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ON THE DIFFERENCE OF 4-GONAL LINEAR SYSTEMS ON SOME CURVES

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Abstract. Let $C = (C, g^1_4)$ be a tetragonal curve. We consider the scrollar invariants e_1, e_2, e_3 of g^1_4. We prove that if $W^1_4(C)$ is a non-singular variety, then every $g^1_4 \in W^1_4(C)$ has the same scrollar invariants.

0. Introduction. Let C be a complete non-singular curve defined over an algebraically closed filed k with $\mathrm{char}(k) \neq 2$. Let $g = g(C)$ be the genus of C and let g^1_d be a base-point-free linear system on C of degree d and projective dimension 1. A pair $C = (C, \mathcal{O}(g^1_d))$ is called a d-gonal curve if C does not admit a linear system of degree $e < d$. If a $C = (C, \mathcal{O}(g^1_d))$ is a d-gonal curve, then g^1_d is a (base-point-free) complete linear system. Now we consider a pair $C = (C, \mathcal{O}(g^1_d))$ such that g^1_d is a complete base-point-free of degree d. Let ω_C be a canonical sheaf on C, let $\mathcal{L} = \mathcal{O}(g^1_d)$ and let $F_i = \Gamma(C, \omega_C \otimes \mathcal{L}^{\otimes -i})$. If $p : C \longrightarrow \mathbf{P^1}$ is the map which corresponds to g^1_d then $p_*\omega_C \cong \mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(e_{d-1}) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ and $F_i \cong \Gamma(\mathbb{P}^1, p_*\omega_C \otimes \mathcal{O}_{\mathbb{P}^1}(-i))$. The modules F_i $(i = 1, 2, \cdots)$ give a filtration,

$$F_0 \supset F_1 \supset \cdots \supset F_n \supset \cdots$$

and by the definition of $\{F_i\}_{i=0}^{\infty}$ we have injective maps

$$F_0/F_1 \hookrightarrow F_1/F_2 \hookrightarrow \cdots \hookrightarrow F_n/F_{n+1} \hookrightarrow \cdots.$$

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By Riemann-Roch's Theorem, $\dim F_0/F_1 = d-1$. Now we define the scrollar invariants $e_i = e_i(\mathcal{L})$ $(i = 1, \cdots d-1)$ by

$$e_i = e_i(\mathcal{L}) = \#\{j \in \mathbb{N}; \dim\,(F_{j-1}/F_j) \geq i\} - 1 \;\; (i = 1, 2, \cdots, d-1)$$

and we put $e_0 = e_0(\mathcal{L}) = 0$. Let $W^r_d(C)$ be a subscheme of a Picard variety $Pic^d(C)$ roughly defined as follows:

$$W^r_d(C) \overset{\text{as a set}}{=} \{\mathcal{L} \in Pic^d(C) | \dim\, \Gamma(C, \mathcal{L}) \geq r+1\}.$$

The precise definition is found in Arbarello, Cornalba, Griffiths, Harris [1] p. 176. If $C = (C, g^1_d)$ is a hyperelliptic curve or a trigonal curve, then the scrollar invariants of any $g^1_d \in W^1_d(C)$ depend only on the curve C. We now assume that $C = (C, g^1_d)$ is a tetragonal curve. If $C = (C, g^1_d)$ is an elliptic-hyperelliptic curve, then there is a $\pi : C \to E$ where E is an elliptic curve and $\deg \pi = 2$. Then $W^1_4(C) = \pi^* W^1_2(E)$. Hence the scrollar invariants of any $g^1_4 \in W^1_4(C)$ depend only on the curve C. If $g \leq 4$, then C is a trigonal curve. So we assume $5 \leq g$ and C is not an elliptic-hyperelliptic curve.

Definition A. *Let $C_1 \subset \mathbb{P}^2$ be a plane curve and let $P \in C_1$ be a double point. We call that P is an r-fold node if P is analytically isomorphic to the singularity at* (0,0) *of the curve $y^2 = x^{2r}$ in $\mathbb{A}^2$ and we call that P is an r-fold cusp if P is analytically isomorphic to the singularity at* (0,0) *of the curve $y^2 = x^{2r+1}$ in $\mathbb{A}^2$ (see Hartshorne* [9] *p.* 38 *Exercise* 5.14(*d*)).

Theorem A (Main Theorem). *Let C be a tetragonal curve of genus g, where $6 \leq g \leq 8$. Assume that C is not elliptic-hyperelliptic and $\#(W^1_4(C)) \geq 2$. For any $g^1_4 \in W^1_4(C)$, there is a divisor $D = D_{g^1_4}$ such that $|K_C - g^1_4 - D|$ gives a birational morphism $\rho = \rho_{g^1_4} : C \to C_1 \subset \mathbb{P}^2$, $\deg(C_1) = 6$ and every singular point of C_1 has multiplicity 2. Let $k^1_4 \in W^1_4(C)$. Then there is a $g^1_4 \in W^1_4(C)$ and a $P \in \mathrm{Sing}(C_1) = \mathrm{Sing}(\rho_{g^1_4}(C))$ such that k^1_4 is given by a cut out of lines which pass through P. And we have the following:*

I) *The following statements are equivalent:*
  1) *$k^1_4 \in W^1_4(C)$ is a reduced point*
  2) *P is an ordinary node or an ordinary cusp*
  3) *k^1_4 is of type* (1,1,1) *if $g = 6$,* (2,1,1) *if $g = 7$ and* (2,2,1) *if $g = 8$.*

II) *The following statements are equivalent:*
  1) *P is a 2-fold node or a 2-fold cusp*
  2) *k^1_4 is of type* (2,1,0) *if $g = 6$,* (2,2,0) *if $g = 7$ and* (3,1,1) *if $g = 8$.*

III) *The following statements are equivalent:*
  1) *P is a 3-fold node or a 3-fold cusp*
  2) *k^1_4 is of type* (3,1,0) *if $g = 7$.*

As a corollary of Theorem A, we have the following:

Corollary A. *Assume that C is a tetragonal curve and C is not an elliptic-hyperelliptic curve. If $g \geq 10$, then C has only one g^1_4. If $5 \leq g \leq 9$ and $W^1_4(C)$ is reduced, then there exist integers $e_1 \geq e_2 \geq e_3 \geq 0$ such that any $g^1_4 \in W^1_4(C)$ has e_1, e_2, e_3 for its scrollar invariants.*

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NOTATIONS

$\mathrm{char}(k)$**:** The characteristic of a field k

$\mathcal{O}_A$**:** The structure sheaf of a variety A

f^***:** The pull back defined by a morphism f

f_***:** The direct image defined by a morphism f

$\deg(f)$**:** The degree of a finite morphism f

$|\mathcal{L}|$**:** The complete linear system defined by an invertible sheaf $\mathcal{L}$

ϕ_V**:** The rational map defined by a linear system V

$\mathcal{O}_A(D)$**:** The invertible sheaf associated with a divisor D

$\Gamma(A, \mathcal{F})$**:** The global sections of a sheaf $\mathcal{F}$

K_A**:** A canonical divisor on a non-singular variety A

ω_A**:** The canonical invertible sheaf on a non-singular variety A

$\mathbb{P}(\mathcal{E})$**:** The projective bundle $\mathrm{Proj}(\bigoplus_{n=0}^{\infty} \mathrm{Sym}^n \mathcal{E})$ defined by a locally free sheaf $\mathcal{E}$ on a variety Y

1. Preliminary and Known Facts. Let C be a non-singular curve of genus g defined over an algebraically closed field k. Let g^1_d be one of a base-point-free linear system on C of degree d and projective dimension 1. We assume that C is a tetragonal curve, i.e. C admits a g^1_4 but does not admit a g^1_e for every $e < 4$. We know that if $g \leq 4$, then C is a hyperelliptic curve or a trigonal curve. So we assume that $g \geq 5$. The following results are proved in [10].

Theorem 1. *If $|(e_{d-1}+2)g_d^1|$ is birationally very ample, then*

$$e_{i-1} \leq e_i + e_{d-1} + 2$$

for any $i \in \mathbb{Z}/d\mathbb{Z}$.

Theorem 2. *Let e_1, e_2, e_3 and $g \geq 5$ be integers such that*

$$e_1 \leq e_2 + e_3 + 2, \quad e_2 \leq 2e_3 + 2, \quad e_1 \geq e_2 \geq e_3, \quad e_1 + e_2 + e_3 = g - 3,$$

then there is a tetragonal curve $C = (C, g_4^1)$ of genus g such that $\mathcal{O}(g_4^1)^{\otimes e_3+2}$ is birationally very ample and $e_1 = e_1(g_4^1), e_2 = e_2(g_4^1), e_3 = e_3(g_4^1)$.

Theorem 3. *Let $C = (C, g_4^1)$ be a tetragonal curve of genus g with scrollar invariants e_1, e_2, e_3. If $\mathcal{O}(g_4^1)^{\otimes e_3+2}$ is not birationally very ample, then there exists a curve $C = (C_1, h_2^1)$ of genus $e_3 + 1$ with a pencil of degree 2 and a map $\pi : C \to C_1$ of degree 2 such that $g_4^1 = \pi^*(h_2^1)$.*

Hence we have the following result.

Corollary 1. *Let e_1, e_2, e_3 and $g \geq 5$ be integers. Then there exists a tetragonal curve $C = (C, g_4^1)$ of genus g such that $e_1 = e_1(g_4^1), e_2 = e_2(g_4^1), e_3 = e_3(g_4^1)$ if and only if*

$$e_1 \leq e_2 + e_3 + 2, \quad e_1 \geq e_2 \geq e_3, \quad e_1 + e_2 + e_3 = g - 3.$$

We now assume that C is not elliptic-hyperelliptic. For g=5, we have the following result. Let $C \hookrightarrow \mathbb{P}^4$ be the canonical embedding. Let $\delta \cong \mathbb{P}^2$ be the linear system of quadrics in $\mathbb{P}^4$ containing C, Γ is the locus of quadrics of rank ≤ 4 and Γ' is the locus of quadrics of rank ≤ 3. We know the following:

Proposition 1. *If C is a tetragonal curve, then a general $Q \in \delta$ is non-singular.*

By Proposition 1, we have that $\Gamma \subset \mathbb{P}^2$ is a plane curve of degree 5. Let $\mathcal{L} \in W_4^1(C)$ and let $Q_{\mathcal{L}} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \mathcal{O}_{\mathbb{P}^1}(e_2) \oplus \mathcal{O}_{\mathbb{P}^1}(e_3)) \subset \mathbb{P}^4$, where $e_i = e_i(\mathcal{L})$. As $Q_{\mathcal{L}}$ is contained in Γ (see E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris [1] p. 240 Theorem 2.1), we have the morphism

$$\phi : W_4^1(C) \to \Gamma$$

given by

$$\phi(\mathcal{L}) = Q_{\mathcal{L}}.$$

Then we know the following Theorem:

Theorem 4. $(W^1_4(C))_{\mathrm{sing}} = \phi^{-1}(\Gamma') = \{\mathcal{L}|\mathcal{L} \in W^1_4(C), \mathcal{L}^{\otimes 2} \cong \omega_C\}$.

As a corollary of Theorem 4, we have the following:

Corollary 2. *Let C be a tetragonal curve of genus 5. If $W^1_4(C)$ is non-singular, then any $g^1_4 \in W^1_4(C)$ has same e_1, e_2, e_3.*

We now assume that C is a tetragonal curve C of genus 6 which is not elliptic-hyperelliptic. Then we know the following results. Let $\mathcal{L}$ be a tetragonal linear system (therefore $\mathcal{L}$ is a base-point-free linear system) on C. Then the line bundle $\omega_C \otimes \mathcal{L}^{\otimes -1}$ defines a base-point-free linear system of degree 6 and of projective dimension 2 on C because if $\omega_C \otimes \mathcal{L}^{\otimes -1}$ has a base point, then $\omega_C \otimes \mathcal{L}^{\otimes -1}$ defines a map $\phi : C \to C_0 \subset \mathbb{P}^2$ such that $\deg(\phi)\deg(C_0) \le 5$. If $\deg(\phi) \ge 2$, then C is a trigonal curve or a hyperelliptic curve. Therefore C has a singular plane curve model of degree ≤ 5. Hence C has a trigonal linear system or a hyperelliptic linear system because C_0 must have a singular point and the lines in $\mathbb{P}^2$ which pass through one of the singular points of C_0 induces a trigonal linear system or a hyperelliptic linear system. This is a contradiction. So $\omega_C \otimes \mathcal{L}^{\otimes -1}$ defines a base-point-free linear system. As we assume that C is not an elliptic-hyperelliptic curve, therefore $\deg(\phi) = 1$. So C has a singular plane curve model C_0 of degree 6, therefore the arithmetic genus $p_a(C_0)$=10. As C is not trigonal and not hyperelliptic, every singular point of C_0 is multiplicity 2. So C_0 has just 4 singular points (including infinitely near singular points).

Proposition 2. *Every member of $W^1_4(C)$ is given by a cut out of the lines in $\mathbb{P}^2$ which pass through one of the singular points of C_0 or is given by a cut out of conics in $\mathbb{P}^2$ which pass through 4 singular points (singular points of C_0 and its infinitely near singular points of C_0).*

Proof. See Griffiths, Harris [7] p. 210. □

Let $g^1_4, k^1_4 \in W^1_4(C)$ and let C_1 be the singular plane curve model defined by $|K_C - g^1_4|$. Assume that k^1_4 is given by a cut out of conics which pass through 4 singular points. Then we consider a singular plane curve model defined by $|K_C - k^1_4|$ and let h^1_4 be a tetragonal linear system given by a cut out of the lines. We consider a singular plain curve model defined by $|K_C - h^1_4|$. Then k^1_4 is given by a cut out of the lines. So we may assume that $k^1_4 \in W^1_4(C)$ is given by a cut out of the lines. Let $P \in \mathrm{Sing}(C_1)$ be a singular point corresponds to k^1_4. By the definition of (e_1, e_2, e_3), we have the following proposition.

Proposition 3. *The following conditions are equivalent:*
1) *$g^1_4 \in W^1_4(C)$ is a non-reduced point*
2) *g^1_4 is type* (2,1,0).
3) *P is a 2-fold node or a 2-fold cusp.*

So we have the following theorem:

Theorem 5. *Let C be of genus 6. If $W^1_4(C)$ is reduced of dimension 0, then every g^1_4 is of type* (1,1,1).

The following result is given by the adjunction formula on $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 1 (Riemann). *Let C be a Riemann surface of genus g, let f_1, f_2 be a meromorphic function on C of $\deg(f_1) = d_1, \deg(f_2) = d_2$ and $k(C) = k(f_1, f_2)$. Then $g \leq (d_1 - 1)(d_2 - 1)$.*

By Lemma 1, we have the following result.

Theorem 6. *Assume that C is a tetragonal curve and it is not an elliptic-hyperelliptic curve of genus g. If $g \geq 10$, then C has only one g^1_4.*

Proof. We assume that $\#(W^1_4(C)) \geq 2$. Then we can take distinct tetragonal (therefore base-point-free) linear systems g^1_4, h^1_4 on C. Let $\phi : C \to \mathbb{P}^1 \times \mathbb{P}^1$ be defined by $\phi = (g^1_4, h^1_4)$. Then $g^1_4 = p_1 \cdot \phi$ and $h^1_4 = p_2 \cdot \phi$ where p_1 and p_2 are projections. Therefore $\deg \phi$ =1,2 or 4. Let $C_0 = \phi(C)$. If $\deg \phi = 4$, then C_0 is a rational curve so $g^1_4 = h^1_4$. If $\deg \phi = 2$, then C_0 is linearly equivalent to $2l + 2m$ where $l = \mathrm{pt} \times \mathbb{P}^1$ and $m = \mathbb{P}^1 \times \mathrm{pt}$. Therefore C_0 is a rational curve or an elliptic curve by the adjunction formula. Therefore we have that $\deg(\phi) = 1$. Therefore we prove Theorem 6 by Lemma 1. □

The following theorem is found in E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris [1] (see p. 189 (4.2) Proposition).

Theorem 7. *Let $\mathcal{L} \in W^r_d(C) \setminus W^{r+1}_d(C)$. Then the tangent space $T_{\mathcal{L}}(W^r_d(C))$ is isomorphic to $(\mathrm{im}\mu_0)^{\perp} \subset H^1(C, \mathcal{O}_C)$ where $\mu : \Gamma(C, \mathcal{L}) \otimes \Gamma(C, \omega_C \otimes \mathcal{L}^{-1}) \to \Gamma(C, \omega_C)$ is the cup product map and $(\mathrm{im}\mu_0)^{\perp}$ denotes the complement space of $\mathrm{im}\mu_0 \subset \Gamma(C, \omega_C)$.*

The following is also found in E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris [1] (see p. 191 (5.1) Theorem and p. 193 (5.2) Theorem).

Theorem 8 (Martens-Mumford). *If $g \geq 3$, $2 \leq d \leq g - 1$ and $0 < 2r \leq d$, then $\dim W^r_d(C) \leq d - 2r$. If C is a non-hyperelliptic curve and $d \leq g - 2$, then $\dim W^r_d(C) \leq d - 2r - 1$ and if there is a component $X \subset W^r_d(C)$ such that $\dim X = d - 2r - 1$, then C is either trigonal, elliptic-hyperelliptic or smooth plane quintic.*

2. The proof of Main Theorem. We now prove Theorem A. By Proposition 3, we may assume that C is a non-singular curve of genus g=7 or 8. We have already assumed that C is a tetragonal curve i.e. neither hyperelliptic nor trigonal and admits a tetragonal (base-point-free) linear system g^1_4. Moreover we may assume that C is not an elliptic-hyperelliptic curve. Let $g^1_4, h^1_4 \in W^1_4(C)$

be such that $g^1_4 \neq h^1_4$. Let $\rho = (g^1_4, h^1_4) : C \to \mathbb{P}^1 \times \mathbb{P}^1$. As C is not an elliptic-hyperelliptic curve, so ρ is a birational morphism to its image. And every singular point of $\rho(C)$ has multiplicity 2 because $\rho(C) \subset \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ is of degree 8 and C is neither hyperelliptic, nor trigonal and nor elliptic-hyperelliptic, i.e. if $\rho(C)$ has a singular point P such that $\mathrm{mult}_P(\rho(C)) \geq 3$, then we have a singular plane curve model of degree ≤ 5 and such curve is hyperelliptic or trigonal or elliptic-hyperelliptic. Let $l = \mathrm{pt} \times \mathbb{P}^1$ and $m = \mathbb{P}^1 \times \mathrm{pt}$ and let $C_0 = \rho(C)$. Take one singular point $P \in C_0$ and take $l_1 \ni P$ and $m_1 \ni P$ such that $l \sim l_1$ and $m \sim m_1$ where $\sim$ means a linear equivalence. We consider a blowing-up $\pi_1 : T_1 \to \mathbb{P}^1 \times \mathbb{P}^1$ at P. Let l_0 be a proper transform of l_1, let m_0 be a proper transform of m_1, let F_0 be the exceptional divisor and let $\tilde{C}$ be a proper transform of C_0. We first assume that g=7. Then $\tilde{C}$ has one singular point Q. So we consider a blowing-up $\pi_2 : T_2 \to T_1$ at Q, let $S = T_2$, $\pi = \pi_1 \cdot \pi_2$, let $E_1 =$ the total transform of l_0, $E_2 =$ the total transform of m_0 and $E_3 =$ the exceptional divisor of π_2 and let $L = \pi_2^*(F_0 + l_0 + m_0)$. Let ϕ be the morphism defined by the complete linear system $|L|$. By the definition of S, the proper transform of C_0 in S is C and $C \sim 6L - 2E_1 - 2E_2 - 2E_3$. Moreover $C_1 = \phi(C)$ is a (singular) plane curve model of C such that $\deg(C_1) = 6$ and $\phi : C \to C_1 \subset \mathbb{P}^2$ is a normalization map. By elementary arguments, we have that g^1_4 is given by a cut out of the lines which pass through one of the singular point of $C_1 \subset \mathbb{P}^2$ because this linear system corresponds to the linear system $|L - E_1|$. As $K_C - g^1_4 \sim K_S + C - (L - E_1)|C$, $K_C - g^1_4 \sim K_S + C - (L - E_1)|C \sim 2L - E_2 - E_3|C$ and $\dim|K_C - g^1_4| = \dim|2L - E_2 - E_3| = 3$. Hence $|K_C - g^1_4|$ is birationally very ample but not very ample because $L - E_1 - E_2$ contracts to one point, so $|K_C - g^1_4 - (L - E_1 - E_2|C)|$ gives $\phi : C \to \mathbb{P}^2$ and $\phi(C)$ is a (singular) plane curve of C such that $\deg(\phi(C)) = 6$. We put $D = L - E_1 - E_2|C$. Let $k^1_4 \in W^1_4(C)$ be such that $g^1_4 \neq k^1_4$. Then $\dim|g^1_4 + k^1_4| = 3$. Hence $\dim|K_C - g^1_4 - k^1_4| = 1$. Therefore $\dim|K_C - g^1_4 - k^1_4 - D| = 0$ by the above. This implies that every $k^1_4 \in W^1_4(C)$ such that $k^1_4 \neq g^1_4$ is given by a cut out of the lines which pass through one of a singular points of $\phi(C)$. Now we assume that $g = 8$. We put $S = T_1$ and $\pi = \pi_1$, let $E_1 = m_0$, $E_2 = l_0$ and let $L = F_0 + l_0 + m_0$. Let ϕ be the morphism defined by the complete linear system $|L|$. By the definition of S, the proper transform of C_0 in S is C and $C \sim 6L - 2E_1 - 2E_2$. Moreover $C_1 = \phi(C)$ is also a (singular) degree 6 plane curve model of C and $\phi : C \to C_1 \subset \mathbb{P}^2$ is a normalization map and we have that g^1_4 is given by a cut out of the lines which pass through one of the singular point of $C_1 \subset \mathbb{P}^2$ because this linear system corresponds to the linear system $|L - E_1|$. As $K_C - g^1_4 \sim K_S + C - (L - E_1)|C$, $K_C - g^1_4 \sim K_S + C - (L - E_1)|C \sim 2L - E_2|C$ and $\dim|K_C - g^1_4| = \dim|2L - E_2| = 4$. Hence $|K_C - g^1_4|$ is birationally very ample but not very ample because E_1 contracts to one point, so $|K_C - g^1_4 - (E_1|C)|$ gives a birational morphism $C \to \mathbb{P}^3$ because C is not hyperelliptic, not trigonal and not elliptic-hyperelliptic. As

$K_C - g^1_4 - (E_1|C) \sim 2L - E_1 - E_2|C$, $\dim|K_C - g^1_4 - (E_1|C)| = \dim|2L - E_1 - E_2| = 3$ and $L - E_1 - E_2$ is contracted to one point by the linear system $|2L - E_1 - E_2|$, $C \to \mathbb{P}^3$ is not very ample. Therefore we have a morphism $\phi : C \to \mathbb{P}^2$ and $\phi(C)$ is a (singular) plane curve of C such that $\deg(\phi(C)) = 6$. We put $D = L - E_1 - E_2|C$. Let $k^1_4 \in W^1_4(C)$ such that $g^1_4 \neq k^1_4$. Then $\dim|g^1_4 + k^1_4| = 3$. Hence $\dim|K_C - g^1_4 - k^1_4| = 2$. Therefore $\dim|K_C - g^1_4 - k^1_4 - D| = 0$ by the above. This implies that every $k^1_4 \in W^1_4(C)$ such that $k^1_4 \neq g^1_4$ is given by a cut out of the lines which pass through one of a singular points of $\phi(C)$. □

We now prove the following lemma:

Lemma 2. *Let $C \to \phi(C) \subset \mathbb{P}^2$ be a singular plane curve model of C constructed as above. Let $P \in \phi(C) \subset \mathbb{P}^2$ be a singular point and let k^1_4 be a tetragonal linear system given by a cut out of the lines which pass through P. Then $(e_1, e_2, e_3) = (2,1,1)$, $(2,2,0)$ or $(3,1,0)$ if $g = 7$ and $(e_1, e_2, e_3) = (3,1,1)$ or $(2,2,1)$ if $g = 8$. And $k^1_4 \in W^1_4(C)$ is reduced point if and only if P is ordinary node or ordinary cusp. Moreover P is ordinary node or ordinary cusp if and only if $(e_1(k^1_4), e_2(k^1_4), e_3(k^1_4)) = (2,1,1)$ if g=7 and $(e_1(k^1_4), e_2(k^1_4), e_3(k^1_4)) = (2,2,1)$ if g=8. P is 2-fold node or 2-fold cusp if and only if $(e_1(k^1_4), e_2(k^1_4), e_3(k^1_4)) = (2,2,0)$ if g=7, $(e_1(k^1_4), e_2(k^1_4), e_3(k^1_4)) = (3,1,1)$ if g=8. P is 3-fold node or 3-fold cusp if and only if $(e_1(k^1_4), e_2(k^1_4), e_3(k^1_4)) = (3,1,0)$ if g=7.*

Proof. Let $e_i = e_i(k^1_4)$ (i=1,2,3). We first assume that g=7. By Corollary 1, the possibilities are $(e_1, e_2, e_3) = (2,2,0), (3,1,0), (2,1,1)$. By Theorem 8, we have that $\dim W^1_4(C) = 0$. Therefore $k^1_4 \in W^1_4(C)$ is reduced point if and only if k^1_4 is of type $(2,1,1)$ by Theorem 7. Hence we only have to prove that $\dim\Gamma(C, \mathcal{O}(2k^1_4)) = 3$ if and only if P is ordinary node or ordinary cusp. Let $\psi = \psi_1\psi_2\psi_3$ where $\psi_1 : S_1 \to \mathbb{P}^2$ is a blowing-up at $P_0 = P \in \mathbb{P}^2$, $\psi_2 : S_2 \to S_1$ is a blowing-up at $P_1 \in S_1$ and $\psi_3 : S \to S_2$ is a blowing-up at $P_2 \in S_2$ such that $F_1 = (\psi_2\psi_3)^*(F'_1)$, $F_2 = \psi_3^*(F'_2)$ where F'_1 is the exceptional divisor of ψ_1, F'_2 is the exceptional divisor of ψ_1 and F_3 is the exceptional divisor of ψ_3. As $k^1_4 = L - F_1|C$, so we have

$$0 \to \mathcal{O}(2L - 2F_1 - C) \to \mathcal{O}(2L - 2F_1) \to \mathcal{O}_C(2k^1_4) \to 0.$$

and $2L - 2F_1 - C \sim -4L + 2F_2 + 2F_3$ because $C \sim 6L - 2F_1 - 2F_2 - 2F_3$. As $4L - 2F_2 - 2F_3$ is linearly equivalent to some effective divisor, we have $h^0(S, \mathcal{O}(2L - 2F_1 - C)) = 0$. By Serre's duality, $h^2(S, \mathcal{O}(2L - 2F_1 - C)) = h^0(S, \mathcal{O}(L + F_1 - F_2 - F_3))$. By Definition A, P is an ordinary node or an ordinary cusp if and only if $P_1 \notin F'_1$, P is a 2-fold node or a 2-fold cusp if and only if $P_1 \in F'_1$ but $P_2 \notin \psi_2^* F'_1$ and P is a 3-fold node or a 3-fold cusp if and only if $P_1 \in F'_1$ and $P_2 \in \psi_2^* F'_1$. Hence we have that P is an ordinary node or ordinary cusp if and only if $h^0(S_1, \mathcal{O}(L + F_1 - F_2 - F_3)) = 1$ and P is an r-fold node or an r-fold cusp $(r = 2, 3)$ if and only if $h^0(S_1, \mathcal{O}(L + F_1 - F_2 - F_3)) = 2$. As $h^0(S, \mathcal{O}(2L - 2F_1)) = 3$

and $h^2(S,\mathcal{O}(2L-2F_1)) = h^0(S,\mathcal{O}(-5L+3F_1+F_2+F_3)) = 0$ (by Seerre's duality and $5L-3F_1-F_2-F_3$ is linearly equivalent to an effective divisor), we have $h^1(S,\mathcal{O}(2L-2F_1)) = 0$ by Riemann-Roch's Theorem (see Hartshorne [9] p. 362 Theorem 1.6). Therefore P is an ordinary node or an ordinary cusp if and only if $(e_1,e_2,e_3) = (2,1,1)$ and P is an r-fold node or an r-fold cusp ($r = 2,3$) if and only if $(e_1,e_2,e_3) = (3,1,0),(2,2,0)$. By the same calculation, we have that $(e_1,e_2,e_3) = (3,1,0)$ if and only if $h^2(S,\mathcal{O}(3L-3F_1-C)) = h^0(S,\mathcal{O}(2F_1-F_2-F_3)) = 1$. Hence $(e_1,e_2,e_3) = (3,1,0)$ if and only if P is a 3-fold node or a 3-fold cusp. We now assume that g=8. By Corollary 1, the possibilities are $(e_1,e_2,e_3) = (3,2,0),(3,1,1),(2,2,1)$. As $\dim W^1_4(C) = 0$. Therefore $k^1_4 \in W^1_4(C)$ is reduced point if and only if k^1_4 is of type (2,2,1) or (3,1,1) by Theorem 7. And $\dim \Gamma(C,\mathcal{O}(2k^1_4)) = 3$ if and only if P is an ordinary node or an ordinary cusp. So we first prove that $\dim \Gamma(C,\mathcal{O}(2k^1_4)) = 3$ if and only if k^1_4 is of type $(2,2,1)$. Let $\psi = \psi_1\psi_2$ where $\psi_1 : S_1 \to \mathbb{P}^2$ is a blowing-up at $P_0 = P \in \mathbb{P}^2$, $\psi_2 : S_2 \to S_1$ is a blowing-up at $P_1 \in S_1$ such that $F_1 = \psi_2^*(F_1')$ where F_1' is the exceptional divisor of ψ_1, F_2 is the exceptional divisor of ψ_1. As $k^1_4 = L - F_1|C$, so we have

$$0 \to \mathcal{O}(3L-3F_1-C) \to \mathcal{O}(3L-3F_1) \to \mathcal{O}_C(3k^1_4) \to 0.$$

and $3L-3F_1-C \sim -3L-F_1+2F_2$ because $C \sim 6L-2F_1-2F_2$. We have $h^0(S,\mathcal{O}(3L-3F_1-C)) = 0$ and by Serre's duality, $h^2(S,\mathcal{O}(3L-3F_1-C)) = h^0(S,\mathcal{O}(2F_1-F_2))$. As P is an ordinary node or an ordinary cusp, $h^0(S,\mathcal{O}(2F_1-F_2)) = 0$. Therefore we have that $h^1(S,\mathcal{O}(3L-3F_1-C)) = 1$. As $h^0(S,\mathcal{O}(3L-3F_1)) = 4$ and $h^2(S,\mathcal{O}(3L-3F_1)) = h^0(S,\mathcal{O}(-6L+4F_1+F_2)) = 0$, we have $h^1(S,\mathcal{O}(3L-3F_1)) = 0$ by Riemann-Roch's Theorem. Therefore we have that $\dim \Gamma(C,\mathcal{O}(3k^1_4)) = 5$. Hence k^1_4 is of type (2,2,1). And $(e_1,e_2,e_3) =$ (3,2,0) if and only if P is a 2-fold node or a 2-fold cusp by the same calculation. □

P r o o f o f C o r o l l a r y A. We now prove Corollary A. By Theorem 6, we may assume that C is a non-singular curve of genus $g = 7, 8$ or 9. We have already assumed that C is a tetragonal curve i.e. neither hyperelliptic nor trigonal and admits a tetragonal (base-point-free) linear system g^1_4. Moreover we assume that C is not an elliptic-hyperelliptic curve. If there is only one g^1_4 on C, there is nothing to prove. So we may assume that there are $g^1_4, h^1_4 \in W^1_4(C)$ such that $g^1_4 \neq h^1_4$. If $g = 7$ or 8, then Corollary A holds by Theorem A. Therefore the remaining case is g=9 case. But in this case, if $\#W^1_4(C) \geq 2$, then we have an embedding $C \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. In this case one easily proves that $h^0(C,\mathcal{O}(K_C - 3g^1_4)) = h^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-l+2m)) = 0$, thus the invariants of g^1_4 are (2,2,2). Therefore the e-numbers of any $k^1_4 \in W^1_4(C)$ are the same and equal to (2,2,2). This proves Corollary A. □

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