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SOME THEORETICAL RESULTS ON THE PROGENY OF A BISEXUAL GALTON-WATSON BRANCHING PROCESS

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Abstract. A Superadditive Bisexual Galton-Watson Branching Process is considered and the total number of mating units, females and males, until the n-th generation, are studied. In particular some results about the stochastic monotony, probability generating functions and moments are obtained. Finally, the limit behaviour of those variables suitably normed is investigated.

1. Introduction. Introduced by Daley [3], the Bisexual Galton-Watson Branching Process (BGWBP), is a two-type branching model with f_n females and m_n males in the n-th generation, $n = 1, 2, \ldots$, which form $Z_n = L(f_n, m_n)$ mating units. These mating units reproduce independently according to the same offspring distribution for each generation. The mating function $L : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is monotonic non-decreasing in each argument, integer-valued for integer-valued arguments and such that $L(0, 0) = 0$. Then, considering $Z_0 = N \geq 1$,

$$(f_{n+1}, m_{n+1}) = \sum_{i=1}^{Z_n} (f_{ni}, m_{ni}) \quad n = 0, 1, \ldots$$

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with the empty sum defined to be $(0,0)$ and where f_{ni} and m_{ni} are, respectively, the number of females and males produced by the i-th mating unit in the n-th generation, being (f_{ni}, m_{ni}), $n = 0,1,\ldots,$ $i = 1,2,\ldots,$ independent and identically distributed non-negative integer-valued bivariate random variables. It is easy to verify that the sequence $\{Z_n : n = 0,1,\ldots\}$ is a Markov chain, stochastically monotone in the sense of Daley [4], with the non-negative integers as state space and with stationary one-step transition probabilities given by

$$Pr\,(Z_{n+1} = k | Z_n = j) = Pr\left(L\left(\sum_{i=1}^{j}(f_{ni}, m_{ni})\right) = k\right).$$

A BGWBP is said to be superadditive if for all positive integer n, the mating function verifies

$$(1.1)\quad L\left(\sum_{i=1}^{n}(x_i, y_i)\right) \geq \sum_{i=1}^{n} L\,(x_i, y_i) \text{ for all } x_i, y_i \text{ in } \mathbb{R}^+,\ i = 1,\ldots,n,\ n \geq 2$$

The problem of the extinction has been studied by Daley [3], Hull [7], [8], Bruss [2] and Daley et al. [5]. The main result is based on the concept of *mean growth rate* defined for all $j = 1,2,\ldots,$ in the form

$$r_j = j^{-1}E\,[Z_n\,|Z_{n-1} = j]\,.$$

Theorem (Daley et al. [5]). *For a superadditive BGWBP the mean growth rates satisfy*

$$r = \lim_{j\to\infty} r_j = \sup_{j>0} r_j$$

and

$$(1.2)\qquad q_j = 1 \ \textit{ for all } j \quad \textit{if and only if} \quad r \leq 1$$

where q_j is the probability of extinction when the process starts with j mating units.

On the other hand, the limit behaviour of the process has been investigated by Bagley [1] and recently, by González and Molina [6].

In this paper, we shall consider a superadditive BGWBP and shall assume the classical condition $Pr\,(Z_n \to 0) + Pr\,(Z_n \to \infty) = 1$ holds. We define the random variables $Y_n = \sum_{i=0}^{n} Z_i$, $n = 0,1,\ldots;$ $F_n = \sum_{i=1}^{n} f_i$ and $M_n = \sum_{i=1}^{n} m_i$, $n = 1,2,\ldots,$ which represent the total number of mating units, females and males until the n-th generation, respectively.

In section 2, some results about the stochastic monotony of the bisexual process are obtained. In section 3, relations between the probability generating functions of these variables are deduced and bounds for their expected values are obtained. Section 4 is devoted to research the asymptotic behaviour of Y_n, F_n and M_n suitably normed.

2. Stochastic monotony.

Lemma 2.1. *Let be* $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ *in* $\mathbb{R}^n$ *such that* $\sum_{i=1}^{k} x_i \le \sum_{i=1}^{k} y_i$, $k = 1, \ldots, n$. *Let be* $(u_1, \ldots, u_n)$ *in* $\mathbb{R}^n$ *such that* $u_1 \ge u_2 \ge \ldots \ge u_n \ge 0$. *Then*

$$\sum_{i=1}^{n} u_i x_i \le \sum_{i=1}^{n} u_i y_i. \tag{2.1}$$

P r o o f. Let $t_i = \sum_{j=1}^{i} x_j$, $s_i = \sum_{j=1}^{i} y_j$, $i = 1, \ldots, n$. Then $t_i \le s_i$, $i = 1, \ldots, n$. It is easy to obtain that (2.1) is equivalent to the inequality

$$\sum_{i=1}^{n-1} (u_i - u_{i+1})\, t_i + u_n t_n \le \sum_{i=1}^{n-1} (u_i - u_{i+1})\, s_i + u_n s_n$$

and this inequality holds because $u_i - u_{i+1} \ge 0$, $i = 1, \ldots, n-1$ and $u_n \ge 0$. □

Lemma 2.2. *Let* $\{X_n\}_n$ *and* $\{Y_n\}_n$ *be two independent sequences of non-negative random variables independent and identically distributed, both with the same common distribution. Let be* X *and* Y *integer-valued and non-negative random variables independent of* $\{X_n\}_n$ *and* $\{Y_n\}_n$*, respectively. Then, if* $X \prec Y$[1]*, it is verified that*

$$\sum_{i=1}^{X} X_i \prec \sum_{i=1}^{Y} Y_i.$$

P r o o f. It is clear for all y that $Pr\left(\sum_{i=1}^{x} X_i \le y\right)$ is non-increasing on x. Then, by Lemma 2.1,

$$Pr\left(\sum_{i=1}^{X} X_i \le y\right) = \sum_{x} Pr\left(\sum_{i=1}^{X} X_i \le y \middle| X = x\right) Pr\,(X = x)$$

[1] A random variable X is said stochastically smaller than other random variable Y, (written $X \prec Y$), if for all $u \in \mathbb{R}$, $Pr\,(Y \le u) \le Pr\,(X \le u)$.

$$
\begin{aligned}
&= \sum_x Pr\left(\sum_{i=1}^{x} X_i \le y\right) Pr\,(X=x) \\
&\ge \sum_x Pr\left(\sum_{i=1}^{x} X_i \le y\right) Pr\,(Y=x) \\
&= \sum_x Pr\left(\sum_{i=1}^{x} Y_i \le y\right) Pr\,(Y=x) = Pr\left(\sum_{i=1}^{Y} Y_i \le y\right)
\end{aligned}
$$

Theorem 2.3. *In a superadditive BGWBP, for all positive integer k, we have*

$$\sum_{i=1}^{Z_k} Z_n^{(i)} \prec Z_{k+n}\,, \quad n = 0, 1, \ldots$$

where $\{Z_n^{(i)}: n = 0, 1, \ldots\}$, $i = 1, \ldots, Z_k$, are independent versions of $\{Z_n: n = 0, 1, \ldots\}$ with the same parameters (i.e. the same mating function, offspring distribution and $Z_0^{(i)} = 1$).

Proof. Let k be a fixed positive integer. Let $X_n = Z_{k+n}$ and $Y_n = \sum_{i=1}^{Z_k} Z_n^{(i)}$, $n = 0, 1, \ldots$ We have to prove that

$$Pr\,(X_n \le y) \le Pr\,(Y_n \le y) \;\; for\ all\ y\,,\ n = 0, 1, \ldots$$

This inequality will be obtained by induction on n, but previously we need to establish:

i) $Pr\,(X_{n+1} \le y | X_n = x)$ is non-increasing on x (for all y).

ii) $Pr\,(X_{n+1} \le y | X_n = x) \le Pr\,(Y_{n+1} \le y | Y_n = x)$ for all x and y.

The first is clear because $\{X_n\}_n$ is stochastically monotone in the sense of Daley [4]. To prove the second, suppose that $Z_k = z_k$. Then

$$
\begin{aligned}
Pr\,(Y_{n+1} \le y | Y_n = x) &= Pr\left(\sum_{i=1}^{z_k} Z_{n+1}^{(i)} \le y \,\middle|\, \sum_{i=1}^{z_k} Z_n^{(i)} = x\right) \\
&\ge \inf_{x_1+\ldots+x_{z_k}=x} Pr\left(\sum_{j=1}^{z_k} L\left(\sum_{i=1}^{x_j}\left(f_{ni}^{(j)}, m_{ni}^{(j)}\right)\right) \le y\right) \\
&\ge \inf_{x_1+\ldots+x_{z_k}=x} Pr\left(L\left(\sum_{j=1}^{z_k}\sum_{i=1}^{x_j}\left(f_{ni}^{(j)}, m_{ni}^{(j)}\right)\right) \le y\right)
\end{aligned}
$$

$$= \; Pr\left(L\left(\sum_{i=1}^{x}(f_{k+n,i}, m_{k+n,i})\right) \leq y\right)$$
$$= \; Pr\left(Z_{k+n+1} \leq y | Z_{k+n} = x\right) = Pr\left(X_{n+1} \leq y | X_n = x\right).$$

Obviously $Pr\left(X_0 \leq y\right) \leq Pr\left(Y_0 \leq y\right)$ for all y and suppose that $Pr\left(X_n \leq y\right) \leq Pr\left(Y_n \leq y\right)$. Then

$$\begin{aligned} Pr\left(X_{n+1} \leq y\right) &= \sum_{x=0}^{\infty} Pr\left(X_{n+1} \leq y | X_n = x\right) Pr\left(X_n = x\right) \\ &\leq \sum_{x=0}^{\infty} Pr\left(X_{n+1} \leq y | X_n = x\right) Pr\left(Y_n = x\right) \\ &\leq \sum_{x=0}^{\infty} Pr\left(Y_{n+1} \leq y | Y_n = x\right) Pr\left(Y_n = x\right) = Pr\left(Y_{n+1} \leq y\right). \end{aligned}$$

The first inequality is obtained considering i), the inductive assumption and Lemma 2.1. The second inequality is deduced from ii). □

Corollary 2.4. *In a superadditive BGWBP, for all positive integer k, it is verified that*

$$\sum_{i=1}^{Z_k} f_n^{(i)} \prec f_{k+n} \quad \text{and} \quad \sum_{i=1}^{Z_k} m_n^{(i)} \prec m_{k+n}\,, \qquad n = 1, 2, \ldots$$

where $\{(f_n^{(i)}, m_n^{(i)}) \colon n = 1, 2, \ldots\}$, $\{Z_n^{(i)} \colon n = 0, 1, \ldots\}$, $i = 1, \ldots, Z_k$, are independent versions of $\{(f_n, m_n) \colon n = 1, 2, \ldots\}$, $\{Z_n \colon n = 0, 1, \ldots\}$ with the same parameters.

The proof of this result is based on the Theorem 2.3 and Lemma 2.2.

3. Probability generating functions and moments. For simplicity we shall assume that $Z_0 = 1$. Obviously, this implies that $Y_0 = 1$.

Proposition 3.1. *For a superadditive BGWBP it is verified*

- *i)* $E\left[Y_n\right] \leq A_n$ *where* $A_n = n+1$ *if* $r = 1$ *or* $(1 - r^{n+1})/(1-r)$ *if* $r \neq 1$, $n = 0, 1, \ldots$
- *ii)* $E\left[F_n\right] \leq B_n^1$ *and* $E\left[M_n\right] \leq B_n^2$, *where* $B_n^i = n\mu_i$ *if* $r = 1$ *or* $\mu_i(1-r^n)/(1-r)$ *if* $r \neq 1$, $i = 1, 2$, $n = 0, 1, \ldots$

being $\mu_1 = E[f_1]$ *and* $\mu_2 = E[m_1]$.

Proof. i) is proved taking into account that $r = \sup_{j>0} r_j$. In fact:

$$E[Z_i] = E[E[Z_i|Z_{i-1}]] = \sum_j E[Z_i|Z_{i-1}=j]\, Pr(Z_{i-1}=j) =$$
$$= \sum_j j r_j Pr(Z_{i-1}=j) \le r \sum_j j Pr(Z_{i-1}=j) = rE[Z_{i-1}] \quad i=1,2,\dots$$

Consequently, by finite induction we deduce $E[Z_i] \le r^i$, $i = 1, 2, \dots$ So,

$$E[Y_n] = 1 + \sum_{i=1}^{n} E[Z_i] \le 1 + \sum_{i=1}^{n} r^i.$$

ii) $E[F_n] = \sum_{i=1}^{n} E[f_i] = \sum_{i=1}^{n} E[E[f_i|Z_{i-1}]] = \mu_1 \sum_{i=1}^{n} E[Z_{i-1}] \le \mu_1 \sum_{i=0}^{n-1} r^i$.
In a similar way we can obtain the corresponding inequality for M_n. □

Theorem 3.2. *Let* H_n *be the two-dimensional probability generating function (p.g.f.) of* (Y_n, Z_n), $n = 0, 1, \dots$ *Then, for* $0 \le s, t \le 1$,

$$H_n(s,t) \le H_{n-1}(s, f(st)), \quad n = 1, 2, \dots \tag{3.1}$$

where f *is the p.g.f. of* Z_1 *and* $H_0(s,t) = st$.

Proof.

$$H_n(s,t) = E\left[s^{Y_n} t^{Z_n}\right] = E\left[s^{Y_{n-1}} (st)^{Z_n}\right] = E\left[E\left[s^{Y_{n-1}}(st)^{Z_n} \middle| Z_0, \dots, Z_{n-1}\right]\right]$$
$$= E\left[s^{Y_{n-1}} E\left[(st)^{Z_n} \middle| Z_0, \dots, Z_{n-1}\right]\right].$$

Now, from (1.1) it is derived that $E\left[u^{Z_n} \middle| Z_{n-1}\right] \le f(u)^{Z_{n-1}}$, $0 \le u \le 1$. Thus

$$H_n(s,t) \le E\left[s^{Y_{n-1}} f(st)^{Z_{n-1}}\right] = H_{n-1}(s, f(st)). \qquad \square$$

Corollary 3.3.

(3.2) $E[Y_n] \ge a_n$, *where* $a_n = n+1$ *if* $m = 1$ *or* $(1-m^{n+1})/(1-m)$ *if* $m \ne 1$, $n = 0, 1, \dots$, *being* $m = E[Z_1]$.

Proof. From (3.1) it deduced that $E\left[s^{Y_n}\right] = H_n(s,1) \le H_{n-1}(s, f(s))$. Then, by differentiation and evaluating on $s = 1$, we obtain

$$E[Y_n] = E[Y_{n-1}] + E[Z_{n-1}]\, E[Z_1] \ge E[Y_{n-1}] + m^n.$$

Therefore $E[Y_n] \geq \sum_{i=0}^{n} m^i$, $n = 0, 1, \ldots$ and consequently (3.2) is true. $\square$

Theorem 3.4. *Let ϕ_n be the two-dimensional p.g.f. of (F_n, M_n), $n = 1, 2, \ldots$ Then for $0 \leq s, t \leq 1$*

$$\phi_{n+1}(s,t) \leq g(s,t,\phi_n(s,t)), \; n = 1, 2, \ldots \tag{3.3}$$

where g is the p.g.f. of (f_1, m_1, Z_1).

Proof. According to the corollary 2.4 we have that $\sum_{j=1}^{Z_1} f_{i-1}^{(j)} \prec f_i$ and $\sum_{j=1}^{Z_1} m_{i-1}^{(j)} \prec m_i$, $i = 2, 3, \ldots$ Then

$$\begin{aligned}
\phi_n(s,t) &= E\left[s^{\sum_{i=1}^n f_i} t^{\sum_{i=1}^n m_i}\right] \leq E\left[s^{f_1 + \sum_{i=2}^n \sum_{j=1}^{Z_1} f_{i-1}^{(j)}} t^{m_1 + \sum_{i=2}^n \sum_{j=1}^{Z_1} m_{i-1}^{(j)}}\right] \\
&= E\left[E\left[s^{f_1} t^{m_1} s^{\sum_{j=1}^{Z_1} F_{n-1}^{(j)}} t^{\sum_{j=1}^{Z_1} M_{n-1}^{(j)}} \middle| f_1, m_1, Z_1\right]\right] \\
&= E\left[s^{f_1} t^{m_1} \phi_{n-1}(s,t)^{Z_1}\right] = g(s,t,\phi_{n-1}(s,t)).
\end{aligned}$$

$\square$

Corollary 3.5. *$E[F_n] \geq b_n^1$ and $E[M_n] \geq b_n^2$, where $b_n^i = n\mu_i$ if $m = 1$ or $\mu_i(1-m^n)/(1-m)$ if $m \neq 1$, $i = 1, 2$, $n = 0, 1, \ldots$*

Proof. From (3.3) and in a similar way to the proof of corollary 3.3, we obtain that

$$E[F_n] \geq \mu_1 + mE[F_{n-1}], \; n \geq 2$$

$$E[M_n] \geq \mu_2 + mE[M_{n-1}], \; n \geq 2$$

From which the proof follows immediately. $\square$

If ρ_n denotes the correlation coefficient between F_n and M_n, we can obtain the following result

Corollary 3.6. *For an additive BGWBP, i.e. such that the mating function verifies*

$$L\left(\sum_{i=1}^{n}(x_i, y_i)\right) = \sum_{i=1}^{n} L(x_i, y_i), \tag{3.4}$$

it is verified that

$$\lim_{n\to\infty} \rho_n = \begin{cases} 1 & \text{if } m \geq 1 \\ \dfrac{\tau(1-m)^2 + \mu_1\mu_2\sigma^2 + (\tau_1\mu_2 + \tau_2\mu_1)(1-m)}{\left[\left(\sigma_1^2(1-m)^2 + \mu_1^2\sigma^2 + 2\tau_1\mu_1(1-m)\right)\left(\sigma_2^2(1-m)^2 + \mu_2^2\sigma^2 + 2\tau_2\mu_2(1-m)\right)\right]^{-1/2}} & \text{if } m < 1 \end{cases}$$

being $\sigma^2 = Var\,[Z_1]$*,* $\tau_1 = Cov\,[f_1, Z_1]$*,* $\tau_2 = Cov\,[m_1, Z_1]$*,* $\sigma_1^2 = Var\,[F_1]$*,* $\sigma_2^2 = Var\,[M_1]$ *and* $\tau = Cov\,[F_1, M_1]$*.*

P r o o f. For the additive case, the mean growth rate, r_j, coincides with m. In fact, for all j

$$r_j{=}j^{-1}E\,[Z_{n+1}|Z_n{=}j]=j^{-1}E\left[L\left(\sum_{i=1}^{j}(f_{ni},m_{ni})\right)\right]=j^{-1}E\left[\sum_{i=1}^{j}L\,(f_{ni},m_{ni})\right]=m.$$

Consequently from (1.2) we have $r = \sup_{j>0} r_j = m$.

Taking into account (3.4) can be proved that Z_n, f_n and m_n have the same distribution that $\sum_{i=1}^{Z_1} Z_{n-1}^{(i)}$, $\sum_{i=1}^{Z_1} f_{n-1}^{(i)}$ and $\sum_{i=1}^{Z_1} m_{n-1}^{(i)}$, respectively, $n = 2, 3, \ldots$ Hence the inequality (3.3) becomes and equality:

$$\phi_n(s,t) = g(s,t,\phi_{n-1}(s,t)) \tag{3.5}$$

From (3.5) it is matter of straightforward computations to prove that for $n \geq 2$

$$\begin{aligned} Var\,[F_n] &= \sigma_1^2 \sum_{i=0}^{n-1} m^i + \sigma^2 \sum_{i=1}^{n-1} m^{i-1} E\,[F_{n-i}]^2 + 2\tau_1 \sum_{i=1}^{n-1} m^{i-1} E\,[F_{n-i}] \\ Var\,[M_n] &= \sigma_2^2 \sum_{i=0}^{n-1} m^i + \sigma^2 \sum_{i=1}^{n-1} m^{i-1} E\,[M_{n-i}]^2 + 2\tau_2 \sum_{i=1}^{n-1} m^{i-1} E\,[M_{n-i}] \\ Cov\,[F_n, M_n] &= \tau \sum_{i=0}^{n-1} m^i + \sigma^2 \sum_{i=1}^{n-1} m^{i-1} E\,[F_{n-i}]\, E\,[M_{n-i}] + \tau_2 \sum_{i=1}^{n-1} m^{i-1} E\,[F_{n-i}] \\ &+ \tau_1 \sum_{i=1}^{n-1} m^{i-1} E\,[M_{n-i}] \end{aligned}$$

from where the result is obtained. □

Remark. The class of additive BGWBP includes many interesting models. For example, includes the *sibling-mating-only process*, that allows the mating of a male and female only when they have been generated by the same mating unit. This model has been used by Hull (1982) in the problem of the extinction.

4. Limit behaviour. In a recent paper, González and Molina [6], has been proved that $\{r^{-n} Z_n\}_n$ is a non-negative and L^1-bounded supermartingale and consequently converges almost surely to a non-negative and finite random variable W, and moreover, for the superadditive case with $r > 1$, has been provided sufficient conditions which guarantee the convergence to a non-degenerate variable W. Obviously, according to (1.2), when $r \leq 1$, W will be 0 a.s. In this section, using the results obtained in the former paper, we study the limit behaviour of the processes $\{Y_n\}_n$, $\{F_n\}_n$ and $\{M_n\}_n$ suitable normed.

Theorem 4.1. *In a superadditve BGWBP, for $r > 1$ and as $n \to \infty$, we have*

i) $r^{-n} Y_n \to r(r-1)^{-1} W$ *a.s.*

ii) $r^{-n} F_n \to \mu_1 (r-1)^{-1} W$ *a.s. and* $r^{-n} M_n \to \mu_2 (r-1)^{-1} W$ *a.s.*

where W *is the a.s. limit of* $r^{-n} Z_n$ *as* $n \to \infty$, *where* $\mu_1 = E[f_1]$ *and* $\mu_2 = E[m_1]$.

Proof. i)

$$r^{-n} \sum_{k=1}^{n} \left(r^k - r^{k-1}\right) r^{-k} Z_k = \left(1 - \frac{1}{r}\right) r^{-n} \sum_{k=1}^{n} Z_k = \frac{r-1}{r} r^{-n} \left(Y_n - 1\right). \tag{4.1}$$

Then, taking into account that $r^n \to \infty$ as $n \to \infty$ and $r^{-n} Z_n \to W < \infty$ a.s. as $n \to \infty$, from (4.1), applying the Cesaro's lemma (see Williams (1992) [9, p. 117]), we obtain

$$((r-1)/r) r^{-n} (Y_n - 1) \to W \text{ a.s. as } n \to \infty$$

whence it follows immediately that $r^{-n} Y_n \to (r/(r-1)) W$ a.s. as $n \to \infty$, which concludes the proof.

ii)

$$r^{-n} \sum_{k=1}^{n} \left(r^k - r^{k-1}\right) r^{-k} f_k = \left(1 - \frac{1}{r}\right) r^{-n} \sum_{k=1}^{n} f_k = \frac{r-1}{r} r^{-n} \left(F_n - 1\right) \tag{4.2}$$

and similarly

$$r^{-n} \sum_{k=1}^{n} \left(r^k - r^{k-1}\right) r^{-k} m_k = \frac{r-1}{r} r^{-n} \left(M_n - 1\right). \tag{4.3}$$

Therefore, taking into account that

$$r^{-n} f_n \to (\mu_1 / r) W \text{ a.s. as } n \to \infty$$

and

$$r^{-n} m_n \to (\mu_2 / r) W \text{ a.s. as } n \to \infty$$

(see González and Molina [6]), from (4.2) and (4.3), applying the Cesaro's lemma we obtain

$$((r-1)/r)r^{-n}(F_n - 1) \to (\mu_1/r)W \ a.s. \ as \ n \to \infty$$

and

$$((r-1)/r)r^{-n}(M_n - 1) \to (\mu_2/r)W \ a.s. \ as \ n \to \infty$$

whence it follows the result. □

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