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ABSTRACT SUBDIFFERENTIAL CALCULUS AND SEMI-CONVEX FUNCTIONS*

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Abstract. We develop an abstract subdifferential calculus for lower semicontinuous functions and investigate functions similar to convex functions. As application we give sufficient conditions for the integrability of a lower semicontinuous function.

1. Introduction. Throughout this paper we develop in an abstract form subdifferential calculus for lower semicontinuous functions on Banach spaces and give a number of applications.

The aim, as stated by Thibault in [13] is to prove at once theorems valid for various subdifferentials.

Related work can be found in [13], [1], [9] and references given there. There is another reason for the recent development of this topic. Starting with the famous paper of Borwein and Preiss [3], attention has been focused on so called smooth subdifferentials. The methods developed provided unified treatment of the locally Lipschitz and the lower semicontinuous functions. They had been successfully applied to the theory of Clarke-Rockafellar's subdifferential.

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Thus it is a natural attempt to refine some properties, satisfied by the smooth subdifferentials and Clarke-Rockafellar's subdifferential, that in turn would provide a reasonable calculus.

The presubdifferentials defined by Thibault in [13] almost do this, but it is not clear whether Gâteaux subdifferential is a presubdifferential if the space admits Gâteaux differentiable norm and is nonseparable.

Aussel, Corvellec and Lassonde in [1] give a definition of subdifferential which is satisfied by the smooth subdifferentials if the norm is appropriately smooth.

In this paper we define a property called Υ. Given a proper lower semicontinuous function, the multivalued operator T from X to X^* satisfies the property Υ for f if the directional subdifferential of f can be approximated by elements of the graph of T. The operator T is to be regarded as some subdifferential of f.

The property Υ is a version of the Smooth Variational Principle with Constraints proved in [7] - an argument which goes back to Borwein and Preiss, see [3], especially the proof of Theorem 3.2 there.

In Section 2 we give the definition of the property Υ and show that if the space is β-smooth (see the definitions at the end of this section) then the subdifferential D_β^- of each proper lower semicontinuous function has the property Υ for this function. The same conclusion holds for Clarke-Rockafellar's subdifferential on arbitrary Banach space.

Section 3 is inspired by an assertion due to Correa, Jofre and Thibault, [5], which characterizes the proper lower semicontinuous functions which have a monotone Clarke-Rockafellar's subdifferential.

We call the proper lower semicontinuous function *semi-convex* if its Clarke-Rockafellar's subdifferential is φ-monotone. We show that such functions have properties similar to those of convex functions. The class of semi-convex functions generalizes the class of primal lower nice functions, introduced by Poliquin in [11], as well as the semi-convex functions considered in [8].

In Section 4 we prove that if the Clarke-Rockafellar's subdifferential of a proper lower semicontinuous function is φ-monotone then it is actually maximal φ-monotone, generalizing the well-known theorem of Rockafellar.

The Section 5 is devoted to the question of integrability of certain lower semicontinuous functions. We prove that the perturbation of a semi-convex function with locally Lipschitz regular function is integrable. As corollaries we obtain some results of Poliquin, [11], and Thibault and Zagrodny, [14].

Notations. X denotes a real Banach space, S_X, B_X, (B_X^0) are respectively the unit sphere, the closed (open) unit ball there. The dual space is X^* while the duality pairing is always written as $\langle\cdot,\cdot\rangle$. $\overline{\mathbb{R}}$ denotes the extended real line, i. e. $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. We say that a function $f : X \to \overline{\mathbb{R}}$ is proper, if $\mathrm{dom} f = \{x \in X : \ f(x) \in \mathbb{R}\}$ is nonempty.

When we consider an operator on X its domain by definition is the set $\mathrm{dom} T = \{x \in X : T(x) \neq \emptyset\}$. With $\partial^c f(x)$ we will denote the usual subdifferential of a convex function f at point x.

For a proper lower semicontinuous function f the Clarke-Rockafellar's subdifferential at the point $x \in \mathrm{dom} f$ is the set

$$\partial^{CR} f(x) = \{p \in X^* : f^0(x;v) \geq \langle p, v\rangle, \ \forall v \in X\},$$

where

$$f^\circ(x;v) = \lim_{\varepsilon\downarrow 0} \limsup_{\substack{y\to_f x\\ t\downarrow 0}} \inf_{w\in v+\varepsilon B} \frac{f(y+tw)-f(y)}{t}$$

and $y \to_f x$ means that $(y, f(y))$ tend to $(x, f(x))$ in $X \times \mathbb{R}$. If $f(x) = \infty$ then $\partial^{CR} f(x) = \emptyset$.

Let β be a bornology on X. A Banach space X is said to be β-smooth if it has a Lipschitz continuous, β differentiable bump function (see [10]). $f \in C^1_\beta$, where $f : X \to \mathbb{R}$, means that f is Gâteaux differentiable and the derivative is a continuous mapping from X to the dual space X^*, equipped with the topology of uniform convergence on the members of the bornology β.

Our essential tool will be the Smooth Variational Principle of Deville, Godefroy and Zizler:

Theorem 1.1 ([6]). *Let X be a β-smooth Banach space. Then for each proper lower semicontinuous and bounded below function $f : X \to \mathbb{R}\cup\{+\infty\}$ and every $\varepsilon > 0$, there exists a function $g \in C^1_\beta$ such that $f + g$ attains its minimum on X and $\|g\|_\infty = \sup\{|g(x)| : x \in X\} < \varepsilon$ and $\|g'\|_\infty < \varepsilon$.*

Finally, If β is some bornology on X and $f : X \to \mathbb{R} \cup \{+\infty\}$ is proper and lower semicontinuous then the smooth subdifferential of f at x is

$$D^-_\beta f(x) = \{u'(x) : \ u \in C^1_\beta \text{ and } f - u \text{ has a local minimum at } x\},$$

if $x \in \mathrm{dom} f$ and $D^-_\beta f(x) = \emptyset$, if $f(x) = \infty$.

2. Υ property. Examples. We are going to introduce in an axiomatic way a class of subdifferential operators. To this end we need some preliminaries.

Let $f : X \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous function, $h \in S_X$ and $\lambda \in \mathbb{R}$. We define the sets

$$\boldsymbol{\delta}^\lambda_h f(x) = \{p \in X^* : \ \exists\tau > 0 \ : \ \forall t : |\, t \,| < \tau, \ f(x+th) - f(x) \geq t\langle p, h\rangle + \lambda t^2\},$$

if $x \in \mathrm{dom} f$, and $\boldsymbol{\delta}_h^\lambda f(x) = \emptyset$, if $x \notin \mathrm{dom} f$. We put

$$\boldsymbol{\delta}_h f(x) = \bigcup_{\lambda \in \mathbb{R}} \delta_h^\lambda f(x).$$

The main fact about the directional subdifferential $\boldsymbol{\delta}_h$ is the following easy mean value inequality:

Lemma 2.1. *Let $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function with $a \in \mathrm{dom} f$ and $f(b) \geq r \in \mathbb{R}$, $b > a$. Then there are $c \in [a, b)$ and sequences $x_n \underset{n\to\infty}{\longrightarrow_f} c$, $p_n \in \boldsymbol{\delta} f(x_n)$, such that $p_n(b-a) \geq r - f(a)$, where $\boldsymbol{\delta}$ stands for $\boldsymbol{\delta}_1$.*

P r o o f. Put $s = \dfrac{r - f(a)}{b-a}$, $g(x) = f(x) - sx$. The proper lower semicontinuous function $g(x)$ attains its minimum on the interval $[a, b]$. If a is a point of minimum of $g(x)$ then for each $n \in \mathbb{N}$ there exists $\alpha_n > 0$ such that for $y_n = a - \dfrac{1}{n}$

$$f(y_n) - sy_n + \frac{\alpha_n}{n^2} > f(a) - sa.$$

Then the lower semicontinuous function $f_n(x) = g(x) + \alpha_n(x-a)^2$ attains its minimum on the interval $[a - \frac{1}{n}, a]$ at some point $x_n \in (a - \frac{1}{n}, a]$ and in fact x_n is a local minimum for the latter function, hence for $t \leq \min\{n^{-1}, b-a\}$

$$f_n(x_n + t) \geq f_n(x_n),$$

i.e.

$$f(x_n + t) - f(x_n) \geq [s - 2\alpha_n(x_n - a)]t - \alpha_n t^2,$$

from where $p_n = s - 2\alpha_n(x_n - a) \in \boldsymbol{\delta} f(x_n)$. Since x_n is a point of minimum of $f_n(x)$ it is easy to see that $f(x_n) \leq f(a) + s(x_n - a)$ and having in mind the lower semicontinuity of $f(x)$ and that $x_n \underset{n\to\infty}{\longrightarrow} a$ we conclude that $f(x_n) \underset{n\to\infty}{\longrightarrow} f(a)$, hence $x_n \underset{n\to\infty}{\longrightarrow_f} a$.

It remains to put $c = a$ and observe that $p_n(b-a) = [s - 2\alpha_n(x_n - a)](b - a) \geq s(b-a) = r - f(a)$.

If b is a point of minimum of $g(x)$ then it is easy to see that a is a point of minimum too and this case is already considered.

In the last case – when a minimal point c of $g(x)$ is such that $c \in (a, b)$ is obvious that $s \in \delta f(c)$, so we put $x_n = c$ and $p_n = s$ and this completes the proof. $\square$

Following the above proof it is easy to show that for the proper lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ $\boldsymbol{\delta}_h$ is graphically dense in dom$f$, i. e. for any $x \in \mathrm{dom} f$ there is a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathrm{dom}\boldsymbol{\delta}_h f$ such that $x_n \underset{n\to\infty}{\longrightarrow_f} x$.

Let us introduce the following class of real valued functions:

$$\begin{array}{c}\mathrm{A} := \{\alpha : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\},\ \alpha(0) = 0,\\ \alpha \text{ strictly increasing, continuous at } 0\}.\end{array}$$

+

For $x \in X$, $f(x) \geq 0$ define $\omega_x f(t) = \sup\{0, -f(y) :\ \|y - x\| \leq t\}$ and for a compact set K, such that $f\,|_K \geq 0$, $\omega_K f(t) = \sup\limits_{x\in K} \omega_x f(t)$.

Proposition 2.2. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. If $f\,|_K \geq 0$, where K is a compact set, then there exists $\alpha \in \mathrm{A}$ such that $\alpha(t) \geq \omega_K f(t)$ for sufficiently small positive t.*

Proof. From the lower semicontinuity of f it is clear that for arbitrary $\varepsilon > 0$ and for every $x \in K$, there exists $\delta_x > 0$ such that from $\|y - x\| < \delta_x$ it follows that $f(y) > -\varepsilon$. Define the open set $U = \bigcup\limits_{x\in K} B^0_X(x, \delta_x) \supset K$ and let $\delta = \dfrac{1}{2}\mathrm{dist}(K, X \setminus U)$. Then $\delta > 0$ since K is compact and for every $x \in K$, such that $\|y - x\| \leq \delta$ we have $y \in U$ and $f(y) > -\varepsilon$, hence $\omega_K f(\delta) \leq \varepsilon$. Since $\omega_K f(0) = 0$ and $\omega_K f(t)$ is an increasing function, it follows that it is continuous at 0. Then the function $\alpha = \omega_K f(t) + t$ has the desired property. $\square$

Definition 2.3. *An operator $T : X \to 2^{X^*}$ has the property Υ for the proper lower semicontinuous function f, written $T \in \Upsilon(f)$, if the following holds: There exists a map $\Phi : \mathrm{A} \to \mathrm{A}$ such that if $p \in \boldsymbol{\delta}^{\lambda}_h f(x)$ and*

$$\alpha(t) \geq \omega_{[x-\delta h, x+\delta h]}(f(\cdot) - f(x) - \langle p, \cdot - x\rangle - \lambda\|P_h(\cdot - x)\|^2)(t) \tag{2.1}$$

where $\delta > 0$ and P_h is a projector of norm one over $x + \mathbb{R}h$, then there exist $x_n \in X$, $x_n \underset{n\to\infty}{\longrightarrow_f} x$, $p_n + \xi_n \in T(x_n)$ such that for $\gamma = \Phi(\alpha)$ are fulfilled

(i) $\|\xi_n\| \underset{n\to\infty}{\longrightarrow} 0$;

(ii) $\langle p_n, h\rangle = \langle p, h\rangle, \quad \forall n \in \mathbb{N}$;

(iii) $d^x_h(x_n) := \mathrm{dist}(x_n, x + \mathbb{R}h) \leq \gamma(\frac{1}{n})$ *for sufficiently large $n \in \mathbb{N}$;*

(iv) $\|p_n\|\gamma(\frac{1}{n}) \underset{n\to\infty}{\longrightarrow} 0$.

Note that Proposition 2.2 implies that there does exist $\alpha \in \mathrm{A}$ satisfying (2.1). Also both (iii) and (iv) imply that $d^x_h(x_n)\|p_n\| \underset{n\to\infty}{\longrightarrow} 0$.

The above notion is well fitted for dealing with mean value properties. We give the first application:

Theorem 2.4 (Zagrodny, [15]). *Let $T \in \Upsilon(f)$, where the function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is proper and lower semicontinuous. Then for every $x \in \operatorname{dom} f$, $y \in \mathbb{R}$, $y \neq x$ and every $r \in \mathbb{R}$, such that $r \leq f(y)$ there exist $c \in [x, y)$, $x_n \underset{n\to\infty}{\longrightarrow_f} c$ and $p_n \in T(x_n)$ such that*

(i) $\dfrac{\|x-y\|}{\|y-c\|} \liminf\limits_{n\to\infty} \langle p_n, y - x_n \rangle \geq r - f(x)$;

(ii) $\liminf\limits_{n\to\infty} \langle p_n, y - x \rangle \geq r - f(x)$.

P r o o f. Let $h = \|x-y\|^{-1}(y-x)$ and $d = d_h^x$. From Lemma 2.1 there are $c \in [x, y)$ and $y_n \in x + \mathbb{R}h$, $y_n \underset{n\to\infty}{\longrightarrow_f} c$ and $q_n \in \boldsymbol{\delta}_h f(y_n)$ such that $\langle q_n, y - x \rangle \geq r - f(x)$. Since $T \in \Upsilon(f)$ we can find for n large $x_n \in X$ and $r_n + \xi_n \in T(x_n)$ such that

a) $\|x_n - y_n\| < n^{-1}$, $|f(x_n) - f(y_n)| < n^{-1}$;

b) $\|\xi_n\| < n^{-1}$, $\langle r_n, h \rangle = \langle q_n, h \rangle$ and $d(x_n)\|r_n\| < n^{-1}$.

Now $x_n \underset{n\to\infty}{\longrightarrow_f} c$, since $y_n \underset{n\to\infty}{\longrightarrow_f} c$. If $x_n' \in x + \mathbb{R}h$ is such that $\|x_n - x_n'\| = d(x_n)$ then $x_n' \underset{n\to\infty}{\longrightarrow} c$ and for $p_n = r_n + \xi_n$

$$\begin{aligned} \langle p_n, y - x_n \rangle &= \langle r_n, y - x_n' \rangle + \langle r_n, x_n' - x_n \rangle + \langle \xi_n, y - x_n \rangle \\ &\geq \langle q_n, y - x_n' \rangle - \|r_n\| d(x_n) - \|\xi_n\| \, \|y - x_n\| \\ &\geq (r - f(x)).\frac{\|y - x_n'\|}{\|y - x\|} - \|r_n\| d(x_n) - \|\xi_n\| \, \|y - x_n\|. \end{aligned}$$

But $\|r_n\| d(x_n)$ tends to zero as long as $\|\xi_n\| \, \|y - x_n\|$, so

$$\liminf_{n\to\infty} \langle p_n, y - x_n \rangle \geq (r - f(x)) \frac{\|y - c\|}{\|y - x\|},$$

which is (i). Similarly $\langle p_n, y - x \rangle \geq \langle q_n, y - x \rangle - \|\xi_n\| \, \|y - x\|$ and (ii) follows. □

It takes some effort to give nontrivial examples of operators having the property Υ for huge classes of functions.

We are going to prove that on a β-smooth space $D_\beta^- f \in \Upsilon(f)$ for each proper lower semicontinuous function f. The following lemma is a variant of Smooth Variational Principle with Constraints (see [7]). Recall that if the Banach space X is β smooth, then there exists Lipschitz continuous function ψ on X (the Leduc's function) such that $\psi \in C_\beta^1$ away from the origin, $\psi(tx) = |\, t \,|\, \psi(x)$, $x \in X$, $t \in \mathbb{R}$ and there is a constant $b > 0$ such that $\|x\| \leq \psi(x) \leq b\|x\|$ for $x \in X$ (see [6]). It is easy to derive that $\psi^2 \in C_\beta^1$, $D_\beta \psi^2(0) = 0$ and $\lim_{x\to 0} \|D_\beta \psi^2(x)\| = 0$.

For $\alpha \in \mathrm{A}$ define the inverse function $\overline{\alpha} \in \mathrm{A}$ as $\overline{\alpha}(\alpha(t)) = t$ and the map $\Phi : \mathrm{A} \to \mathrm{A}$ as $\Phi(\alpha)(t) = \overline{\alpha}(\dfrac{t^2}{2})$.

Lemma 2.5. *Let X be a β smooth Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function such that $f(0) = 0$, $f \geq 0$ over $[-\delta h, \delta h]$ and $\alpha \geq \omega_{[-\delta h, \delta h]} f$, where $\delta > 0$ and $\alpha \in \mathrm{A}$. Then there exist $x_n \underset{n\to\infty}{\longrightarrow_f} 0$, $p_n + \xi_n \in D^-_\beta f(x_n)$, such that*
(i) $\|\xi_n\| \underset{n\to\infty}{\longrightarrow} 0$;
(ii) $\langle p_n, h\rangle = 0, \ \forall n \in \mathbb{N}$;
(iii) $d^0_h(x_n) \leq \gamma(\frac{1}{n})$ *for sufficiently large* $n \in \mathbb{N}$;
(iv) $\|p_n\|\gamma(\frac{1}{n}) \underset{n\to\infty}{\longrightarrow} 0$,
where $\gamma = \Phi(\alpha)$ and Φ is as above.

P r o o f. Let us define for $n \in \mathbb{N}$ the closed sets

$$U_n = \{x : \|P_h x\| \leq \frac{1}{n}, \ \|x - P_h x\| \leq \gamma(\frac{1}{n})\},$$

set $c_n = [n^2\gamma^2(\dfrac{1}{n})]^{-1}$ and consider the lower semicontinuous functions

$$e_n(x) = \begin{cases} f(x) + \psi^2(P_h x) + c_n\psi^2(x - P_h x), & x \in U_n \\ +\infty, & x \notin U_n. \end{cases}$$

For sufficiently large n and $x \in U_n$, we have that $P_h x \in [-\delta h, \delta h]$ and $\|x - P_h x\| \leq \gamma(\frac{1}{n})$. Then $e_n(x) \geq f(x) \geq -\alpha(d^0_h(x))$ and $d^0_h(x) \leq \|x - P_h x\| \leq \gamma(\frac{1}{n})$. So, using that α is increasing and the definition of γ

$$e_n(x) \geq f(x) \geq -\alpha(\gamma(\frac{1}{n})) = -\alpha(\overline{\alpha}(\frac{1}{2n^2})) = -\frac{1}{2n^2}.$$

When x is at the boundary of U_n (∂U_n), then it is possible that $\|P_h x\| = \dfrac{1}{n}$ and then $e_n(x) \geq -\dfrac{1}{2n^2} + \psi^2(P_h x) \geq -\dfrac{1}{2n^2} + \dfrac{1}{n^2} \geq \dfrac{1}{2n^2}$, or $\|x - P_h x\| = \gamma(\dfrac{1}{n})$ and then $e_n(x) \geq -\dfrac{1}{2n^2} + c_n\psi^2(x - P_h x) \geq -\dfrac{1}{2n^2} + \dfrac{1}{n^2\gamma^2(\frac{1}{n})}\gamma^2(\dfrac{1}{n}) \geq \dfrac{1}{2n^2}$.

According to the Smooth Variational Principle of Deville, Godefroy and Zizler, Theorem 1.1, there exist $g_n \in C^1_\beta$ such that $\max\{\|g_n\|_\infty, \ \|g'_n\|_\infty\} < \dfrac{1}{4n^2}$ and such that $e_n + g_n$ attains its minimum at x_n. Observe that

$$(e_n + g_n)\mid_{\partial U_n} > \frac{1}{2n^2} - \frac{1}{4n^2} = \frac{1}{4n^2} > (e_n + g_n)(0),$$

which insures that $x_n \in intU_n$. Obviously $d_h^0(x_n) \leq \|x_n - P_h x_n\| \leq \gamma(\frac{1}{n})$ and $\|x_n\| \leq \dfrac{1}{n} + \gamma(\dfrac{1}{n})$, hence $x_n \underset{n\to\infty}{\longrightarrow} 0$. Also, $f(x_n) - e_n(0) \leq g_n(0) - g_n(x_n)$, since $(e_n + g_n)(x_n) \leq (e_n + g_n)(0)$. Then $f(x_n) \leq \frac{1}{2n^2}$ and, since $f(0) = 0$ and f is lower semicontinuous, $f(x_n) \underset{n\to\infty}{\longrightarrow} 0$.

Define $p_n = -c_n D_\beta \psi^2(\cdot - P_h\cdot)(x_n)$, $\xi_n = -g_n'(x_n) - D_\beta\psi^2(P_h\cdot)(x_n)$, where the derivatives exist from the chain rule. From the definition $p_n + \xi_n \in D_\beta^- f(x_n)$.

We need here an easy consequence of the chain rule for differentiation.

Fact: If $f \in C_\beta^1(X)$, $T : X \to X$ is a bounded linear operator, then $\langle D_\beta f(T\cdot)(x), y\rangle = \langle D_\beta f(Tx), Ty\rangle$ and in particular $\|D_\beta f(T\cdot)(x)\| \leq \|T\|.\|D_\beta f(Tx)\|$.

In our case:

$$\Big\|D_\beta\psi^2(P_h\cdot)(x_n)\Big\| \leq 2b\|x_n\|\Big\|D_\beta\psi(P_h x_n)\Big\|,$$

and using that $x_n \underset{n\to\infty}{\longrightarrow} 0$ we have that $\|\xi_n\| \underset{n\to\infty}{\longrightarrow} 0$, which is (i).

Also $\langle p_n, h\rangle = -c_n\langle D_\beta\psi^2(\cdot - P_h\cdot), h - P_h h\rangle = 0$, which is (ii) and

$$\|p_n\| \leq c_n\Big\|D_\beta\psi^2(\cdot - P_h\cdot)(x_n)\Big\| \leq 2c_n\psi(x_n - P_h x_n)\Big\|D_\beta\psi(\cdot - P_h\cdot)(x_n)\Big\| \leq$$

$$2c_n b\|x_n - P_h x_n\|\Big\|D_\beta\psi(\cdot - P_h\cdot)(x_n)\Big\| \leq \frac{2b}{n^2\gamma^2(\frac{1}{n})}\gamma(\frac{1}{n})\Big\|D_\beta\psi(\cdot - P_h\cdot)(x_n)\Big\|,$$

therefore $\|p_n\|\gamma(\frac{1}{n}) \underset{n\to\infty}{\longrightarrow} 0$, which is (iv). $\square$

Theorem 2.6. *Let X be a β-smooth Banach space and $f : X \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous function. Then $D_\beta^- f \in \Upsilon(f)$.*

Proof. Fix $x \in X$, $h \in S_X$, $p \in \boldsymbol{\delta}_h^\lambda f(x)$ and $\delta > 0$, such that for $|t| \leq \delta$ $f(x + th) \geq f(x) + \langle p, th\rangle + \lambda t^2$. Consider the function

$$g(y) = f(x + y) - f(x) - \langle p, y\rangle - \lambda\psi^2(P_h y).$$

Note that if $\alpha \in \mathrm{A}$ satisfies (2.1), then $\alpha \geq \omega_{[-\delta h,\delta h]}g$ and apply the previous Lemma 2.5 to g to obtain $y_n \underset{n\to\infty}{\longrightarrow_g} 0$, $p_n^1 + \xi_n^1 \in D_\beta^- g(y_n)$ with the listed there properties.

Set $x_n = x + y_n$. Since $y_n \underset{n\to\infty}{\longrightarrow} 0$ and $g(y_n) \underset{n\to\infty}{\longrightarrow} 0$ we have $x_n \underset{n\to\infty}{\longrightarrow_f} x$. Also it is fulfilled that

$$d_h^x(x_n) = \mathrm{dist}(x_n, x + \mathbb{R}h) = \mathrm{dist}(y_n, \mathbb{R}h) = d_h^0(y_n) \leq \gamma(\frac{1}{n}).$$

If $p_n = p + p_n^1$ then, of course, $\langle p_n, h\rangle = \langle p, h\rangle$ and $\|p_n\|\gamma(\frac{1}{n}) \underset{n\to\infty}{\longrightarrow} 0$, since $\gamma(\frac{1}{n}) \underset{n\to\infty}{\longrightarrow} 0$.

Let $\xi_n = \xi_n^1 + \xi_n^2$, where $\xi_n^2 = \lambda D_\beta \psi^2(P_h\cdot)(y_n)$ (the existence of the derivative follows from the chain rule). We have as well $\|\xi_n^2\| \le 2|\lambda|\|y_n\|\|D_\beta(P_h\cdot)(y_n)\| \underset{n\to\infty}{\longrightarrow} 0$, hence $\|\xi_n\| \underset{n\to\infty}{\longrightarrow} 0$.

Finally, it is easy to see that $p_n + \xi_n \in D_\beta^- f(x_n)$, which completes the proof. $\square$

We define a notion, which we need in the sequel, and which only slightly differs from the subdifferentials defined by Thibault and Zagrodny in [14].

A class $\mathcal{F}$ of lower semicontinuous proper functions, defined on the Banach space X is called *admissible* (for our purpose) if it contains all convex continuous functions and for all $f \in \mathcal{F}$ and g convex continuous, the function $f + g \in \mathcal{F}$.

Definition 2.7. *An operator $\partial f : X \to 2^{X^*}$, defined on an admissible class $\mathcal{F}$ is called absubdifferential for $\mathcal{F}$ if the following conditions hold:*
1) $\forall f \in \mathcal{F}$ $\operatorname{dom}\partial f \subset \operatorname{dom} f$.
2) $\forall f,\ h \in \mathcal{F}$ $\partial f(x) = \partial h(x)$, *whenever f and h coincide on a neighborhood of x.*
3) $0 \in \partial f(x)$, *whenever x is a local minimum of f.*
4) $0 \in \partial f(x) + \partial^c g(x)$, *whenever g is convex continuous and for $f \in \mathcal{F}$ the function $f + g$ has a local minimum at x.*

Having in mind the properties of the Clarke-Rockafellar's subdifferential (see for example [4], Theorem 2.9.8) it is easy to derive the following:

Proposition 2.8. *The operator $\partial^{CR} f : X \to 2^{X^*}$ is absubdifferential for the class of all proper lower semicontinuous functions.*

When restricted to certain admissible classes, the Clarke-Rockafellar's subdifferential is written in more convenient form (see Corollary 3.4 below).

In fact the proofs of Lemma 2.5 and Theorem 2.6 could be easily adapted to any Banach space (if the square of the Leduc's function be changed to the square of the norm) to get the following:

Theorem 2.9. *If ∂ is absubdifferential for the admissible class of functions $\mathcal{F}$, defined on arbitrary Banach space, then for each $f \in \mathcal{F}$ it follows that $\partial f \in \Upsilon(f)$.*

Proof. One uses Ekeland's Variational Principle instead of the Smooth Variational Principle of Deville, Godefroy and Zizler. The elementary sum rule used at certain steps (i.e. $0 \in D_\beta^- f(x) + D_\beta^- g(x)$ if g is β-smooth and x is a local minimum for $f + g$) is replaced by axiom 4) from Definition 2.7 and this is legal since the functions that appear are convex (as they are sums of square of the norm and the perturbation). For more details see for example [13], [1]. $\square$

We just mention that the presubdifferentials considered by Thibault, see [13], also have Υ property for each proper lower semicontinuous function.

Aussel, Corvellec and Lassonde in their paper [1] recently defined a very general notion of subdifferential, including the Clarke-Rockaffelar's and the smooth subdifferentials. If the Banach space X has an equivalent ∂-smooth norm (see Definition 2.1 in [1]), then $\partial f \in \Upsilon(f)$ for each proper lower semicontinuous f. (Apply the Smooth Variational Principle of Deville, Godefroy and Zizler instead of Proposition 2.3 from [1], that is actually a version of Borwein-Preiss Smooth Variational Principle).

3. Semi-convex functions. It seems that until now the word "semi-convex" has no fixed meaning, though this way are called different classes of functions, possessing some of the intrinsic properties of the convex functions. In this paper we take as a device the following assertion, established by Correa, Jofre and Thibault (see [5], also Corollary 3.10 bellow):

The proper lower semicontinuous function is convex if and only if its Clarke-Rockafellar's subdifferential is a monotone operator.

Therefore we study functions with φ-monotone ∂^{CR} and show that they really deserve the title "semi-convex".

Definition 3.1. *Let $\varphi : X \to \mathbb{R}$ be even, convex and Gâteaux differentiable at the origin function satisfying $\varphi(0) = 0$ and $\varphi'(0) = 0$.*

An operator $T : X \to 2^{X^*}$ is said to be φ-monotone, if for every $x,\ y \in X$

$$\langle Tx - Ty, x - y\rangle \geq -\varphi(x - y).$$

The above inequality means

$$\forall p \in Tx,\ q \in Ty \Rightarrow \langle p - q, x - y\rangle \geq -\varphi(x - y).$$

So, it is trivially satisfied if Tx or Ty is empty. All inequalities including sets, that we meet below, have the same meaning.

For sake of simplicity, we assume from now on that we have some fixed function φ with the above properties.

Definition 3.2. *The proper lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be of class $\mathcal{C}_\varphi$ (written $f \in \mathcal{C}_\varphi$) if $\partial^{CR} f$ is φ-monotone.*

For $f \in \mathcal{C}_\varphi$ define

$$\partial^{\varphi} f(x) = \{p \in X^* : f(y) \geq f(x) + \langle p, y - x\rangle - \varphi(y - x),\ \forall y \in X\}. \tag{3.1}$$

The following lemma which is due essentially to Correa, Jofre and Thibault ([5], Theorem 2.2) plays a crucial role.

Lemma 3.3 *If* $T \in \Upsilon(f)$, *where* $f : X \to \mathbb{R} \cup \{+\infty\}$ *is proper and lower semicontinuous, is* $\varphi-$*monotone, then* $T \subset \partial^{\varphi} f$.

Proof. Take any $p \in T(x)$. Since $T \in \Upsilon(f)$ from Zagrodny's Theorem 2.4 it follows that for every $y \in \mathrm{dom} f$, $y \neq x$ (we have nothing to prove if there is no such y) there exist $x_n \underset{n\to\infty}{\longrightarrow_f} c \in (x, y]$ and $p_n \in T(x_n)$ such that

$$f(x) - f(y) \le \frac{\|x-y\|}{\|x-c\|} \liminf_{n\to\infty} \langle p_n, x - x_n\rangle.$$

From the φ-monotonicity of T we have

$$\langle p_n, x - x_n\rangle - \langle p, x - x_n\rangle \le \varphi(x - x_n)$$

and then

$$\begin{aligned} f(x) - f(y) &\le \frac{\|x-y\|}{\|x-c\|} \lim_{n\to\infty} \langle p, x - x_n\rangle + \frac{\|x-y\|}{\|x-c\|} \lim_{n\to\infty} \varphi(x - x_n) \\ &= \frac{\|x-y\|}{\|x-c\|} \langle p, x - c\rangle + \frac{\|x-y\|}{\|x-c\|} \varphi(x - c) \\ &\le \langle p, x - y\rangle + \varphi(x - y), \end{aligned}$$

where for the last inequality we use that φ is convex and $\varphi(0) = 0$. Finally $f(y) - f(x) \ge \langle p, y - x\rangle - \varphi(y - x)$, which means that $p \in \partial^{\varphi} f(x)$. □

Since $\partial^{CR} f \in \Upsilon(f)$ (see Theorem 2.9) and, clearly, $\partial^{\varphi} \subset \partial^{CR}$ we obtain immediately the following:

Corollary 3.4. *If* $f \in \mathcal{C}_{\varphi}$ *then* $\partial^{CR} f \equiv \partial^{\varphi} f$.

Lemma 3.5. *If* $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *is a proper lower semicontinuous function and* $\boldsymbol{\delta} f$ *is* $\varphi-$*monotone then*
a) $\mathrm{dom} f$ *is an interval, in the interior of which* f *is locally Lipschitz.*
b) For every $a,\ b \in \mathbb{R}$, $\lambda \in (0, 1)$

$$f(\lambda a + (1-\lambda) b) \le \lambda f(a) + (1-\lambda) f(b) + 2\lambda(1-\lambda)\varphi(a - b). \tag{3.2}$$

c) The directional derivatives $f'(x, \pm 1)$ *exist for each* $x \in \mathrm{dom} f$ *and belong to* $\overline{\mathbb{R}}$. *Moreover, if we define the operator* $Df : \mathbb{R} \to 2^{\mathbb{R}}$ *with* $\mathrm{dom} Df \subset \mathrm{dom} f$ *by*

$$Df(x) = \{p \in \mathbb{R} : \ p \le f'(x, 1),\ p \ge -f'(x, -1)\},$$

then Df is $\varphi-$monotone.

P r o o f. If dom$f$ is singleton we have nothing to prove, so we assume that it is not.

a) Let $\varepsilon > 0$ and $\{x - 2\varepsilon, y + 2\varepsilon\} \subset \mathrm{dom} f$ for some $x < y$. We will show that f is Lipschitz continuous on $[x, y]$, in particular $[x, y] \subset \mathrm{dom} f$. Assume the contrary, then by Lemma 2.1 there exist $p_n \in \boldsymbol{\delta} f(x_n),\ x_n \in [x-\varepsilon, y+\varepsilon]$, so that $| p_n | \to \infty$. By taking a subsequence we may assume without loss of generality that $p_n \to \infty$. But then, since obviously $\boldsymbol{\delta} f \in \Upsilon(f)$, by Lemma 3.3 it follows that

$$\begin{aligned} f(y + 2\varepsilon) &\geq f(x_n) + p_n(y - x_n + 2\varepsilon) - \varphi(y - x_n + 2\varepsilon) \\ &\geq f(x_n) + p_n\varepsilon - \varphi(y - x + 4\varepsilon). \end{aligned}$$

So, $f(x_n) \underset{n\to\infty}{\longrightarrow} -\infty$. But the lower semicontinuous function f is bounded below on the compact set $[x-2\varepsilon, y + 2\varepsilon]$, contradiction.

b) If a or b is not in dom$f$ then (3.2) is trivial, so let $\{a, b\} \in \mathrm{dom} f$. Since $[a, b] \subset \mathrm{dom} f$ and dom$\boldsymbol{\delta} f$ is graphically dense in dom$f$ it is enough to prove the inequality for the case when $c = \lambda a + (1 - \lambda)b \in \mathrm{dom}\boldsymbol{\delta} f$. Let then $p \in \boldsymbol{\delta} f(c) \subset \partial^{\varphi} f(c)$, which means that

$$\begin{aligned} f(a) &\geq f(c) + p(a - c) - \varphi(a - c) \\ &\geq f(c) + (1 - \lambda)(a - b)p - (1 - \lambda)\varphi(a - b) \end{aligned}$$

(note that since φ is convex and $\varphi(0) = 0,\ \varphi((1 - \lambda)(a - b)) \leq (1 - \lambda)\varphi(a - b))$ and similarly

$$f(b) \geq f(c) + \lambda(b - a)p - \lambda\varphi(b - a).$$

Multiply the first inequality by λ and the second by $(1 - \lambda)$ and add both, using that $\varphi(a - b) = \varphi(b - a)$:

$$\lambda f(a) + (1 - \lambda)f(b) \geq f(c) - 2\lambda(1 - \lambda)\varphi(a - b),$$

which is (3.2).

c) Take arbitrary $x \in \mathrm{dom} f$ and let the sequence $\{x_n\}$ decrease in such a way that

$$\lim_{n\to\infty} \frac{f(x_n) - f(x)}{x_n - x} = \liminf_{t\downarrow 0} \frac{f(x + t) - f(x)}{t}.$$

We apply (3.2) to derive for $\lambda \in (0, 1)$

$$f(\lambda x + (1 - \lambda)x_n) \leq \lambda f(x) + (1 - \lambda)f(x_n) + 2\lambda(1 - \lambda)\varphi(x_n - x) \Rightarrow$$

$$f(\lambda x + (1 - \lambda)x_n) - f(x) \leq (1 - \lambda)(f(x_n) - f(x)) + 2\lambda(1 - \lambda)\varphi(x_n - x).$$

Multiply the last inequality by $[\lambda x+(1-\lambda)x_n-x]^{-1}=(1-\lambda)^{-1}(x_n-x)^{-1}>0$ and take supremum over $\lambda\in(0,1)$ to obtain

$$\sup_{t\le x_n-x}\frac{f(x+t)-f(x)}{t}\le\frac{f(x_n)-f(x)}{x_n-x}+2\frac{\varphi(x_n-x)}{x_n-x}.$$

Take now limit as $n\to\infty$ and use that $\varphi'(0)=0$:

$$\limsup_{t\downarrow 0}\frac{f(x+t)-f(x)}{t}\le\liminf_{t\downarrow 0}\frac{f(x+t)-f(x)}{t},$$

which means the existence of $f'(x,1)$. The existence of $f'(x,-1)$ can be proved in a similar way.

To conclude take arbitrary $x,y\in\operatorname{dom}f$ with $x<y$ and $p\in Df(x)$, $q\in Df(y)$. Fix $\varepsilon>0$ smaller than $2^{-1}(y-x)$. By the definition $p\le f'(x,1)$, so we can find $z>x$ with $z-x<\varepsilon$ and $r\in\mathbb{R}$ such that $r\le f(z)$ and $r-f(x)>(p-\varepsilon)(z-x)$. By Lemma 2.1 there are x_1 with $\mid x_1-x\mid<\varepsilon$ and $p_1\in\boldsymbol{\delta}f(x_1)$ so that $p_1>p-\varepsilon$.

In the same way, using that $q\ge -f'(y,-1)$ we obtain y_1 such that $|y_1-y|<\varepsilon$, $q_1\in\boldsymbol{\delta}f(y_1)$, $q_1<q+\varepsilon$. Applying the φ-monotonicity of $\boldsymbol{\delta}f$ we write

$$-\varphi(y_1-x_1)\le(q_1-p_1)(y_1-x_1)<(q-p+2\varepsilon)(y_1-x_1).$$

The left hand side tends to $-\varphi(y-x)$ as $\varepsilon\to 0$ by the continuity of φ, while the right hand side obviously tends to $(q-p)(y-x)$. Thereby the φ-monotonicity of Df is established. $\square$

Proposition 3.6. *Assume that the operator $T\in\Upsilon(f)$ and T is φ-monotone, where the function $f:X\to\mathbb{R}\cup\{+\infty\}$ is proper and lower semicontinuous. Then for every $x\in X$, $h\in S_X$ and $t\in\mathbb{R}$ we have*

$$\langle\boldsymbol{\delta}_h f(x+th)-\boldsymbol{\delta}_h f(x),th\rangle\ge-\varphi(th). \tag{3.3}$$

P r o o f. Denote $y=x+th$ and let $d=d_h^x$. Take any $p\in\boldsymbol{\delta}_h f(x)$, $q\in\boldsymbol{\delta}_h f(y)$.
Since $T\in\Upsilon(f)$ if denote $\alpha(t)=\max\{\alpha_x(t),\,\alpha_y(t)\}\in\mathrm{A}$, $\gamma(t)=\Phi(\alpha(t))$ we have by the definition of the property Υ that there exist $x_n\underset{n\to\infty}{\longrightarrow_f}x$, $p_n+\xi_n\in T(x_n)$ such that
$(*)$ (i) $\|\xi_n\|\underset{n\to\infty}{\longrightarrow}0$, (ii) $\langle p_n,h\rangle=\langle p,h\rangle$, (iii) $d(x_n)\le\gamma(\frac{1}{n})$, (iv) $\|p_n\|\gamma(\frac{1}{n})\underset{n\to\infty}{\longrightarrow}0$

and there exist $y_n \xrightarrow[n\to\infty]{}_f y,\ q_n + \eta_n \in T(y_n)$ such that
(**) (i) $\|\eta_n\| \xrightarrow[n\to\infty]{} 0$, (ii) $\langle q_n, h\rangle = \langle q, h\rangle$, (iii) $d(y_n) \le \gamma(\frac{1}{n})$, (iv) $\|q_n\|\gamma(\frac{1}{n}) \xrightarrow[n\to\infty]{} 0$.

From φ-monotonicity of T it follows that

$$\langle x_n - y_n, p_n + \xi_n - q_n - \eta_n\rangle \ge -\varphi(x_n - y_n), \tag{3.4}$$

i.e. $\langle x_n - y_n, \xi_n - \eta_n\rangle + \langle x_n - y_n, p_n - q_n\rangle \ge -\varphi(x_n - y_n)$, and using (i) we see that the first term tends to zero. Take $x'_n,\ y'_n \in x + \mathbb{R}h$ so that $\|x'_n - x_n\| = d(x_n)$, $\|y'_n - y_n\| = d(y_n)$ and note that $x'_n \xrightarrow[n\to\infty]{} x,\ y'_n \xrightarrow[n\to\infty]{} y$. Then

$$\langle x_n - y_n, p_n - q_n\rangle = \langle x_n - x, p_n - q_n\rangle + \langle y - y_n, p_n - q_n\rangle + \langle x - y, p_n - q_n\rangle.$$

Note that the last term is equal to $\langle x - y, p - q\rangle$ from (ii), while

$$\begin{aligned} |\langle x_n - x, p_n - q_n\rangle| &\le \|x_n - x'_n\|\,\|p_n - q_n\| + \|x - x'_n\|\,|\langle h, p - q\rangle| \\ &\le d(x_n)(\|p_n\| + \|q_n\|) + \|x - x'_n\|\,|\langle h, p - q\rangle|. \end{aligned}$$

Then $\|x - x'_n\| \xrightarrow[n\to\infty]{} 0$ and the conditions (iii) and (iv) from $(*)$ and $(**)$ imply $\langle x_n - x, p_n - q_n\rangle \xrightarrow[n\to\infty]{} 0$. Similarly, $\langle y - y_n, p_n - q_n\rangle \xrightarrow[n\to\infty]{} 0$. So, taking limit as $n \to \infty$ in both sides of (3.4) and using the continuity of φ we derive $\langle x - y, p - q\rangle \ge -\varphi(x - y)$. $\square$

Proposition 3.7. *Assume that the function f is proper lower semicontinuous and the condition* (3.3) *is satisfied for all $x \in X$ and $h \in S_X$. Then $f \in \mathcal{C}_\varphi$.*

P r o o f. We first mention that for $x \in \mathrm{dom} f$ the function $f'(x,\cdot) : X \to \overline{\mathbb{R}}$, which is well defined by Lemma 3.5, is convex. Indeed, for $u,\ v \in X,\ \alpha \in (0,1)$, from (3.2) it follows that

$$f'(x, \alpha u + (1-\alpha)v) = \lim_{t\downarrow 0} \frac{f(x + \alpha t u + (1-\alpha)tv) - f(x)}{t} =$$

$$\lim_{t\downarrow 0} \frac{f(\alpha(x + tu) + (1-\alpha)(x + tv)) - f(x)}{t} \le$$

$$\lim_{t\downarrow 0} \frac{\alpha f(x + tu) + (1-\alpha) f(x + tv) - f(x) + c\varphi(t(u - v))}{t} \le$$

$$\alpha \lim_{t\downarrow 0} \frac{f(x + tu) - f(x)}{t} + (1-\alpha) \lim_{t\downarrow 0} \frac{f(x + tv) - f(x)}{t} + c \lim_{t\downarrow 0} \frac{\varphi(t(u - v))}{t},$$

where $c = 2\alpha(1-\alpha)$. But then, since $\varphi'(0) = 0$ we can write

$$f'(x, \alpha u + (1-\alpha)v) \leq \alpha f'(x, u) + (1-\alpha) f'(x, v).$$

Next, for g convex and continuous we define the operator

$$D(f+g)(x) = \{p \in X^*,\ \langle p, h\rangle \leq f'(x,h) + g'(x,h), \quad \forall h \in X\}.$$

We assert that D is an absubdifferential for the class $f + Conv$, where $Conv = \{g : X \to \mathbb{R} : g \text{ convex and continuous}\}$. To this end let $g \in Conv$ and $f+g$ attains its local minimum at some point x. This means that $0 \in \partial^c(f'(x,\cdot) + g'(x,\cdot))(0)$. By the well known formula from convex analysis $0 \in \partial^c f'(x,\cdot)(0) + \partial^c g'(x,\cdot)(0) = Df(x) + \partial^c g(x)$, which is axiom 4) in the definition of absubdifferentials. The other axioms are even more obvious.

From part c) of Lemma 3.5 and condition (3.3), which means that $\boldsymbol{\delta}_h f$ is $\varphi(\cdot\, h)$-monotone we deduce that Df is φ-monotone. By Lemma 3.3 $Df \subset \partial^\varphi f$. Since obviously $\partial^\varphi f \subset Df$ we have

$$Df = \partial^\varphi f.$$

To conclude we show that $Df = \partial^{CR} f$, so the latter is φ-monotone and $f \in \mathcal{C}_\varphi$. It is enough to prove that $f^\circ(x,v) \leq f'(x,v)$ for each $x \in \mathrm{dom} f$ and $v \in X$. Recall that

$$f^\circ(x,v) = \lim_{\varepsilon \to 0} \limsup_{\substack{y \to_f x \\ t \downarrow 0}} \inf_{w \in v + \varepsilon B} \frac{f(y+tv) - f(y)}{t}.$$

Let $M \in (-\infty, +\infty]$ and $M \geq f'(x,v)$. Fix arbitrary $\varepsilon_1 > 0$. We can find $t > 0$ so small that

$$\frac{f(x+tv) - f(x)}{t} < M + \varepsilon_1, \quad \frac{\varphi(tv)}{t} < \frac{\varepsilon_1}{2} \text{ (since } \varphi'(0) = 0).$$

By continuity of φ there is $\delta > 0$ so that $\|v' - tv\| < \delta$ imply $t^{-1}\varphi(v') < \varepsilon_1$. For any $0 < \varepsilon < \varepsilon_1$ and $y \in X$ such that it is graphically close enough to x, i.e. $\|x - y\| < \min\{t\varepsilon,\ \delta\}$ and $|\ f(y) - f(x)\ | < t\varepsilon$ (and hence $y \in \mathrm{dom} f$), for any positive $s < t$ and $w \in v + \varepsilon B$ in virtue of (3.2) we have

$$f(y+sw) \leq \frac{s}{t} f(y+tw) + \frac{t-s}{t} f(y) + \frac{2s(t-s)}{t^2}\varphi(tw).$$

Consequently

$$\frac{f(y+sw) - f(y)}{s} \leq \frac{f(y+tw) - f(y)}{t} + \frac{2(t-s)}{t^2}\varphi(tw).$$

Since $\frac{1}{t}(x+tv-y) \in v+\varepsilon B$ and $t-s \le t$

$$\begin{aligned}\inf_{w\in v+\varepsilon B}\frac{f(y+sw)-f(y)}{s} &\le \inf_{w\in v+\varepsilon B}\{\frac{f(y+tw)-f(y)}{t}+\frac{2}{t}\varphi(tw)\}\\ &\le \frac{f(x+tv)-f(x)}{t}+\frac{2}{t}\varphi(tv+x-y)+\varepsilon,\end{aligned}$$

where we used that $| f(x)-f(y) |< t\varepsilon$. But $\|x-y\|<\delta$ and hence $\frac{\varphi(tv+x-y)}{t}<\varepsilon_1$. Finally

$$\inf_{w\in v+\varepsilon B}\frac{f(y+sw)-f(y)}{s} \le \frac{f(x+tv)-f(x)}{t}+3\varepsilon_1 \le M+4\varepsilon_1$$

if $s<t$ and y is graphically close enough to x. So

$$\limsup_{\substack{y\to_f x\\ s\downarrow 0}}\inf_{w\in v+\varepsilon B}\frac{f(y+sw)-f(y)}{s} \le M+4\varepsilon_1$$

for all ε small enough. Letting $\varepsilon\downarrow 0$ we obtain $f^0(x,v)\le M+4\varepsilon_1$. Since $M\ge f'(x,v)$ was arbitrary as long as ε_1:

$$f^\circ(x,v)\le f'(x,v). \qquad \square$$

Theorem 3.8. *If some φ-monotone operator has the property Υ for the proper lower semicontinuous function f, then $f\in\mathcal{C}_\varphi$.*

P r o o f. Follows immediately from Proposition 3.6 and Proposition 3.7. $\square$

We now put together the above properties:

Theorem 3.9. *Let X be a Banach space, $f: X\to\mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous function and φ be as in Definition 3.1. Then*

a) $f\in\mathcal{C}_\varphi$ if and only if for all $x\in \operatorname{dom} f$, $h\in S_X$ the function $g_{x,h}(t)=f(x+th)\in\mathcal{C}_{\varphi_h}$, where $t\in\mathbb{R}$ and $\varphi_h(t)=\varphi(th)$;

b) If $f\in\mathcal{C}_\varphi$ then $\operatorname{dom} f$ is a convex set. The function f is locally Lipschitz and regular in the (probably empty) interior of $\operatorname{dom} f$. Moreover, $\forall x,y\in\operatorname{dom} f$ and $\lambda\in(0,1)$ one has

$$f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y)+2\lambda(1-\lambda)\varphi(x-y). \tag{3.5}$$

P r o o f. If $g_{x,h} \in \mathcal{C}_{\varphi_h}$ for all $x \in \mathrm{dom} f$ and $h \in S_X$ then, since $\boldsymbol{\delta} g_{x,h}(t) = \langle \boldsymbol{\delta}_h f(x+th), h\rangle$ and $\boldsymbol{\delta} g_{x,h} \subset \partial^{CR} g_{x,h}$, the condition (3.3) is satisfied for all $x \in X$, $h \in S_X$ and Proposition 3.7 implies that f is in $\mathcal{C}_\varphi$.

If $f \in \mathcal{C}_\varphi$ then for every $x \in \mathrm{dom} f$ and $h \in S_X$ it follows that $\boldsymbol{\delta} g_{x,h}$ is φ_h-monotone and consequently $g_{x,h} \in \mathcal{C}_{\varphi_h}$. The inequality (3.5) follows from (3.2). It is obvious then that $\mathrm{dom} f$ is a convex set. The proof of the local Lipschitz continuity of f on $int\mathrm{dom} f$ goes through the same patterns as in the case of convex lower semicontinuous function, see [10]. Using Baire Category Theorem one shows that f is bounded on some open set. After that an application of the inequality (3.5) and the continuity of φ shows that f is locally bounded on $int\mathrm{dom} f$. Then it is straightforward to see that $\partial^\varphi f$ is locally bounded on $int\mathrm{dom} f$, which means that f is locally Lipschitz there.

The regularity of f is a part of the proof of Proposition 3.7. $\square$

Corollary 3.10 ([5], Theorem 2.4)**.** *The proper and lower semicontinuous function f is convex if and only if $\partial^{CR} f$ is monotone.*

P r o o f. As it is easy to check, ∂^c is an absubdifferential for the class of all convex proper lower semicontinuous functions, hence if f is such function then $\partial^c f \in \Upsilon(f)$ and $\partial^c f$ is monotone by definition. So by Corollary 3.4 (note that $\partial^c = \partial^0$) $\partial^{CR} = \partial^c$ and the former is monotone.

If $\partial^{CR} f$ is monotone then we apply Theorem 3.9 with $\varphi \equiv 0$. $\square$

If $\partial^c f \in \Upsilon(f)$ then f is convex, since $\partial^c f$ is monotone for each f. In contrast by definition ∂^φ is only 2φ-monotone, so $\partial^\varphi f \in \Upsilon(f)$ implies only that $f \in \mathcal{C}_{2\varphi}$. Considering function on the real line $f(x) = -x^\alpha$, $1 < \alpha < 2$, for which $\partial^\varphi f$ coincides with its usual derivative if $\varphi(x) = -x^\alpha$, one can see that nevertheless $f \notin \mathcal{C}_\varphi$, so one can not expect that $f \in \mathcal{C}_\varphi$ if $\partial^\varphi f \in \Upsilon(f)$ for a general function φ satisfying the conditions of Definition 3.1 and non identically equal to 0.

4. Maximality.

The well known Rockafellar's theorem asserts that the subdifferential of each convex lower semicontinuous function is a maximal monotone operator. It is natural to ask whether something similar holds true for the functions in $\mathcal{C}_\varphi$. In this section we answer in the affirmative by slightly modifying in Lemma 4.2 the Simons' proof (see [10], Section 3) of Rockafellar's theorem.

Definition 4.1. *The φ-monotone operator $T : X \to 2^{X^*}$ is called maximal if for each φ-monotone $S : X \to 2^{X^*}$, such that $T \subset S$, it follows that $T = S$.*

Of course, $T \subset S$ means that $\mathrm{Gr}T \subset \mathrm{Gr}S$, where $\mathrm{Gr}T = \{(x, x^*) : x^* \in T(x)\}$. Equivalently, the φ-monotone T is maximal when the following relation holds:

$$\forall (x,x^*) \notin \mathrm{Gr}T \Rightarrow \exists (z,z^*) \in \mathrm{Gr}T : \langle z-x, z^*-x^*\rangle < -\varphi(z-x).$$

Lemma 4.2. *If $f \in \mathcal{C}_\varphi$, $x \in X$ and $r < f(x)$ is such that $r > \inf_{y\in X}\{f(y)+\varphi(y-x)\}$, then there exist $y \in X$ and $z^* \in \partial^{CR} f(z)$ such that*

$$\langle z^*, x-z\rangle > \varphi(x-z).$$

P r o o f. There is $y_1 \in X$ such that $r > f(y_1)+\varphi(y_1-x)$. We can choose $\lambda \in (0,1)$ in such a way that for the convex function $\psi(x) := \lambda\varphi(\lambda^{-1}x)$ we still have $r > f(y_1)+\psi(y_1-x)$. Then if

$$K = \sup_{y\in X,\ y\neq x} \frac{r-f(y)-\psi(y-x)}{\|y-x\|}$$

it is clear that $0 < K$.

Our next aim is to show that $K < \infty$. To this end, let $F = \{y \in X : f(y)+\psi(y-x) \le r\}$, so F is closed, non-empty and $x \notin F$. If $y \notin F$, then $\dfrac{r-f(y)-\psi(y-x)}{\|y-x\|} \le 0$. Let now $y \in F$. There exist $z \in \mathrm{dom}f$, $p \in \partial^{CR} f(z) = \partial^\varphi f(z)$, so

$$\begin{aligned} f(y) &\ge f(z)+\langle p, y-z\rangle - \varphi(y-z) \\ &\ge f(z)+\langle p, x-z\rangle - \|p\|\,\|y-x\| - \varphi(y-z). \end{aligned}$$

Hence

$$\frac{r-f(y)-\psi(y-x)}{\|y-x\|} \le \frac{r-f(z)-\langle p,x-z\rangle+\varphi(y-z)-\psi(y-x)}{\mathrm{dist}(x,F)} + \|p\|$$

(note that $\mathrm{dist}(x,F) > 0$). From the other side the convexity of φ implies

$$\begin{aligned} \varphi(y-z) &= \varphi(\lambda(\lambda^{-1}(y-x)) + (1-\lambda)(1-\lambda)^{-1}(x-z)) \\ &\le \lambda\varphi(\lambda^{-1}(y-x)) + (1-\lambda)\varphi((1-\lambda)^{-1}(x-z)) \\ &= \psi(y-x)+C. \end{aligned}$$

Consequently

$$\frac{r-f(y)-\psi(y-x)}{\|y-x\|} \le \frac{r-f(z)-\langle p,x-z\rangle+C}{\mathrm{dist}(x,F)} + \|p\|,$$

where the right hand side does not depend on y and is an upper bound of K.

Let now $\varepsilon \in (0,1)$. Then $(1-\varepsilon)K < K$ and by the definition of K there exists $x_0 \in X$ such that $x_0 \neq x$ and

$$\frac{r - f(x_0) - \psi(x_0 - x)}{\|x_0 - x\|} > (1-\varepsilon)K.$$

For $z \in X$, let $N(z) = K\|z-x\|$. From the above inequality and the the definition of K it follows that

$$r \leq \inf\{N(y) + f(y) + \psi(y-x) : y \in X\}$$

and also $(N+f+\psi(\cdot - x))(x_0) < r + \varepsilon N(x_0)$. Then from the Ekeland's Variational Principle it follows that there exists $z \in \mathrm{dom}(N + f + \psi(\cdot - x)) = \mathrm{dom} f$ such that $\|z - x_0\| < \|x - x_0\|$ and z is the minimal point of the function $N + f + \psi(\cdot - x) + \varepsilon K\|\cdot - z\|$. It follows that $\|z - x\| > 0$. Since N and $\psi(\cdot - x)$ are convex continuous and ∂^{CR} is an absubdifferential, we have

$$0 \in \partial^{CR} f(z) + \partial^c N(z) + \partial^c \psi(z-x) + \varepsilon K \partial^c \|\cdot\|(0),$$

so there exist $y^* \in \partial^c N(z)$, $z^* \in \partial^{CR} f(z)$ and $p^* \in \partial^c \psi(z-x)$ such that if we put $w^* = y^* + z^* + p^*$ then $\|w^*\| \leq \varepsilon K$. Since $y^* \in \partial^c N(z)$, we must have $\langle y^*, z-x\rangle \geq N(z) - N(x) = K\|z-x\|$. The same trick for ψ ($\psi(0) = 0$) yields

$$\begin{array}{rcl}\langle z^*, x-z\rangle & = & \langle y^*, z-x\rangle + \langle p^*, z-x\rangle + \langle w^*, x-z\rangle \\ & \geq & K\|z-x\| - \|w^*\|.\|z-x\| + \psi(z-x) \\ & \geq & (1-\varepsilon)K\|z-x\| + \psi(z-x) \\ & > & \psi(z-x).\end{array}$$

We are left only to notice that $\varphi(x) = \varphi(\lambda(\lambda^{-1}x) + (1-\lambda).0) \leq \psi(x)$. □

Theorem 4.3. *Let $f \in \mathcal{C}_\varphi$. Then $\partial^{CR} f = \partial^\varphi f$ is a maximal φ-monotone operator.*

P r o o f. Suppose that $x \in X$, $x^* \in X^*$ and $x^* \notin \partial^\varphi f(x)$ and consider $g(u) = f(u) - \langle x^*, u\rangle$. The function $g + \varphi(\cdot - x)$ does not attain its minimum at the point x (otherwise $x^* \in \partial^\varphi f(x)$), so we can find $r < g(x)$ such that

$$r > \inf_{y \in X}\{g(y) + \varphi(y-x)\}.$$

It's easy to see that $\partial^{CR} g(y) = \partial^{CR} f(y) - x^*$ for all $y \in X$ and in particular $g \in \mathcal{C}_\varphi$. Therefore we are able to apply Lemma 4.2 to the function g at the point

x with r as above. We find $z \in X$, $z_1^* \in \partial^{CR} g(z)$, such that $\langle z_1^*, x - z\rangle > \varphi(x - z)$. But $z_1^* = z^* - x^*$, where $z^* \in \partial^{CR} f(z)$. Then $\langle z^* - x^*, z - x\rangle < -\varphi(z - x)$ and then $\mathrm{Gr}\partial^{CR} f \cup (x, x^*)$ is not φ-monotone, hence $\partial^\varphi f$ is maximal φ-monotone. $\square$

5. Integrability. In this section we show how the techniques previously developed can be applied to question of integrability of certain lower semicontinuous functions.

Definition 5.1. *The proper lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, where X is a Banach space, is called integrable if for each proper lower semicontinuous $g : X \to \mathbb{R} \cup \{+\infty\}$ such that $\partial^{CR} g \subset \partial^{CR} f$ there exists a constant $c \in \mathbb{R}$ such that*

$$f(x) = g(x) + c, \quad \forall x \in X.$$

Various topics connected with the integrability of locally Lipschitz functions are presented in the paper of Borwein and Moors [2].

Theorem 5.2. *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Assume that there exists a locally Lipschitz regular function $g : X \to \mathbb{R}$ such that $f + g \in \mathcal{C}_\varphi$. Then f is integrable.*

Before proceeding with the proof we list some straight corollaries.

Letting $g \equiv 0$ we see that each function in $\mathcal{C}_\varphi$ is integrable. For $\varphi \equiv 0$, i. e. f convex, this is a well known result of Rockafellar, see [12]. For $\varphi = \|\cdot\|^2$ this was established by Poliquin when X is finitedimensional and by Thibault and Zagrodny when X is an Hilbert space, see [11], [14].

As we showed (Theorem 3.9) each $g \in \mathcal{C}_\varphi$ with $\mathrm{dom} g = X$ is locally Lipschitz and regular, so each function in $\mathcal{C}_\varphi - \mathcal{C}_{\varphi_1}^{\mathrm{cont}}$, where $\mathcal{C}_{\varphi_1}^{\mathrm{cont}} = \{g \in \mathcal{C}_{\varphi_1} : \mathrm{dom} g = X\}$, is integrable. The case $\varphi = \varphi_1 = 0$, i. e. f is the difference of convex lower semicontinuous and convex continuous functions, was proved in [14] under the additional requirement X to be weak Asplund.

P r o o f o f T h e o r e m 5.2. We are going to derive the result from the simple case $X = \mathbb{R}$.

Lemma 5.3. *Let $f = v - g$, where $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and regular and $v : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ belongs to $\mathcal{C}_\varphi$. Assume that the proper lower semicontinuous function $u : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ satisfies*

$$\boldsymbol{\delta} u \subset \partial^{CR} v + \partial^{CR}(-g). \tag{5.1}$$

Then there is $c \in \mathbb{R}$ such that $f = u + c$.

Assume for a while that the Lemma 5.3 is proved and suppose that the statement of Theorem 5.2 is false. Then we can find a proper lower semicontinuous function $u : X \to \mathbb{R} \cup \{+\infty\}$ such that $\partial^{CR} u \subset \partial^{CR} f$ and $f - u$ is not constant. Fix $x_0 \in \mathrm{dom}\partial^{CR} u \subset \mathrm{dom}\partial^{CR} f \subset \mathrm{dom} f$ and put $c = f(x_0) - u(x_0)$. There is by our assumption $y_0 \neq x_0$ such that $f(y_0) \neq u(y_0) + c$. Let $h = \|y_0 - x_0\|^{-1}(y_0 - x_0)$ and for $t \in \mathbb{R}$

$$f_0\ , u_0\ , g_0,\ v_0\,(t) = f, u,\ g,\ v(x_0 + th)$$

respectively, where $v = f + g \in \mathcal{C}_\varphi$. Due to Theorem 3.9 $v_0 \in \mathcal{C}_{\varphi_h}$. There are $t_0 \in \mathbb{R}$, $p \in \boldsymbol{\delta} u_0(t_0)$ such that

$$p \notin \partial^{CR} v_0(t_0) + \partial^{CR}(-g_0)(t_0) \tag{5.2}$$

(if not Lemma 5.3 would imply $f_0(\|y_0 - x_0\|) = u(\|y_0 - x_0\|) + c$, i. e. $f(y_0) = u(y_0) + c$).

Since $\partial^{CR} u \in \Upsilon(u)$, there are $x_n \underset{n\to\infty}{\longrightarrow}_u z = x_0 + t_0 y_0$ and $p_n \in \partial^{CR} u(x_n)$ such that $\langle p_n, h\rangle \underset{n\to\infty}{\longrightarrow} p$ and $\|p_n\| d(x_n) \underset{n\to\infty}{\longrightarrow} 0$, where $d = d_h^{x_0}$. For each n

$$\partial^{CR} u(x_n) \subset \partial^{CR} f(x_n) \subset \partial^{CR} v(x_n) + \partial^{CR}(-g)(x_n)$$

(the right hand inclusion comes from Theorem 2.9.8 in [4]). So, there are $q_n \in \partial^{CR} v(x_n)$ and $r_n \in \partial^{CR}(-g)(x_n)$ such that $p_n = q_n + r_n$. The sequence $\{\langle r_n, h\rangle\}_{n=1}^\infty$ is bounded (recall that g is locally Lipschitz) and extracting if necessary a subsequence of $\{r_n\}$ we can assume that $\langle r_n, h\rangle \underset{n\to\infty}{\longrightarrow} r \in \mathbb{R}$. Since $\langle r_n, h\rangle \le (-g)^\circ(x_n, h)$ and the function $(-g)^\circ(\cdot, h)$ is upper semicontinuous, $r \le (-g)^\circ(z, h)$. But Proposition 2.1.1 from [4] claims that $(-g)^\circ(z, h) = g^\circ(z, -h)$. Since g is regular, $g^\circ(z, -h) = g'(z, -h)$. Similarly, $g'(z, -h) = (-g_0)^\circ(t_0, 1)$. So, $(-g)^\circ(z, h) = (-g_0)^\circ(t_0, 1)$ and we have

$$r \le (-g_0)^\circ(t_0, 1). \tag{5.3}$$

We use $\partial^{CR} v = \partial^\varphi v$ to write for $t > 0$:

$$\begin{aligned} v(z + th) &\ge v(x_n) + \langle q_n, z + th - x_n\rangle - \varphi(z + th - x_n) \\ &\ge v(x_n) + \langle q_n, z + th - x_n'\rangle - \|q_n\| d(x_n) - \varphi(z + th - x_n), \end{aligned}$$

where $x_n' \in x_0 + \mathbb{R}h$ is such that $\|x_n - x_n'\| = d(x_n)$. Note that $\|q_n\| d(x_n) \underset{n\to\infty}{\longrightarrow} 0$ because $\{\|r_n\|\}_{n=1}^\infty$ is a bounded sequence. Also, $\liminf_{n\to\infty} v(x_n) \ge v(z)$ since v is lower semicontinuous, $\varphi(z + th - x_n) \underset{n\to\infty}{\longrightarrow} \varphi(th)$ and for n large enough

$$\langle q_n, z + th - x_n'\rangle = \|z + th - x_n'\| \langle q_n, h\rangle \underset{n\to\infty}{\longrightarrow} qt,$$

where $q = p - r$. So, passing to the limit, $v_0(t_0 + t) \geq v_0(t_0) + qt - \varphi_h(t)$ and $(v_0)^\circ(t_0, 1) \geq q$. Compare with (5.3) to get

$$p = q + r \leq (v_0)^\circ(t_0, 1) + (-g_0)^\circ(t_0, 1).$$

Replacing h with $-h$ in the above computations one obtains

$$-p \leq (v_0)^\circ(t_0, -1) + (-g_0)^\circ(t_0, -1)$$

but both these mean that $p \in \partial^{CR} v_0(t_0) + \partial^{CR}(-g_0)(t_0)$, which is in contradiction with (5.2). $\square$

Proof of Lemma 5.3. Let $a = \inf \operatorname{dom} f$ and $b = \sup \operatorname{dom} f$. Then by (5.1) $\operatorname{dom}\boldsymbol{\delta} u \subset \operatorname{dom}\partial^{CR} f \subset [a, b]$ and $\operatorname{dom} u \subset [a, b]$, since $\operatorname{dom}\boldsymbol{\delta} u$ is dense in $\operatorname{dom} u$. If $a = b$ then $\operatorname{dom} f = \operatorname{dom} u = \{a\}$ (recall that both these are proper) and the conclusion is trivial.

Let now $a < b$. Since $\operatorname{dom} f = \operatorname{dom} v$, by Lemma 3.5 it follows that $(a, b) \subset \operatorname{dom} v$ and v is locally Lipschitz and regular on (a, b). Then the right hand side of (5.1) is locally bounded, so by Lemma 2.1 u is locally Lipschitz on (a, b).

Let U be a subset of full measure of (a, b) on which both v and g are differentiable. From the regularity of these functions it follows that $\forall x \in U \Rightarrow \partial^{CR} v(x) = v'(x)$ and $\partial^{CR}(-g)(x) = -g'(x)$. Also, the functions $v^\circ(\cdot, 1)$ and $(-g)^\circ(\cdot, 1)$ are continuous at each point of U, see [4].

One can see from Lemma 2.1 that for each $x \in (a, b)$

$$u^\circ(x, 1) = \lim_{\varepsilon \to 0} \sup\{p : p \in \boldsymbol{\delta} u(y),\ |y - x| < \varepsilon\}.$$

If now $x \in U$ then

$$\begin{aligned} u^\circ(x, 1) &\leq \limsup_{y \to x} \left(v^\circ(y, 1) + (-g)^\circ(y, 1)\right) \\ &= v'(x) - g'(x), \end{aligned}$$

where the condition (5.1) was used at the first step and the continuity of the generalized derivatives at the second. In a similar way we prove that $u^\circ(x, -1) \leq -v'(x) + g'(x)$, and then for each $x \in U$ we have $\partial^{CR} u(x) = v'(x) - g'(x)$. Finally, we see that $u'(x) = f'(x)$ almost everywhere on (a, b), which means that $f(x) - u(x) = c \in \mathbb{R}$ on (a, b).

The proof is finished if $(a, b) = \mathbb{R}$, so assume that $a, b \in \mathbb{R}$. Suppose that

$$u(b) < \liminf_{x \uparrow b} u(x).$$

Then, as it is easy to check, $\boldsymbol{\delta} u(b) = \mathbb{R}$ and using (5.1) and the boundedness of $\partial^{CR}(-g)(b)$ we can find $p_n \in \partial^{CR} v(b)$, such that $p_n \underset{n\to\infty}{\longrightarrow} -\infty$. But $p_n \in \partial^{\varphi} v(b)$, since $v \in \mathcal{C}_\varphi$, and then $u(2^{-1}(a+b)) \geq u(b) - p_n r - \varphi(r) \underset{n\to\infty}{\longrightarrow} \infty$, where $r = 2^{-1}(b-a)$. This contradiction and the semicontinuity of u implies that $u(b) = \liminf\limits_{x\uparrow b} u(x)$. Now it is clear how to prove that $f(b) = \liminf\limits_{x\uparrow b} f(x)$. Finally,

$$\begin{aligned} u(b) &= \liminf_{x\uparrow b} u(x) \\ &= \liminf_{x\uparrow b} (f(x) - c) \\ &= f(b) - c. \end{aligned}$$

In the same way we show that $f(a) = u(a) + c$ and the proof is completed. $\square$

We note that the above proof works if we only assume that g is locally Lipschitz and for any $x \in X, h \in S_X$ the map $\langle \partial^{CR} g(x+th), h\rangle$ is minimal cusco from $\mathbb{R}$ to $2^{\mathbb{R}}$, see for the definitions [2].

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