *if* $\gamma_k^2 \geq \alpha^2 \gamma_{k-1}\gamma_{k+1}$ *for all* $k$*, where* $\alpha \geq \max(2, (1+\sqrt{1+\gamma_1})/\sqrt{2})$ *(see* [6]*).*

In order to provide some background information about the next problem, we note that rapidly decreasing sequences are multiplier sequences ([6, Theorem 4.3 and Corollary 4.4]). Furthermore, in general, rapidly decreasing sequences cannot be interpolated by any function in $\mathcal{L}$-$\mathcal{P}(-\infty, 0)$ [6, p. 438].

**Example 4.3.** Set $\gamma_k = \dfrac{1}{(k+1)^{(k+1)^{k+1}}}$. It is rapidly decreasing and Boas [22, p. 140] has shown that if $\mu_0 \geq 1$, $\mu_n \geq (n\mu_{n-1})^n$, then $\mu_n = \int_0^\infty t^n\, d\mu(t)$ with $\mu'(t) \geq 0$. It follows that $\{\mu_k\}_{k=0}^\infty$, $\mu_k = \dfrac{1}{\gamma_k}$, is a positive definite sequence and hence that $\{1/\gamma_k\}_{k=0}^\infty$, $\dfrac{1}{\gamma_k} = (k+1)^{(k+1)^{k+1}}$, is a $\lambda$-sequence. Indeed, if $\mu_k = \int_0^\infty t^k\, d\mu(t)$ and $P(x) = \sum\limits_{k=0}^{n} a_k x^= k \geq 0$ for all $x$, then $\Lambda[P(x)] = \int_0^\infty P(xt)\, d\mu(t) \geq 0$ for all $x$.

**Example 4.4.** $\{e^{k^2}\}$ is a $\lambda$-sequence since it is the reciprocal of a CZDS, hence $\{e^{e^{k^2}}\}$ is a $\lambda$-sequence [20, Part VII, no. 36].

The foregoing discussion leads to the following problem.

*Problem 10.* Let $\{\gamma_k\}_{k=0}^\infty$, $\gamma_k > 0$, be a rapidly decreasing sequence. Is $\{1/\gamma_k\}_{k=0}^\infty$ a $\lambda$-sequence? Is $\{1/\gamma_k\}_{k=0}^\infty$ a CZDS?

For results related to Problem 10 we refer to [6, §4]. We also remark that we have recently shown that most rapidly decreasing sequences of the form $\{a^{-k^m}\}_{k=0}^\infty$ are in fact $\lambda$-sequences.

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