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# $M$-SOLID SUBVARIETIES OF SOME VARIETIES OF COMMUTATIVE SEMIGROUPS

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Abstract. The basic concepts are $M$-hyperidentities, where $M$ is a monoid of hypersubstitutions. The set of all $M$-solid varieties of semigroups forms a complete sublattice of the lattice of all varieties of semigroups. We fix some specific varieties $V$ of commutative semigroups and study the set of all $M$-solid subvarieties of $V$, in particular, if $V$ is nilpotent.

**1. Introduction.** The purpose of this work is to study the lattice $\mathtt{L}(V)$ of all subvarieties of some varieties $V$ of commutative semigroups and the sublattices of $\mathtt{L}(V)$. Our basic concepts are $M$-hyperidentities and the stronger concept of a hyperidentity ([7]). A mapping $\sigma$ from the binary operation symbol $f$ into the set $W(X)$ is called a hypersubstitution, where $W(X)$ denotes the set of all terms over a fixed alphabet $X$. For a term $t \in W(X)$ let $\sigma_t$ be the hypersubstitution defined by $\sigma_t(f) := t$. For a hypersubstitution $\sigma$ we define the extension $\sigma^\wedge$ of $\sigma$ as a mapping from $W(X)$ into $W(X)$ inductively:

(i) $\sigma^\wedge[x] := x$ for $x \in X$;

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(ii) $\sigma^\wedge[f(s,t)] := \sigma(f)^{W(X)}(\sigma^\wedge[s], \sigma^\wedge[t])$, where $\sigma(f)^{W(X)}$ denotes the term operation generated by the term $\sigma(f)$.
*Hyp* denotes the set of all hypersubstitutions. Clearly, for two hypersubstitutions $\sigma_1$, $\sigma_2$ the product $\sigma_1 \circ_h \sigma_2$ defined by $(\sigma_1 \circ_h \sigma_2)^\wedge[t] := \sigma_1^\wedge[\sigma_2^\wedge[t]]$ for $t \in W(X)$ is again a hypersubstitution. Thus *Hyp* is a monoid under $\circ_h$ and the identity element $\sigma_{xy}$ (see [3]).

Let $M$ be a submonoid of *Hyp* and let $V$ be a variety of semigroups. An identity $u \approx v$ in $V$ is called an $M$-hyperidentity in $V$ if $\sigma^\wedge[u] \approx \sigma^\wedge[v]$ is an identity in $V$ for each $\sigma \in M$. The variety $V$ is called $M$-solid if each identity in $V$ is also an $M$-hyperidentity in $V$. By [4] the collection $S_M$ of all $M$-solid varieties of semigroups forms a complete sublattice of the lattice $S$ of all varieties of semigroups.

The lattice of all *Hyp*-solid varieties (or only solid varieties) of semigroups is studied in [2], [5] and [6]. $M$-solid varieties for other submonoids $M$ of *Hyp* have been studied; see for example [1] and [3]. In this paper we will study lattices of $M$-solid varieties of some commutative semigroups for all submonoids $M$ of *Hyp*.

**2. Basic concepts.** We fix a specific variety $V$ of semigroups. The collection of all subsets of the lattice $\mathtt{L}(V)$ of all subvarieties of $V$ will be denoted by $P(\mathtt{L}(V))$. The collection of all subsets of *Hyp* will be denoted by $P(Hyp)$.

We define a relation $R_V \subseteq Hyp \times \mathtt{L}(V)$ as follows: For $\sigma \in Hyp$ and $Y \in \mathtt{L}(V)$ set $(\sigma, Y) \in R_V$ iff for any identity $u \approx v$ in $Y$, $Y$ satisfies $\sigma^\wedge[u] \approx \sigma^\wedge[v]$. Now we define two mappings $\alpha_V^*$ and $\beta_V^*$ on $P(Hyp)$ and $P(\mathtt{L}(V))$, respectively, as follows:

For $M \in P(Hyp)$ set $\alpha_V^*(M) := \{Y : Y \in \mathtt{L}(V), (\sigma, Y) \in R_V$ for all $s \in M\}$;

for $L \in P(\mathtt{L}(V))$ set $\beta_V^*(L) := \{\sigma : \sigma \in Hyp, (\sigma, Y) \in R_V$ for all $Y \in L\}$.
Obviously, $(\alpha_V^*, \beta_V^*)$ forms a GALOIS-connection.

$L(\mathtt{L}(V))$ denotes the collection of all complete sublattices of $\mathtt{L}(V)$. Further we define a relation $\sim_V$ on *Hyp* as follows: For $\sigma_1, \sigma_2 \in Hyp$ we have $\sigma_1 \sim_V \sigma_2$ iff $\sigma_1^\wedge[xy] \approx \sigma_2^\wedge[xy]$ is an identity in $V$. Obviously $\sim_V$ is an equivalence relation and $[\sigma]_V$ denotes the equivalence class of $\sigma \in Hyp$. For a submonoid $M$ of *Hyp* by $M_V$ we put $M_V := \{[\sigma]_V : \sigma \in M\}$ and for $\sigma \in Hyp$ we define $[\sigma]_V[t] := \sigma^\wedge[t]$ for $t \in W(X)$. $S_V(Hyp)$ denotes the collection of all $M_V$ where $M$ is a submonoid of *Hyp*. Now define a map $\alpha_V$ (a map $\beta_V$) on $S_V(Hyp)$ (on $L(\mathtt{L}(V))$) by $\alpha_V(M_V) := \alpha_V^*(M)$ $(\beta_V(L) := (\beta_V^*(L))_V)$.

Clearly, for $M_V \in S_V(Hyp)$, $\alpha_V^*(M)$ is the collection of all $M$-solid subvarieties of $V$, that means, $\alpha_V^*(M) = S_M \cap \mathtt{L}(V)$. As $S_M$ is a complete sublattice of $S$ and $\mathtt{L}(V)$ is a complete lattice, $\alpha_V(M_V) = a_V^*(M) = S_M \cap \mathtt{L}(V)$ forms a complete sublattice of $\mathtt{L}(V)$.

Let $L \in L(\mathtt{L}(V))$ and $V^* \in L$. Then for $\sigma_1, \sigma_2 \in \beta_V^*(L)$ we have $(\sigma_1, V^*) \in R_V$ and $(\sigma_2, V^*) \in R_V$. From this it follows if $u \approx v$ an identity in $V^*$ then $\sigma_2^{\wedge}[u] \approx \sigma_2^{\wedge}[v]$ is an identity in $V^*$ and $\sigma_1^{\wedge}[\sigma_2^{\wedge}[u]] \approx \sigma_1^{\wedge}[\sigma_2^{\wedge}[v]]$ is an identity in $V^*$. Thus $(\sigma_1 \circ_h \sigma_2, V^*) \in R_V$. Clearly, $(\sigma_{xy}, V^*) \in R_V$. Altogether $\beta_V^*(L)$ forms a submonoid of *Hyp*, that means, $\beta_V(L) \in S_V(Hyp)$.

We have now mappings $\alpha_V : S_V(Hyp) \to L(\mathtt{L}(V))$ and $\beta_V : L(\mathtt{L}(V)) \to S_V(Hyp)$. Since $(\alpha_V^*, \beta_V^*)$ forms a GALOIS-connection it is easy to check that $(\alpha_V, \beta_V)$ has the properties of a GALOIS-connection. For $M \in S_V(Hyp)$ we put $\underline{\underline{M}}:=\beta_V(\alpha_V(M))$ and for $L \in L(\mathtt{L}(V))$ we put $\underline{\underline{L}} := \alpha_V(\beta_V(L))$. An $M \in S_V(Hyp)$ (an $L \in L(\mathtt{L}(V))$) is called closed if $\underline{\underline{M}} = M$ ($\underline{\underline{L}} = L$).

Now want to use the kernels of $\alpha_V$ and $\beta_V$ (denoted by $\ker \alpha_V$ and $\ker \beta_V$, respectively) to define maps on the closed monoids and on the closed sublattices, respectively. We define a map $\underline{\underline{\alpha}}_V$ on $S_V(Hyp)_{/\ker \alpha_V}$ by $\underline{\underline{\alpha}}_V([M]_{\ker \alpha_V}) := [a_V(M)]_{\ker \alpha_V}$ and we define a map $\underline{\underline{\beta}}_V$ on $L(\mathtt{L}(V))_{/\ker \beta_V}$ by $\underline{\underline{\beta}}_V([L]_{\ker \beta_V}) := [\beta_V(L)]_{\ker \beta_V}$. Then $\underline{\underline{\alpha}}_V$ and $\underline{\underline{\beta}}_V$ are bijections between $S_V(Hyp)_{/\ker \alpha_V}$ and $L(\mathtt{L}(V))_{/\ker \beta_V}$. Clearly, all members of each $\ker \alpha_V$ class ($\ker \beta_V$ class) have the same closure, so we can label an equational class as $[\underline{\underline{M}}]_{\ker \alpha_V}$ for any $M$ (as $[\underline{\underline{L}}]_{\ker \beta_V}$ for any $L$) in the class. We could also think of $\underline{\underline{\alpha}}_V$ (of $\underline{\underline{\beta}}_V$) as the restriction of $\alpha_V$ (of $\beta_V$) to the closed members of $S_V(Hyp)$ (of $L(\mathtt{L}(V))$).

In this paper we will now determine the closed members of $L(\mathtt{L}(V))$ for varieties of specific commutative semigroups, in particular, if $V$ is nilpotent. Note that a variety $V$ of semigroups is called nilpotent if there exists a natural number $k \geq 2$ such that $x^k \approx x$ is satisfied by $V$.

**3. Varieties of commutative nilpotent semigroups.** In the next by $\sigma$ we mean $[\sigma]_V$ for a variety $V$.

**Theorem 3.1.** *Let $V$ be a variety of nilpotent commutative semigroups and $M \in S_V(Hyp)$ with $M \cap (\{\sigma_{x^i} : 1 \leq i \in \mathbb{N}\} \cup \{\sigma_{y^i} : 1 \leq i \in \mathbb{N}\}) = \emptyset$. Then $\alpha_V(M) = \{V' : V' \in \mathtt{L}(V), V' \subseteq \vartheta_V(M)\}$ where $\vartheta_V(M) :=$* Mod$\{x^i \approx x : i \in$

*$I_V(M)\}$ and $I_V(M)$ denotes the set of all natural numbers $i \geq 1$ such that there exists a natural number $j \geq 1$ with $M \cap \{\sigma_{x^iy^j}, \sigma_{x^jy^i}\} \neq \emptyset$.*

Proof. Let $V' \in \alpha_V(M)$ and $i \in I_V(M)$. Clearly, $V' \in \mathtt{L}(V)$ and there exists a natural number $j \geq 1$ with $M \cap \{\sigma_{x^iy^j}, \sigma_{x^jy^i}\} \neq \emptyset$. Suppose that $\sigma_{x^iy^j} \in M$. Then from $xy \approx yx$ it follows $\sigma_{x^iy^j}[xy] \approx \sigma_{x^iy^j}[yx]$, that means, $x^iy^j \approx y^ix^j \approx x^jy^i$ is an identity in $V'$. Since $V$ is nilpotent, there exists a natural number $k \geq 2$ such that $x^k \approx x$ is an identity in $V$ and thus in $V'$. From this it follows that $x^{3k-2} \approx x$ and there exists a natural number $t$ with $x^{3k-2} \approx (x^2)^i x^j x^t$. From $x^iy^j \approx x^jy^i$ it follows $(x^2)^i x^j x^t \approx (x^2)^j x^i x^t$. Clearly, $(x^2)^j x^i x^t = x^{2j+i+t} = x^{2i+j+t+(j-i)} = x^{3k-2+(j-i)}$. Therefore $x \approx x^{3k-2+(j-i)}$. From this is follows $x^i \approx x^{3k-3+j}$. From $x^{3k-2} \approx x$ it follows $x^j \approx x^{3k-3+j}$. Thus $x^i \approx x^j$ is an identity in $V'$. From $(xy)z \approx x(yz)$ it follows $\sigma_{x^iy^j}[(xy)z] \approx \sigma_{x^iy^j}[x(yz)]$ where $\sigma_{x^iy^j}[(xy)z] \approx (x^iy^j)^i z^j$ and $\sigma_{x^iy^j}[x(yz)] \approx x^i(y^iz^j)^j$. Because of $x^i \approx x^j$ and the commutative law we have $(x^iy^j)^iz^j \approx x^{i^2}y^{i^2}z^i$ and $x^i(y^iz^j)^j \approx x^iy^{i^2}z^{i^2}$, that means, $x^iy^{i^2}z^{i^2} \approx x^{i^2}y^{i^2}z^i$. By substitution $(y \to x)$ we obtain $x^az^i \approx x^bz^{i^2}$ where $a = i^2 + i^2$ and $b = i + i^2$. By substitution $(x \to z^{k-1})$ we obtain $z^{a(k-1)}z^i \approx z^{b(k-1)}z^{i^2}$. Because of $x^k \approx x$ we have $z^i \approx z^{i^2}$. Thus $x^i \approx x^{i^2}$ is an identity in $V'$. From $x^k \approx x$ it follows $\sigma_{x^iy^j}[x] \approx \sigma_{x^iy^j}[x^k]$. Using $x^i \approx x^j$ and $x^i \approx x^{i^2}$ we obtain $\sigma_{x^iy^j}[x^k] \approx x^{ki}$ and thus $x^{ki} \approx x$ is an identity in $V'$. From $x^k \approx x$ it follows $x^{ki+1-i} \approx x$ and thus $x^{ki} \approx x^{1+i-1}$, that means, $x^{ki} \approx x^i$. Using $x^{ki} \approx x$ we obtain that $x^i \approx x$ is an identity in $V'$. Suppose that $\sigma_{x^jy^i} \in M$ then similarly as above we obtain that $x^i \approx x$ is an identity in $V'$. Altogether $V' \subseteq \mathrm{Mod}\{x^i \approx x : i \in I_V(M)\} = \vartheta_V(M)$.

Conversely let $V' \in \mathtt{L}(V)$ with $V' \subseteq \vartheta_V(M)$. Further let $\sigma \in M$ and let $u \approx v$ be an identity in $V'$. Because of $M \cap (\{\sigma_{x^i} : 1 \leq i \in \mathbb{N}\} \cup \{\sigma_{y^i} : 1 \leq i \in \mathbb{N}\}) = \emptyset$ and the commutative law $\sigma \in \{\sigma_{x^iy^j} : 1 \leq i, j \in \mathbb{N}\}$. Therefore there are natural numbers $i, j \geq 1$ with $\sigma = \sigma_{x^iy^j}$, where $i, j \in I_V(M)$. Therefore $x^i \approx x$ and $x^j \approx x$ are identities in $V'$. From this it follows $\sigma_{x^iy^j} \sim_V \sigma_{xy}$. By [4] then $\sigma_{x^iy^j}[t] \approx \sigma_{xy}[t]$ for any $t \in W(X)$. Therefore $\sigma_{x^iy^j}[u] \approx \sigma_{xy}[u] \approx \sigma_{xy}[v] \approx \sigma_{x^iy^j}[v]$, that means, $\sigma[u] \approx \sigma[v]$ is an identity in $V'$. Thus $(\sigma, V') \in R_V$. Altogether $(\sigma', V') \in R_V$ for all $\sigma' \in M$, that means, $V'$ is $M$-solid and thus $V' \in \alpha_V(M)$. □

**Theorem 3.2.** *Let $V$ be a variety of nilpotent commutative semigroups and $M \in S_V(Hyp)$ such that $M \cap (\{\sigma_{x^i} : 1 \leq i \in \mathbb{N}\} \cup \{\sigma_{y^i} : 1 \leq i \in \mathbb{N}\}) \neq \emptyset$. Then $\alpha_V(M) = \{T\}$ where $T$ denotes the trivial variety.*

P r o o f. Clearly, $T \in \alpha_V(M)$ and thus $\{T\} \subseteq \alpha_V(M)$.

Conversely, let $V' \in \alpha_V(M)$. There exists a natural number $i \geq 1$ with $M \cap \{\sigma_{x^i}, \sigma_{y^i}\} \neq \emptyset$. From $xy \approx yx$ it follows $\sigma_{x^i}[xy] \approx \sigma_{x^i}[yx]$ and $\sigma_{y^i}[xy] \approx \sigma_{y^i}[yx]$, respectively. Thus $x^i \approx y^i$ is an identity in $V'$. Since $V$ is nilpotent there exists a natural number $k \geq 2$ such that $x^k \approx x$ is an identity in $V$ and thus also in $V' \in \alpha_V(M) \subseteq \mathtt{L}(V)$. Therefore $\sigma_{x^i}[x] \approx \sigma_{x^i}[x^k]$ and $\sigma_{y^i}[x] \approx \sigma_{y^i}[x^k]$, respectively, are identities in $V'$. Using the commutative law from this it follows that $x^i \approx x$ is an identity in $V'$. Consequently, $x \approx x^i \approx y^i \approx y$. Hence $V'$ is the trivial variety $T$. Altogether $\alpha_V(M) \subseteq \{T\}$. $\square$

The following examples illustrate Theorem 3.1 and Theorem 3.2. The varieties $V_k := \text{Mod}\{(xy)z \approx x(yz), xy \approx yx, x^k \approx x\}$ for $k \in \{2,3,4,5\}$ are used. Obviously, $V_2$ is the variety $SL$ of all semilattices.

**Example 3.3.** Obviously, $Hyp_{SL} = \{\sigma_x, \sigma_y, \sigma_{xy}\}$ and $S_{SL}(Hyp) = \{M_1, M_2, M_3, M_4\}$ with $M_1 = \{\sigma_x, \sigma_{xy}\}$, $M_2 = \{\sigma_y, \sigma_{xy}\}$, $M_3 = \{\sigma_x, \sigma_y, \sigma_{xy}\}$ and $M_4 = \{\sigma_{xy}\}$.

By Theorem 3.2 we have $\alpha_{SL}(M_i) = \{T\}$ for $i \in \{1,2,3\}$.

Because of $\vartheta_{SL}(M_4) = \text{Mod}\{x \approx x\}$ and $\mathtt{L}(SL) = \{T, SL\}$ we have $\alpha_{SL}(M_4) = \{V' : V' \in \mathtt{L}(SL), V' \subseteq \vartheta_{SL}(M_4)\} = \{T, SL\}$ by Theorem 3.1.

**Example 3.4.** We have $Hyp_{V_3} = \{\sigma_x, \sigma_y, \sigma_{x^2}, \sigma_{y^2}, \sigma_{xy}, \sigma_{x^2y}, \sigma_{xy^2}, \sigma_{x^2y^2}\}$. Let $M \in S_{V_3}(Hyp)$.

If $M \cap \{\sigma_x, \sigma_y, \sigma_{x^2}, \sigma_{y^2}\} \neq \emptyset$ then by Theorem 3.2 we have $\alpha_{V_3}(M) = \{T\}$.

If $M = \{\sigma_{xy}\}$ then it is easy to check that $\alpha_{V_3}(M) = \mathtt{L}(V_3)$.

If $\{\sigma_{xy}, \sigma_{x^2y}, \sigma_{xy^2}, \sigma_{x^2y^2}\} \supseteq M \neq \{\sigma_{xy}\}$ then $2 \in I_{V_3}(M)$, that means, $\vartheta_{V_3}(M) \subseteq \text{Mod}\{x^2 \approx x\}$. We have $\{V' : V' \in \mathtt{L}(V_3), V' \subseteq \vartheta_{V_3}(M)\} \subseteq \{V' : V' \in \mathtt{L}(V_3), V' \subseteq \text{Mod}\{x^2 \approx x\}\} \subseteq \{T, SL\}$ because of the commutative law. Obviously $T, SL \in \{V' : V' \in \mathtt{L}(V_3), V' \subseteq \vartheta_{V_3}(M)\}$. By Theorem 3.1 we have $\alpha_{V_3}(M) = \{T, SL\}$.

**Example 3.5.** We have $Hyp_{V_4} = \{\sigma_x, \sigma_y, \sigma_{x^2}, \sigma_{y^2}, \sigma_{x^3}, \sigma_{y^3}, \sigma_{xy}, \sigma_{x^2y}, \sigma_{xy^2}, \sigma_{x^2y^2}, \sigma_{x^3y}, \sigma_{x^3y^2}, \sigma_{x^3y^3}, \sigma_{x^2y^3}, \sigma_{xy^3}\}$. Let $M \in S_{V_4}(Hyp)$.

If $M \cap \{\sigma_x, \sigma_y, \sigma_{x^2}, \sigma_{y^2}, \sigma_{x^3}, \sigma_{y^3}\} \neq \emptyset$ then by Theorem 3.2 we have $\alpha_{V_4}(M) = \{T\}$.

If $M = \{\sigma_{xy}\}$ then it is easy to check that $\alpha_{V_4}(M) = \mathtt{L}(V_4)$.

If $\{\sigma_{xy}, \sigma_{x^2y}, \sigma_{xy^2}, \sigma_{x^2y^2}, \sigma_{x^3y}, \sigma_{x^3y^2}, \sigma_{x^3y^3}, \sigma_{x^2y^3}, \sigma_{xy^3}\} \supseteq M \neq \{\sigma_{xy}\}$ then $2 \in I_{V_4}(M)$ or $3 \in I_{V_4}(M)$, that means $\vartheta_{V_4}(M) \subseteq \text{Mod}\{x^2 \approx x\}$ or $\vartheta_{V_4}(M) \subseteq$

$\mathrm{Mod}\{x^3 \approx x\}$. Suppose that $\vartheta_{V_4}(M) \subseteq \mathrm{Mod}\{x^2 \approx x\}$ then similarly as in Example 3.4 we obtain that $\alpha_{V_4}(M) = \{T, SL\}$. Suppose that $\vartheta_{V_4}(M) \subseteq \mathrm{Mod}\{x^3 \approx x\}$ then we note that from $x^3 \approx x$ and $x^4 \approx x$ it follows $x^2 \approx x$ (using that from $x^3 \approx x$ it follows $x^2 \approx x^4$). Thus also $\alpha_{V_4}(M) = \{T, SL\}$.

**Example 3.6.** We have $Hyp_{V_5} = \{\sigma_x, \sigma_y, \sigma_{x^2}, \sigma_{y^2}, \sigma_{x^3}, \sigma_{y^3}, \sigma_{x^4}, \sigma_{y^4}, \sigma_{xy}, \sigma_{x^2y}, \sigma_{xy^2}, \sigma_{x^2y^2}, \sigma_{x^3y}, \sigma_{x^3y^2}, \sigma_{x^3y^3}, \sigma_{x^2y^3}, \sigma_{xy^3}, \sigma_{x^4y}, \sigma_{x^4y^2}, \sigma_{x^4y^3}, \sigma_{x^4y^4}, \sigma_{x^3y^4}, \sigma_{x^2y^4}, \sigma_{xy^4}\}$. Let $M \in S_{V_5}(Hyp)$.

If $M \cap \{\sigma_x, \sigma_y, \sigma_{x^2}, \sigma_{y^2}, \sigma_{x^3}, \sigma_{y^3}, \sigma_{x^4}, \sigma_{y^4}\} \neq \emptyset$ then by Theorem 3.2 we have $\alpha_{V_5}(M) = \{T\}$.

If $M = \{\sigma_{xy}\}$ then it is easy to check that $\alpha_{V_5}(M) = \mathtt{L}(V_5)$.

If $M \cap \{\sigma_x, \sigma_y, \sigma_{x^2}, \sigma_{y^2}, \sigma_{x^3}, \sigma_{y^3}, \sigma_{x^4}, \sigma_{y^4}\} = \emptyset$ and $M \cap \{\sigma_{xy}, \sigma_{x^2y}, \sigma_{xy^2}, \sigma_{x^2y^2}, \sigma_{x^3y^2}, \sigma_{x^2y^3}, \sigma_{x^4y}, \sigma_{x^4y^2}, \sigma_{x^4y^3}, \sigma_{x^4y^4}, \sigma_{x^3y^4}, \sigma_{x^2y^4}, \sigma_{xy^4}\} \neq \emptyset$ then $2 \in I_{V_5}(M)$ or $4 \in I_{V_5}(M)$ and we note that from $x^5 \approx x$ and $x^4 \approx x$ it follows $x^2 \approx x$ (using that from $x^4 \approx x$ it follows $x^5 \approx x^2$). Similarly as in Example 3.5 we obtain that $\alpha_{V_5}(M) = \{T, SL\}$.

If $\{\sigma_{x^3y}, \sigma_{x^3y^3}, \sigma_{xy^3}\} \supseteq M \neq \{\sigma_{xy}\}$ then $3 \in I_{V_5}(M)$ and we have $\{V' : V' \in \mathtt{L}(V_5), V' \subseteq \vartheta_{V_5}(M)\} \subseteq \{V' : V' \in \mathtt{L}(V_5), V' \subseteq \mathrm{Mod}\{x^3 \approx x\}\} \subseteq \mathtt{L}(V_3)$. It is easy to check that $\mathtt{L}(V_3) \subseteq \{V' : V' \in \mathtt{L}(V_5), V' \subseteq \vartheta_{V_5}(M)\}$. By Theorem 3.1 we have altogether $\alpha_{V_5}(M) = \mathtt{L}(V_3)$.

**4. Other "closed" lattices.** In the following we study varieties $V$ of commutative semigroups where $V$ satisfies an identity $x_0 \dots x_k \approx y_0 \dots y_k$ for a natural number $k$. We will determine all closed sublattices of $\mathtt{L}(V)$. $S_f$ denotes the collection of all varieties $V$ of commutative semigroups such that there exists a natural number $k$ with $V$ satisfies $x_0 \dots x_k \approx y_0 \dots y_k$. Clearly, the variety $Z$ of all zero-semigroups ($Z := \mathrm{Mod}\{xy \approx zw\}$) is a member of $S_f$. The closed sublattices of $\mathtt{L}(Z)$ and $\mathtt{L}(Z_3)$ where $Z_3$ denotes the variety $Z_3 := \mathrm{Mod}\{(xy)z \approx x(yz), xy \approx yx, x_0x_1x_2 \approx y_0y_1y_2\}$ will be given in two examples shortly. At first we characterize the lattices $\alpha_V(M)$ for any $V \in S_f$ and all $M \in S_V(Hyp)$.

**Theorem 4.1.** *Let $V \in S_f$ and let $M \in S_V(Hyp)$ then $\alpha_V(M) = \{V' : V' \in \mathtt{L}(V), V' \subseteq \mathrm{Mod}I_V(M)\}$ if $M \cap (\{\sigma_{x^i} : 1 \leq i \in \mathbb{N}\} \cup \{\sigma_{y^i} : 1 \leq i \in \mathbb{N}\}) = \emptyset$ and $\alpha_V(M) = \{V' : V' \in \mathtt{L}(V), V' \subseteq \mathrm{Mod}(I_V(M) \cup \{x^{k(M)} \approx x^{k(M)+1}\})\}$ otherwise where $k(M)$ denotes the least natural number $i$ with $M \cap \{\sigma_{x^i}, \sigma_{y^i}\} \neq \emptyset$ and $I_V(M) := \{x^iy^j \approx x^jy^i : 1 \leq i, j \in \mathbb{N}, \sigma_{x^iy^j} \in M\} \cup \{x^iy^{i^2}z^{i^2} \approx x^{i^2}y^{i^2}z^i : 1 \leq i \in \mathbb{N}, \sigma_{x^iy^i} \in M\}$.*

P r o o f. Since $V \in S_f$ there exists a natural number $k$ with $V$ satisfies $x_0 \dots x_k \approx y_0 \dots y_k$. Let $V' \in \alpha_V(M)$. Obviously then $V' \in \mathtt{L}(V)$. Suppose that $M \cap (\{\sigma_{x^i} : 1 \leq i \in \mathbb{N}\} \cup \{\sigma_{y^i} : 1 \leq i \in \mathbb{N}\}) \neq \emptyset$. Then there exists a least natural number $i$ with $\sigma_{x^i} \in M$ or $\sigma_{y^i} \in M$. From $xy \approx yx$ it follows $\sigma_{x^i}[xy] \approx \sigma_{x^i}[yx]$ and $\sigma_{y^i}[xy] \approx \sigma_{y^i}[yx]$, respectively. Thus $x^i \approx y^i$ is an identity in $V'$. From $x^i \approx y^i$ it follows $x^i \approx x^t$ for a natural number $t > k$. Then $x^{i+1} \approx x^{t+1}$. From $x_0 \dots x_k \approx y_0 \dots y_k$ and $t > k$ it follows $x^t \approx x^{t+1}$ and $x^i \approx x^t \approx x^{t+1} \approx x^{i+1}$, that means, $x^i \approx x^{i+1}$ is an identity in $V'$. Note that $i = k(M)$. Let $1 \leq i, j \in \mathbb{N}$ with $\sigma_{x^i y^j} \in M$. From $xy \approx yx$ it follows $\sigma_{x^i y^j}[xy] \approx \sigma_{x^i y^j}[yx]$, that means, $x^i y^j \approx x^j y^i$ is an identity in $V'$. Let $1 \leq i \in \mathbb{N}$ with $\sigma_{x^i y^i} \in M$. From the associative law it follows $\sigma_{x^i y^i}[(xy)z] \approx \sigma_{x^i y^i}[x(yz)]$, that means, $(x^i y^i)^i z^i \approx x^i (y^i z^i)^i$. Using the commutative law we obtain that $x^i y^{i^2} z^{i^2} \approx x^{i^2} y^{i^2} z^i$ is an identity in $V'$. Altogether $V' \subseteq \mathrm{Mod} I_V(M)$ and $V' \subseteq \mathrm{Mod}(I_V(M) \cup \{x^{k(M)} \approx x^{k(M)+1}\})$, respectively.

Conversely, let $V' \in \mathtt{L}(V)$ with $V' \subseteq \mathrm{Mod} I_V(M)$ and $V' \subseteq \mathrm{Mod}(I_V(M) \cup \{x^{k(M)} \approx x^{k(M)+1}\})$, respectively. Further let $\sigma \in M$ and let $u \approx v$ be a nontrivial identity in $V'$. Because of the commutative law $\sigma \in \{\sigma_{x^i} : 1 \leq i \in \mathbb{N}\} \cup \{\sigma_{y^i} : 1 \leq i \in \mathbb{N}\} \cup \{\sigma_{x^i y^j} : 1 \leq i, j \in \mathbb{N}\}$. At first we show that if $\sigma \in \{\sigma_{x^i} : 1 \leq i \in \mathbb{N}\} \cup \{\sigma_{y^i} : 1 \leq i \in \mathbb{N}\}$ then $\sigma[u] \approx \sigma[v]$. Then there exists a natural number $i \geq 1$ with $\sigma_{x^i} = \sigma$ or $\sigma_{y^i} = \sigma$. With loss of generality we assume that $\sigma_{x^i} = \sigma$. Then we have $\sigma_{x^i}[u] \approx (u_0)^a$ and $\sigma_{x^i}[v] \approx (v_0)^b$ with $a, b \in \{i^n : 1 \leq n \in \mathbb{N}\}$ where $u_0$ and $v_0$ denote the first variable in $u$ and $v$, respectively. From $x^{k(M)} \approx x^{k(M)+1}$ it follows $x^{k(M)} \approx x^t$ and $x^{k(M)} y \approx x^t y$ for a natural number $t \geq k$. Using $x_0 \dots x_k \approx y_0 \dots y_k$ we have $x^{k(M)} y \approx x_0 \dots x_k$. Clearly, $i \geq k(M)$. Thus $(u_0)^a \approx x_0 \dots x_k \approx (v_0)^b$. Therefore $\sigma_{x^i}[u] \approx \sigma_{x^i}[v]$ is an identity in $V'$. If $\sigma = \sigma_{xy}$ then obviously $\sigma[u] \approx \sigma[v]$ is an identity in $V'$. If now $\sigma \in \{\sigma_{x^i y^j} : 1 \leq i, j \in \mathbb{N}\} \setminus \{\sigma_{xy}\}$ then there are natural numbers $m$, $n$ and $i, j \geq 1$ and $u_0, \dots, u_m, v_0, \dots, v_n \in X$ with $u = u_0 \dots u_m$, $v = v_0 \dots v_n$ and $\sigma = \sigma_{x^i y^j}$. Thus $x^i y^j \approx x^j y^i$ is an identity in $V'$. Now we show that $\sigma[u] \approx \sigma[v]$ is an identity in $V'$. Here the following cases are possible:

(a) Suppose that $m = n = 0$. Obviously then $\sigma[u] \approx \sigma[v]$.

(b) Suppose that $m = 0$ and $n \geq 1$ (or $n = 0$ and $m \geq 1$). By substitution ($w \to x$ for $w \in X$) from $u \approx v$ it follows $x \approx x^t$ for a natural number $t \geq 2$. From $x_0 \dots x_k \approx y_0 \dots y_k$ it follows $x^{k+2} \approx x^{k+1}$ and using $x \approx x^t$ we obtain $x \approx x^2$. Then $\sigma[u] \approx u_0 \dots u_m \approx v_0 \dots v_n \approx \sigma[v]$.

(c) Suppose that $m = n = 1$ and $\{u_0, u_1\} = \{v_0, v_1\}$. Since $u \approx v$ is a nontrivial identity in $V'$ it is easy to check that $u \approx v$ is the commutative law, that means $u_0 = v_1$ and $u_1 = v_0$. From $x^i y^j \approx x^j y^i$ it follows $\sigma[u] \approx u_0^i u_1^j \approx u_0^j u_1^i \approx u_1^i u_0^j \approx v_0^i v_1^j \approx \sigma[v]$.

(d) Suppose that $m = n = 1$ and $\{u_0, u_1\} \neq \{v_0, v_1\}$. Then there exists a natural number $t \geq 3$ such that $x^2 \approx x^t$ is an identity in $V'$. From $x_0 \dots x_k \approx y_0 \dots y_k$ and $x^2 \approx x^t$ we obtain $x^2 w \approx x_0 \dots x_k$. Because of $\sigma \neq \sigma_{xy}$ we have $i \geq 2$ or $j \geq 2$. Using $x^2 w \approx w x^2 \approx x_0 \dots x_k$ we have $\sigma[u] \approx u_0^i u_1^j \approx x_0 \dots x_k \approx v_0^i v_1^j \approx \sigma[v]$.

(e) Suppose that $m = 1$ and $n \geq 2$ (or $n = 1$ and $m \geq 2$). Then there exists an identity $x^2 \approx x^t$ in $V'$ for a natural number $t \geq 3$. Similarly as in case (d) we obtain that $\sigma[u] \approx \sigma[v]$ is an identity in $V'$.

(f) Suppose that $m \geq 2$ and $n \geq 2$ and $i \neq j$. With out loss of generality we assume that $i < j$. At first we show that from $x^i y^j \approx x^j y^i$ it follows $x^i (y^i)^j z \approx x_0 \dots x_k$. We have $x^i (y^i)^j \approx x^i y^j y^t$ with $t = (i \cdot j) - j$. Using $x^i y^j \approx x^j y^i$ we have $x^i y^j y^t \approx x^j y^i y^t \approx x^j (y^i)^i y^s \approx x^i (y^i)^j y^s$ with $s = (i \cdot j) - i^2 + i - j$. It is easy to check that from $i < j$ it follows $s \geq 1$. Altogether we have $x^i (y^i)^j \approx x^i (y^i)^j y^s$ with $s \geq 1$. Hence there exists a natural number $r \geq k$ with $x^i (y^i)^j \approx x^i (y^i)^j y^r$ and $x^i (y^i)^j z \approx x^i (y^i)^j y^r z$. Using $x_0 \dots x_k \approx y_0 \dots y_k$ we obtain $x^i (y^i)^j z \approx x_0 \dots x_k$. By the commutative law we obtain $\sigma[u] \approx u_0^i (u_1^i)^j w_u$ and $\sigma[v] \approx v_0^i (v_1^i)^j w_v$ with $w_u, w_v \in W(X)$. Using $x^i (y^i)^j z \approx x_0 \dots x_k$ we have $\sigma[u] \approx u_0^i (u_1^i)^j w_u \approx x_0 \dots x_k \approx v_0^i (v_1^i)^j w_v \approx \sigma[v]$.

In the next cases we have $i = j$ and thus $x^{i^2} y^{i^2} z^i \approx x^i y^{i^2} z^{i^2}$ is an identity in $V'$.

(g) Suppose that $m = n = 2$ and $\{u_0, u_1, u_2\} = \{v_0, v_1, v_2\}$. Since $u \approx v$ is a nontrivial identity in $V'$ we have $|\{u_0, u_1, u_2\}| \geq 2$. Without loss of generality let $u_0 \neq u_1$ and $u_0 = v_0$ and $u_1 = v_1$. By substitution ($w \to w^i$ for $w \in X$) from $u \approx v$ it follows $u_0^i u_1^i u_2^i \approx v_0^i v_1^i v_2^i$ and $u_0^{i(i-1)} u_1^{i(i-1)} u_0^i u_1^i u_2^i \approx u_0^{i(i-1)} u_1^{i(i-1)} v_0^i v_1^i v_2^i$ and $u_0^{i^2} u_1^{i^2} u_2^i \approx v_0^{i^2} v_1^{i^2} v_2^i$. By $x^{i^2} y^{i^2} z^i \approx x^i y^{i^2} z^{i^2}$ and the commutative law we obtain $\sigma[u] \approx u_0^{i^2} u_1^{i^2} u_2^i \approx v_0^{i^2} v_1^{i^2} v_2^i \approx \sigma[v]$.

(h) Suppose that $m = n = 2$ and $\{u_0, u_1, u_2\} \neq \{v_0, v_1, v_2\}$. Similarly as in case (d) we obtain $x^3 w \approx x_0 \dots x_k$. From $\sigma \neq \sigma_{xy}$ it follows $i \geq 2$ and $i^2 \geq 3$. Using the commutative law and $x^3 w \approx w x^3 \approx x_0 \dots x_k$ we obtain $\sigma[u] \approx u_0^{i^2} u_1^{i^2} u_2^i \approx x_0 \dots x_k \approx v_0^{i^2} v_1^{i^2} v_2^i \approx \sigma[v]$.

(i) Suppose that $m = 2$ and $n \geq 3$ (or $m \geq 3$ and $n = 2$). Similarly as in case (e) we obtain that $\sigma[u] \approx \sigma[v]$ is an identity in $V'$.

(j) Suppose that $m \geq 3$ and $n \geq 3$. At first we show that from $x^{i^2}y^{i^2}z^i \approx x^iy^{i^2}z^{i^2}$ it follows $x^iy^{i^2}z^{i^3}w \approx x_0 \ldots x_k$. We have $x^iy^{i^2}z^{i^3} \approx x^iy^{i^2}z^{i^2}z^t$ with $t = i^3 - i^2$. Using $x^{i^2}y^{i^2}z^i \approx x^iy^{i^2}z^{i^2}$ we have $x^iy^{i^2}z^{i^2}z^t \approx x^{i^2}y^{i^2}z^iz^t \approx x^{i^2}y^{i^2}(z^i)^iz^s \approx x^iy^{i^2}z^{i^3}z^s$ with $s = i^3 - 2i^2 + i$. Altogether we have $x^iy^{i^2}z^{i^3} \approx x^iy^{i^2}z^{i^3}z^s$ with $s \geq 1$ because of $i \geq 2$. Then there exists a natural number $r \geq k$ with $x^iy^{i^2}z^{i^3} \approx x^iy^{i^2}z^{i^3}z^r$ and $x^iy^{i^2}z^{i^3}w \approx x^iy^{i^2}z^{i^3}z^rw$. Using $x_0 \ldots x_k \approx y_0 \ldots y_k$ we obtain $x^iy^{i^2}z^{i^3}w \approx x_0 \ldots x_k$. Using the commutative law and by $x^{i^2}y^{i^2}z^i \approx x^iy^{i^2}z^{i^2}$ we have then $\sigma[u] \approx u_0^iu_1^{i^2}u_2^{i^3}w_u \approx x_0 \ldots x_k \approx v_0^iv_1^{i^2}v_2^{i^3}w_v \approx \sigma[v]$, for $w_u, w_v \in W(X)$. Altogether $(\sigma, V') \in R_V$. Consequently, $(\sigma', V') \in R_V$ for each $\sigma' \in M$, that means, $V' \in \alpha_V(M)$. □

We can illustrate Theorem 4.1 by the following two examples.

**Example 4.2.** Obviously, $Hyp_Z = \{\sigma_x, s_y, \sigma_{xy}\}$ and $S_Z(Hyp) = \{M_1, M_2, M_3, \;\; M_4\}$ with $M_1 = \{\sigma_x, \sigma_{xy}\}$, $M_2 = \{\sigma_y, \sigma_{xy}\}$, $M_3 = \{\sigma_x, \sigma_y, \sigma_{xy}\}$ and $M_4 = \{\sigma_{xy}\}$. Then we have $\mathrm{Mod}I_Z(M_4) = \mathrm{Mod}\{xy \approx xy\}$ and $\mathrm{Mod}(I_Z(M) \cup \{x^{k(M)} \approx x^{k(M)+1}\}) = \mathrm{Mod}\{x \approx x^2\}$ for $M \in \{M_1, M_2, M_3\}$. Then by Theorem 4.1 it is easy to check that $\alpha_Z(M_4) = \mathtt{L}(Z) = \{T, Z\}$ and $\alpha_Z(M_i) = \{T\}$ for $i \in \{1, 2, 3\}$.

**Example 4.3.** We have $Hyp_{Z_3} = \{\sigma_x, \sigma_y, \sigma_{x^2}, \sigma_{y^2}, \sigma_{x^3}, \sigma_{xy}\}$. Let $M \in S_{Z_3}(Hyp)$. If $M \cap \{\sigma_x, \sigma_y\} \neq \emptyset$ then $k(M) = 1$. It is easy to check that $\{V' : V' \in \mathtt{L}(Z_3), V' \subseteq \mathrm{Mod}\{x \approx x^2\}\} = \{T\}$. Thus $\alpha_{Z_3}(M) = \{T\}$ by Theorem 4.1. If $M \cap \{\sigma_x, \sigma_y\} = \emptyset$ and $M \cap \{\sigma_{x^2}, \sigma_{y^2}\} \neq \emptyset$ then $k(M) = 2$ and we have $\alpha_{Z_3}(M) = \{V' : V' \in \mathtt{L}(Z_3), V' \subseteq \mathrm{Mod}\{x^2 \approx x^3\}\}$ by Theorem 4.1. If $M \cap \{\sigma_x, \sigma_y, \sigma_{x^2}, \sigma_{y^2}\} = \emptyset$ then $M = \{\sigma_{xy}\}$ or $M = \{\sigma_{x^3}, \sigma_{xy}\}$. If $M = \{\sigma_{xy}\}$ then by Theorem 4.1 we have $\alpha_{Z_3}(M) = \mathtt{L}(Z_3)$. If $M = \{\sigma_{x^3}, \sigma_{xy}\}$ then $k(M) = 3$ and by Theorem 4.1 we obtain $\alpha_{Z_3}(M) = \{V' : V' \in \mathtt{L}(Z_3), V' \subseteq \mathrm{Mod}\{x^3 \approx x^4\}\}$. Since from $x_0x_1x_2 \approx y_0y_1y_2$ it follows $x^3 \approx x^4$ we have $\alpha_{Z_3}(M) = \mathtt{L}(Z_3)$.

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