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ON THE DIFFERENCE OF 4-GONAL LINEAR SYSTEMS ON SOME CURVES

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Abstract. Let $C = (C, g^1_d)$ be a tetragonal curve. We consider the scrollar invariants $e_1, e_2, e_3$ of $g^1_d$. We prove that if $W^1_d(C)$ is a non-singular variety, then every $g^1_d \in W^1_d(C)$ has the same scrollar invariants.

0. Introduction. Let $C$ be a complete non-singular curve defined over an algebraically closed field $k$ with $\text{char}(k) \neq 2$. Let $g = g(C)$ be the genus of $C$ and let $g^1_d$ be a base-point-free linear system on $C$ of degree $d$ and projective dimension 1. A pair $C = (C, \mathcal{O}(g^1_d))$ is called a $d$-gonal curve if $C$ does not admit a linear system of degree $e < d$. If a $C = (C, \mathcal{O}(g^1_d))$ is a $d$-gonal curve, then $g^1_d$ is a (base-point-free) complete linear system. Now we consider a pair $C = (C, \mathcal{O}(g^1_d))$ such that $g^1_d$ is a complete base-point-free of degree $d$. Let $\omega_C$ be a canonical sheaf on $C$, let $\mathcal{L} = \mathcal{O}(g^1_d)$ and let $F_i = \Gamma(C, \omega_C \otimes \mathcal{L}^\otimes -i)$. If $p : C \to \mathbb{P}^1$ is the map which corresponds to $g^1_d$ then $p_* \omega_C \cong \mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(e_{d-1}) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ and $F_i \cong \Gamma(\mathbb{P}^1, p_* \omega_C \otimes \mathcal{O}_{\mathbb{P}^1}(-i))$. The modules $F_i$ $(i = 1, 2, \cdots)$ give a filtration, $F_0 \supset F_1 \supset \cdots \supset F_n \supset \cdots$ and by the definition of $\{F_i\}_{i \geq 0}$ we have injective maps

$$F_0/F_1 \hookrightarrow F_1/F_2 \hookrightarrow \cdots \hookrightarrow F_n/F_{n+1} \hookrightarrow \cdots.$$ 

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By Riemann-Roch’s Theorem, \( \dim F_0/F_1 = d - 1 \). Now we define the scrollar invariants \( e_i = e_i(\mathcal{L}) \) \( (i = 1, \ldots, d - 1) \) by

\[
e_i = e_i(\mathcal{L}) = \# \{ j \in \mathbb{N}; \dim (F_{j-1}/F_j) \geq i \} - 1 \quad (i = 1, 2, \ldots, d - 1)
\]

and we put \( e_0 = e_0(\mathcal{L}) = 0 \). Let \( W^r_d(C) \) be a subscheme of a Picard variety \( \text{Pic}^d(C) \) roughly defined as follows:

\[
W^r_d(C) \text{ as a set} = \{ \mathcal{L} \in \text{Pic}^d(C) | \dim \Gamma(C, \mathcal{L}) \geq r + 1 \}.
\]

The precise definition is found in Arbarello, Cornalba, Griffiths, Harris [1] p. 176. If \( C = (C, g_1^1) \) is a hyperelliptic curve or a trigonal curve, then the scrollar invariants of any \( g_1^1 \in W^1_d(C) \) depend only on the curve \( C \). We now assume that \( C = (C, g_2^1) \) is a tetragonal curve. If \( C = (C, g_3^1) \) is an elliptic-hyperelliptic curve, then there is a \( \pi : C \rightarrow E \) where \( E \) is an elliptic curve and \( \deg \pi = 2 \). Then \( W^1_d(C) = \pi^* W^1_2(E) \). Hence the scrollar invariants of any \( g_4^1 \in W^1_d(C) \) depend only on the curve \( C \). If \( g \leq 4 \), then \( C \) is a trigonal curve. So we assume \( 5 \leq g \) and \( C \) is not an elliptic-hyperelliptic curve.

**Definition A.** Let \( C_1 \subset \mathbb{P}^2 \) be a plane curve and let \( P \in C_1 \) be a double point. We call that \( P \) is an \( r \)-fold node if \( P \) is analytically isomorphic to the singularity at \((0,0)\) of the curve \( y^2 = x^2r \) in \( \mathbb{A}^2 \) and we call that \( P \) is an \( r \)-fold cusp if \( P \) is analytically isomorphic to the singularity at \((0,0)\) of the curve \( y^2 = x^{2r+1} \) in \( \mathbb{A}^2 \) (see Hartshorne [9] p. 38 Exercise 5.14(d)).

**Theorem A (Main Theorem).** Let \( C \) be a tetragonal curve of genus \( g \), where \( 6 \leq g \leq 8 \). Assume that \( C \) is not elliptic-hyperelliptic and \( \#(W^1_4(C)) \geq 2 \). For any \( g_4^1 \in W^1_4(C) \), there is a divisor \( D = D_{g_4^1} \) such that \( |K_C - g_4^1 - D| \) gives a birational morphism \( \rho = \rho_{g_4^1} : C \rightarrow C_1 \subset \mathbb{P}^2 \), \( \deg(C_1) = 6 \) and every singular point of \( C_1 \) has multiplicity 2. Let \( k_4^1 \in W^1_4(C) \). Then there is a \( g_4^1 \in W^1_4(C) \) and a \( P \in \text{Sing}(C_1) = \text{Sing}(\rho_{g_4^1}(C)) \) such that \( k_4^1 \) is given by a cut out of lines which pass through \( P \). And we have the following:

I) The following statements are equivalent:

1) \( k_4^1 \in W^1_4(C) \) is a reduced point
2) \( P \) is an ordinary node or an ordinary cusp
3) \( k_4^1 \) is of type \((1,1,1)\) if \( g = 6 \), \((2,1,1)\) if \( g = 7 \) and \((2,2,1)\) if \( g = 8 \).

II) The following statements are equivalent:

1) \( P \) is a 2-fold node or a 2-fold cusp
2) \( k_4^1 \) is of type \((2,1,0)\) if \( g = 6 \), \((2,2,0)\) if \( g = 7 \) and \((3,1,1)\) if \( g = 8 \).

III) The following statements are equivalent:

1) \( P \) is a 3-fold node or a 3-fold cusp
2) \( k_4^1 \) is of type \((3,1,0)\) if \( g = 7 \).
As a corollary of Theorem A, we have the following:

**Corollary A.** Assume that $C$ is a tetragonal curve and $C$ is not an elliptic-hyperelliptic curve. If $g \geq 10$, then $C$ has only one $g^1_4$. If $5 \leq g \leq 9$ and $W^1_4(C)$ is reduced, then there exist integers $e_1 \geq e_2 \geq e_3 \geq 0$ such that any $g^1_4 \in W^1_4(C)$ has $e_1, e_2, e_3$ for its scrollar invariants.

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**Notations**

- $\text{char}(k)$: The characteristic of a field $k$
- $\mathcal{O}_A$: The structure sheaf of a variety $A$
- $f^*$: The pull back defined by a morphism $f$
- $f_*$: The direct image defined by a morphism $f$
- $\deg(f)$: The degree of a finite morphism $f$
- $|\mathcal{L}|$: The complete linear system defined by an invertible sheaf $\mathcal{L}$
- $\phi_V$: The rational map defined by a linear system $V$
- $\mathcal{O}_A(D)$: The invertible sheaf associated with a divisor $D$
- $\Gamma(A, \mathcal{F})$: The global sections of a sheaf $\mathcal{F}$
- $K_A$: A canonical divisor on a non-singular variety $A$
- $\omega_A$: The canonical invertible sheaf on a non-singular variety $A$
- $\mathbb{P}(\mathcal{E})$: The projective bundle $\text{Proj}(\bigoplus_{n=0}^{\infty} \text{Sym}^n \mathcal{E})$ defined by a locally free sheaf $\mathcal{E}$ on a variety $Y$

1. **Preliminary and Known Facts.** Let $C$ be a non-singular curve of genus $g$ defined over an algebraically closed field $k$. Let $g^1_d$ be one of a base-point-free linear system on $C$ of degree $d$ and projective dimension 1. We assume that $C$ is a tetragonal curve, i.e. $C$ admits a $g^1_4$ but does not admit a $g^1_e$ for every $e < 4$. We know that if $g \leq 4$, then $C$ is a hyperelliptic curve or a trigonal curve. So we assume that $g \geq 5$. The following results are proved in [10].
Theorem 1. If \(|(e_{d-1} + 2)g_d^1|\) is birationally very ample, then
\[e_{i-1} \leq e_i + e_{d-1} + 2\]
for any \(i \in \mathbb{Z}/d\mathbb{Z}\).

Theorem 2. Let \(e_1, e_2, e_3\) and \(g \geq 5\) be integers such that
\[e_1 \leq e_2 + e_3 + 2, \quad e_2 \leq 2e_3 + 2, \quad e_1 \geq e_2 \geq e_3, \quad e_1 + e_2 + e_3 = g - 3,\]
then there is a tetragonal curve \(C = (C, g_1^1)\) of genus \(g\) such that \(O(g_1^1)^{\otimes e_3 + 2}\) is birationally very ample and \(e_1 = e_1(g_1^1), e_2 = e_2(g_1^1), e_3 = e_3(g_1^1)\).

Theorem 3. Let \(C = (C, g_1^1)\) be a tetragonal curve of genus \(g\) with scrollar invariants \(e_1, e_2, e_3\). If \(O(g_1^1)^{\otimes e_3 + 2}\) is not birationally very ample, then there exists a curve \(C = (C_1, h_2^1)\) of genus \(e_3 + 1\) with a pencil of degree 2 and a map \(\pi : C \to C_1\) of degree 2 such that \(g_1^1 = \pi^*(h_2^1)\).

Hence we have the following result.

Corollary 1. Let \(e_1, e_2, e_3\) and \(g \geq 5\) be integers. Then there exists a tetragonal curve \(C = (C, g_1^1)\) of genus \(g\) such that \(e_1 = e_1(g_1^1), e_2 = e_2(g_1^1), e_3 = e_3(g_1^1)\) if and only if
\[e_1 \leq e_2 + e_3 + 2, \quad e_1 \geq e_2 \geq e_3, \quad e_1 + e_2 + e_3 = g - 3.\]

We now assume that \(C\) is not elliptic-hyperelliptic. For \(g=5\), we have the following result. Let \(C \hookrightarrow \mathbb{P}^4\) be the canonical embedding. Let \(\delta \cong \mathbb{P}^2\) be the linear system of quadrics in \(\mathbb{P}^4\) containing \(C\), \(\Gamma\) is the locus of quadrics of rank \(\leq 4\) and \(\Gamma'\) is the locus of quadrics of rank \(\leq 3\). We know the following:

Proposition 1. If \(C\) is a tetragonal curve, then a general \(Q \in \delta\) is non-singular.

By Proposition 1, we have that \(\Gamma \subset \mathbb{P}^2\) is a plane curve of degree 5. Let \(\mathcal{L} \in W_1^1(C)\) and let \(Q_\mathcal{L} = \mathbb{P}(O_{\mathbb{P}^1}(e_1) \oplus O_{\mathbb{P}^1}(e_2) \oplus O_{\mathbb{P}^1}(e_3)) \subset \mathbb{P}^4\), where \(e_i = e_i(\mathcal{L})\). As \(Q_\mathcal{L}\) is contained in \(\Gamma\) (see E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris [1] p. 240 Theorem 2.1), we have the morphism
\[\phi : W_1^1(C) \to \Gamma\]
given by
\[\phi(\mathcal{L}) = Q_\mathcal{L}.\]
Then we know the following Theorem:
On the difference of 4-gonal linear systems on some curves

Theorem 4. \((W_4^1(C))_{\text{sing}} = \phi^{-1}(\Gamma') = \{\mathcal{L} | \mathcal{L} \in W_4^1(C), \mathcal{L}^\otimes 2 \cong \omega_C\}\).

As a corollary of Theorem 4, we have the following:

Corollary 2. Let \(C\) be a tetragonal curve of genus 5. If \(W_4^1(C)\) is non-singular, then any \(g_4^1 \in W_4^1(C)\) has same \(e_1, e_2, e_3\).

We now assume that \(C\) is a tetragonal curve \(C\) of genus 6 which is not elliptic-hyperelliptic. Then we know the following results. Let \(\mathcal{L}\) be a tetragonal linear system (therefore \(\mathcal{L}\) is a base-point-free linear system) on \(C\). Then the line bundle \(\omega_C \otimes \mathcal{L}^{\otimes -1}\) defines a base-point-free linear system of degree 6 and of projective dimension 2 on \(C\) because if \(\omega_C \otimes \mathcal{L}^{\otimes -1}\) has a base point, then \(\omega_C \otimes \mathcal{L}^{\otimes -1}\) defines a map \(\phi : C \to C_0 \subset \mathbb{P}^2\) such that \(\deg(\phi)\deg(C_0) \leq 5\). If \(\deg(\phi) \geq 2\), then \(C\) is a trinodal curve or a hyperelliptic curve. Therefore \(C\) has a singular plane curve model of degree \(\leq 5\). Hence \(C\) has a trinodal linear system or a hyperelliptic linear system because \(C_0\) must have a singular point and the lines in \(\mathbb{P}^2\) which pass through one of the singular points of \(C_0\) induces a trinodal linear system or a hyperelliptic linear system. This is a contradiction. So \(\omega_C \otimes \mathcal{L}^{\otimes -1}\) defines a base-point-free linear system. As we assume that \(C\) is not an elliptic-hyperelliptic curve, therefore \(\deg(\phi) = 1\). So \(C\) has a singular plane curve model \(C_0\) of degree 6, therefore the arithmetic genus \(p_a(C_0) = 10\). As \(C\) is not trinodal and not hyperelliptic, every singular point of \(C_0\) is multiplicity 2. So \(C_0\) has just 4 singular points (including infinitely near singular points).

Proposition 2. Every member of \(W_4^1(C)\) is given by a cut out of the lines in \(\mathbb{P}^2\) which pass through one of the singular points of \(C_0\) or is given by a cut out of conics which pass through 4 singular points (singular points of \(C_0\) and its infinitely near singular points of \(C_0\)).


Let \(g_4^1, k_4^1 \in W_4^1(C)\) and let \(C_1\) be the singular plane curve model defined by \(|K_C - g_4^1|\). Assume that \(k_4^1\) is given by a cut out of conics which pass through 4 singular points. Then we consider a singular plane curve model defined by \(|K_C - k_4^1|\) and let \(h_4^1\) be a tetragonal linear system given by a cut out of the lines. We consider a singular plain curve model defined by \(|K_C - h_4^1|\). Then \(k_4^1\) is given by a cut out of the lines. So we may assume that \(k_4^1 \in W_4^1(C)\) is given by a cut out of the lines. Let \(P \in \text{Sing}(C_1)\) be a singular point corresponds to \(k_4^1\). By the definition of \((e_1, e_2, e_3)\), we have the following proposition.

Proposition 3. The following conditions are equivalent:
1) \(g_4^1 \in W_4^1(C)\) is a non-reduced point
2) \(g_4^1\) is type \((2,1,0)\).
3) \(P\) is a 2-fold node or a 2-fold cusp.
So we have the following theorem:

**Theorem 5.** Let $C$ be of genus 6. If $W_4^1(C)$ is reduced of dimension 0, then every $g_4^1$ is of type $(1,1,1)$.

The following result is given by the adjunction formula on $\mathbb{P}^1 \times \mathbb{P}^1$.

**Lemma 1 (Riemann).** Let $C$ be a Riemann surface of genus $g$, let $f_1, f_2$ be a meromorphic function on $C$ of $\deg(f_1) = d_1, \deg(f_2) = d_2$ and $k(C) = k(f_1, f_2)$. Then $g \leq (d_1 - 1)(d_2 - 1)$.

By Lemma 1, we have the following result.

**Theorem 6.** Assume that $C$ is a tetragonal curve and it is not an elliptic-hyperelliptic curve of genus $g$. If $g \geq 10$, then $C$ has only one $g_4^1$.

**Proof.** We assume that $\#(W_4^1(C)) \geq 2$. Then we can take distinct tetragonal (therefore base-point-free) linear systems $g_4^1, h_4^1$ on $C$. Let $\phi : C \to \mathbb{P}^1 \times \mathbb{P}^1$ be defined by $\phi = (g_4^1, h_4^1)$. Then $g_4^1 = p_1 \cdot \phi$ and $h_4^1 = p_2 \cdot \phi$ where $p_1$ and $p_2$ are projections. Therefore $\deg \phi = 1, 2$ or 4. Let $C_0 = \phi(C)$. If $\deg \phi = 4$, then $C_0$ is a rational curve so $g_4^1 = h_4^1$. If $\deg \phi = 2$, then $C_0$ is linearly equivalent to $2l + 2m$ where $l = pt \times \mathbb{P}^1$ and $m = \mathbb{P}^1 \times pt$. Therefore $C_0$ is a rational curve or an elliptic curve by the adjunction formula. Therefore we have that $\deg(\phi) = 1$. Therefore we prove Theorem 6 by Lemma 1. □

The following theorem is found in E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris [1] (see p. 189 (4.2) Proposition).

**Theorem 7.** Let $\mathcal{L} \in W_d^r(C) \setminus W_d^{r+1}(C)$. Then the tangent space $T_{\mathcal{L}}(W^r_d(C))$ is isomorphic to $(\text{im} \mu_0)^{\perp} \subset H^1(C, \mathcal{O}_C)$ where $\mu : \Gamma(C, \mathcal{L}) \otimes \Gamma(C, \omega_C \otimes \mathcal{L}^{-1}) \to \Gamma(C, \omega_C)$ is the cup product map and $(\text{im} \mu_0)^{\perp}$ denotes the complement space of $\text{im} \mu_0 \subset \Gamma(C, \omega_C)$.

The following is also found in E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris [1] (see p. 191 (5.1) Theorem and p. 193 (5.2) Theorem).

**Theorem 8 (Martens-Mumford).** If $g \geq 3, 2 \leq d \leq g - 1$ and $0 < 2r \leq d$, then $\dim W_d^r(C) \leq d - 2r$. If $C$ is a non-hyperelliptic curve and $d \leq g - 2$, then $\dim W_d^r(C) \leq d - 2r - 1$ and if there is a component $X \subset W_d^r(C)$ such that $\dim X = d - 2r - 1$, then $C$ is either trigonal, elliptic-hyperelliptic or smooth plane quintic.

2. The proof of Main Theorem. We now prove Theorem A. By Proposition 3, we may assume that $C$ is a non-singular curve of genus $g = 7$ or 8. We have already assumed that $C$ is a tetragonal curve i.e. neither hyperelliptic nor trigonal and admits a tetragonal (base-point-free) linear system $g_4^1$. Moreover we may assume that $C$ is not an elliptic-hyperelliptic curve. Let $g_4^1, h_4^1 \in W_4^1(C)$
be such that \( g_4^1 \neq h_4^1 \). Let \( \rho = (g_4^1, h_4^1) : C \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \). As \( C \) is not an elliptic-hyperelliptic curve, so \( \rho \) is a birational morphism to its image. And every singular point of \( \rho(C) \) has multiplicity 2 because \( \rho(C) \subset \mathbb{P}^1 \times \mathbb{P}^1 \leftarrow \mathbb{P}^3 \) is of degree 8 and \( C \) is neither hyperelliptic, nor trigonal and nor elliptic-hyperelliptic, i.e. if \( \rho(C) \) has a singular point \( P \) such that \( \text{mult}_P(\rho(C)) \geq 3 \), then we have a singular plane curve model of degree \( \leq 5 \) and such curve is hyperelliptic or trigonal or elliptic-hyperelliptic. Let \( l = pt \times \mathbb{P}^1 \) and \( m = \mathbb{P}^1 \times pt \) and let \( C_0 = \rho(C) \). Take one singular point \( P \in C_0 \) and take \( l_1 \ni P \) and \( m_1 \ni P \) such that \( l \sim l_1 \) and \( m \sim m_1 \) where \( \sim \) means a linear equivalence. We consider a blowing-up \( \pi_1 : T_1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \) at \( P \). Let \( l_0 \) be a proper transform of \( l_1 \), let \( m_0 \) be a proper transform of \( m_1 \), let \( F_0 \) be the exceptional divisor and let \( \tilde{C} \) be a proper transform of \( C_0 \). We first assume that \( g = 7 \). Then \( \tilde{C} \) has one singular point \( Q \). So we consider a blowing-up \( \pi_2 : T_2 \rightarrow T_1 \) at \( Q \), let \( S = T_2 \), \( \pi_1 = \pi_1 \cdot \pi_2 \), let \( E_1 = \text{the total transform of } l_0, E_2 = \text{the total transform of } m_0 \) and \( E_3 = \text{the exceptional divisor of } \pi_2 \) and let \( L = \pi_2^*(F_0 + l_0 + m_0) \). Let \( \phi \) be the morphism defined by the complete linear system \([L]|\). By the definition of \( S \), the proper transform of \( C_0 \) in \( S = C \) and \( C \sim 6L - 2E_1 - 2E_2 - 2E_3 \). Moreover \( C_1 = \phi(C) \) is a (singular) plane curve model of \( C \) such that \( \text{deg}(C_1) = 6 \) and \( \phi : C \rightarrow C_1 \subset \mathbb{P}^2 \) is a normalization map. By elementary arguments, we have that \( g_4^1 \) is given by a cut out of the lines which pass through one of the singular point of \( C_1 \subset \mathbb{P}^2 \) because this linear system corresponds to the linear system \([L - E_1]|\). As \( K_C - g_4^1 \sim K_S + C - (L - E_1)|C, K_C - g_4^1 \sim K_S + C - (L - E_1)|C \sim 2L - E_2 - E_3|C \) and \( \dim[K_C - g_4^1] = \dim[2L - E_2 - E_3] = 3 \). Hence \( |K_C - g_4^1| \) is birationally very ample but not very ample because \( L - E_1 - E_2 \) contracts to one point, so \( |K_C - g_4^1 - (L - E_1 - E_2)|C) \) gives \( \phi : C \rightarrow \mathbb{P}^2 \) and \( \phi(C) \) is a (singular) plane curve of \( C \) such that \( \text{deg}(\phi(C)) = 6 \). We put \( D = L - E_1 - E_2|C \). Let \( k_4^1 \in W_4^1(C) \) be such that \( g_4^1 \neq k_4^1 \). Then \( \dim[g_4^1 + k_4^1] = 3 \). Hence \( \dim[K_C - g_4^1 - k_4^1] = 1 \). Therefore \( \dim[K_C - g_4^1 - k_4^1 - D] = 0 \) by the above. This implies that every \( k_4^1 \in W_4^1(C) \) such that \( k_4^1 \neq g_4^1 \) is given by a cut out of the lines which pass through one of a singular points of \( \phi(C) \). Now we assume that \( g = 8 \). We put \( S = T_1 \) and \( \pi = \pi_1 \), let \( E_1 = m_0, E_2 = l_0 \) and let \( L = F_0 + l_0 + m_0 \). Let \( \phi \) be the morphism defined by the complete linear system \([L]|\). By the definition of \( S \), the proper transform of \( C_0 \) in \( S = C \) and \( C \sim 6L - 2E_1 - 2E_2 \). Moreover \( C_1 = \phi(C) \) is also a (singular) degree 6 plane curve model of \( C \) and \( \phi : C \rightarrow C_1 \subset \mathbb{P}^2 \) is a normalization map and we have that \( g_4^1 \) is given by a cut out of the lines which pass through one of the singular point of \( C_1 \subset \mathbb{P}^2 \) because this linear system corresponds to the linear system \([L - E_1]|\). As \( K_C - g_4^1 \sim K_S + C - (L - E_1)|C, K_C - g_4^1 \sim K_S + C - (L - E_1)|C \sim 2L - E_2|C \) and \( \dim[K_C - g_4^1] = \dim[2L - E_2] = 4 \). Hence \( |K_C - g_4^1| \) is birationally very ample but not very ample because \( E_1 \) contracts to one point, so \( |K_C - g_4^1 - (E_1)|C \) gives a birational morphism \( C \rightarrow \mathbb{P}^3 \) because \( C \) is not hyperelliptic, not trigonal and not elliptic-hyperelliptic. As
$K_C - g_1^1 - (E_1|C) \sim 2L - E_1 - E_2|C$, \( \dim[K_C - g_1^1 - (E_1|C)] = \dim[2L - E_1 - E_2] = 3 \) and \( L - E_1 - E_2 \) is contracted to one point by the linear system \( |2L - E_1 - E_2| \). \( C \to \mathbb{P}^3 \) is not very ample. Therefore we have a morphism \( \phi : C \to \mathbb{P}^2 \) and \( \phi(C) \) is a (singular) plane curve of \( C \) such that \( \deg(\phi(C)) = 6 \). We put \( D = L - E_1 - E_2|C \). Let \( k_4^1 \in W_4^1(C) \) such that \( g_4^1 \neq k_4^1 \). Then \( \dim[g_4^1 + k_4^1] = 3 \). Hence \( \dim[K_C - g_4^1 - k_4^1] = 2 \). Therefore \( \dim[K_C - g_4^1 - k_4^1 - D] = 0 \) by the above. This implies that every \( k_4^1 \in W_4^1(C) \) such that \( k_4^1 \neq g_4^1 \) is given by a cut out of the lines which pass through one of a singular points of \( \phi(C) \). □

We now prove the following lemma:

**Lemma 2.** Let \( C \to \phi(C) \subset \mathbb{P}^2 \) be a singular plane curve model of \( C \) constructed as above. Let \( P \in \phi(C) \subset \mathbb{P}^2 \) be a singular point and let \( k_4^1 \) be a tetragonal linear system given by a cut out of the lines which pass through \( P \). Then \( (e_1, e_2, e_3) = (2,1,1), (2,2,0) \) or \( (3,1,0) \) if \( g = 7 \) and \( (e_1, e_2, e_3) = (3,1,1) \) or \( (2,2,1) \) if \( g = 8 \). And \( k_4^1 \in W_4^1(C) \) is reduced point if and only if \( P \) is ordinary node or ordinary cusp. Moreover \( P \) is ordinary node or ordinary cusp if and only if \( (e_1(k_4^1), e_2(k_4^1), e_3(k_4^1)) = (2,1,1) \) if \( g = 7 \) and \( (e_1(k_4^1), e_2(k_4^1), e_3(k_4^1)) = (2,2,1) \) if \( g = 8 \). \( P \) is 2-fold node or 2-fold cusp if and only if \( (e_1(k_4^1), e_2(k_4^1), e_3(k_4^1)) = (2,2,0) \) if \( g = 7 \), \( (e_1(k_4^1), e_2(k_4^1), e_3(k_4^1)) = (3,1,1) \) if \( g = 8 \). \( P \) is 3-fold node or 3-fold cusp if and only if \( (e_1(k_4^1), e_2(k_4^1), e_3(k_4^1)) = (3,1,0) \) if \( g = 7 \).

**Proof.** Let \( e_i = e_i(k_4^1) \) \( (i=1,2,3) \). We first assume that \( g = 7 \). By Corollary 1, the possibilities are \( (e_1, e_2, e_3) = (2,2,0), (3,1,0), (2,1,1) \). By Theorem 8, we have that \( \dim W_4^1(C) = 0 \). Therefore \( k_4^1 \in W_4^1(C) \) is reduced point if and only if \( k_4^1 \) is of type \( (2,1,1) \) by Theorem 7. Hence we only have to prove that \( \dim \Gamma(C, \mathcal{O}(2k_4^1)) = 3 \) if and only if \( P \) is ordinary node or ordinary cusp. Let \( \psi = \psi_1\psi_2\psi_3 \) where \( \psi_1 : S_1 \to \mathbb{P}^2 \) is a blowing-up at \( P_0 = P \in \mathbb{P}^2 \), \( \psi_2 : S_2 \to S_1 \) is a blowing-up at \( P_1 \in S_1 \) and \( \psi_3 : S \to S_2 \) is a blowing-up at \( P_2 \in S_2 \) such that \( F_1 = (\psi_1(\psi_3)^*(F'_1)), F_2 = \psi_2(\psi_3)^*(F'_2) \) where \( F'_1 \) is the exceptional divisor of \( \psi_1 \), \( F'_2 \) is the exceptional divisor of \( \psi_1 \) and \( F_3 \) is the exceptional divisor of \( \psi_3 \). As \( k_4^1 = L - F_1|C \), so we have

\[
0 \to \mathcal{O}(2L - 2F_1 - C) \to \mathcal{O}(2L - 2F_1) \to \mathcal{O}_C(2k_4^1) \to 0.
\]

and \( 2L - 2F_1 - C \sim -4L + 2F_2 + 2F_3 \) because \( C \sim 6L - 2F_1 - 2F_2 - 2F_3 \). As \( 4L - 2F_2 - 2F_3 \) is linearly equivalent to some effective divisor, we have \( h^0(S, \mathcal{O}(2L - 2F_1 - C)) = 0 \). By Serre’s duality, \( h^2(S, \mathcal{O}(2L - 2F_1 - C)) = h^0(S, \mathcal{O}(L + F_1 - F_2 - F_3)) \). By Definition A, \( P \) is an ordinary node or an ordinary cusp if and only if \( P \notin F_1' \), \( P \) is a 2-fold node or a 2-fold cusp if and only if \( P \in F_1' \) but \( P \neq \psi_2F_1' \) and \( P \) is a 3-fold node or a 3-fold cusp if and only if \( P \in F_1' \) and \( P \neq \psi_2F_1' \). Hence we have that \( P \) is an ordinary node or ordinary cusp if and only if \( h^0(S_1, \mathcal{O}(L + F_1 - F_2 - F_3)) = 1 \) and \( P \) is an \( r \)-fold node or an \( r \)-fold cusp \( (r = 2, 3) \) if and only if \( h^0(S_1, \mathcal{O}(L + F_1 - F_2 - F_3)) = 2 \). As \( h^0(S, \mathcal{O}(2L - 2F_1)) = 3 \)
and $h^2(S, \mathcal{O}(2L - 2F_1)) = h^0(S, \mathcal{O}(-5L + 3F_1 + F_2 + F_3)) = 0$ (by Serre’s duality and $5L - 3F_1 - F_2 - F_3$ is linearly equivalent to an effective divisor), we have $h^1(S, \mathcal{O}(2L - 2F_1)) = 0$ by Riemann-Roch’s Theorem (see Hartshorne [9] p. 362 Theorem 1.6). Therefore $P$ is an ordinary node or an ordinary cusp if and only if $(e_1, e_2, e_3) = (2,1,1)$ and $P$ is an $r$-fold node or an $r$-fold cusp ($r = 2$, 3) if and only if $(e_1, e_2, e_3) = (3,1,0), (2,2,0)$. By the same calculation, we have that $(e_1, e_2, e_3) = (3,1,0)$ if and only if $h^2(S, \mathcal{O}(3L - 3F_1 - C)) = h^0(S, \mathcal{O}(2F_1 - F_2 - F_3)) = 1$. Hence $(e_1, e_2, e_3) = (3,1,0)$ if and only if $P$ is a 3-fold node or a 3-fold cusp. We now assume that $g = 8$. By Corollary 1, the possibilities are $(e_1, e_2, e_3) = (3,2,0), (3,1,1), (2,2,1)$. As $\dim \Gamma^1(C, \mathcal{O}(k_4^1)) = 0$. Therefore $k_4^1 \in \mathcal{W}^1_4(C)$ is reduced point if and only if $k_4^1$ is of type $(2,2,1)$ or $(3,1,1)$ by Theorem 7. And $\dim \Gamma^1(C, \mathcal{O}(2k_4^1)) = 3$ if and only if $P$ is an ordinary node or an ordinary cusp. So we first prove that $\dim \Gamma^1(C, \mathcal{O}(2k_4^1)) = 3$ if and only if $k_4^1$ is of type $(2,2,1)$. Let $\psi = \psi_1 \psi_2$ where $\psi_1 : S_1 \rightarrow \mathbb{P}^2$ is a blowing-up at $P_0 = P \in \mathbb{P}^2$, $\psi_2 : S_2 \rightarrow S_1$ is a blowing-up at $P_1 \in S_1$ such that $F_1 = \psi_2^*(F_1')$ where $F_1'$ is the exceptional divisor of $\psi_1$, $F_2$ is the exceptional divisor of $\psi_1$. As $k_4^1 = L - F_1 | C$, so we have

$$0 \rightarrow \mathcal{O}(3L - 3F_1 - C) \rightarrow \mathcal{O}(3L - 3F_1) \rightarrow \mathcal{O}_C(3k_4^1) \rightarrow 0.$$  

and $3L - 3F_1 - C \sim -3L - F_1 + 2F_2$ because $C \sim 6L - 2F_1 - 2F_2$. We have $h^0(S, \mathcal{O}(3L - 3F_1 - C)) = 0$ and by Serre’s duality, $h^2(S, \mathcal{O}(3L - 3F_1 - C)) = h^0(S, \mathcal{O}(2F_1 - F_2)) = 0$. Therefore we have that $h^1(S, \mathcal{O}(3L - 3F_1 - C)) = 1$. As $h^0(S, \mathcal{O}(3L - 3F_1)) = 4$ and $h^2(S, \mathcal{O}(3L - 3F_1)) = h^0(S, \mathcal{O}(-6L + 4F_1 + F_2)) = 0$, we have $h^1(S, \mathcal{O}(3L - 3F_1)) = 0$ by Riemann-Roch’s Theorem. Therefore we have that $\dim \Gamma^1(C, \mathcal{O}(3k_4^1)) = 5$. Hence $k_4^1$ is of type $(2,2,1)$. And $(e_1, e_2, e_3) = (3,2,0)$ if and only if $P$ is a 2-fold node or a 2-fold cusp by the same calculation. }

**Proof of Corollary A.** We now prove Corollary A. By Theorem 6, we may assume that $C$ is a non-singular curve of genus $g = 7, 8$ or 9. We have already assumed that $C$ is a tetragonal curve i.e. neither hyperelliptic nor trigonal and admits a tetragonal (base-point-free) linear system $g_1^4$. Moreover we assume that $C$ is not an elliptic-hyperelliptic curve. If there is only one $g_1^4$ on $C$, there is nothing to prove. So we may assume that there are $g_1^4, h_4^1 \in \mathcal{W}^1_4(C)$ such that $g_1^4 \neq h_4^1$. If $g = 7$ or 8, then Corollary A holds by Theorem A. Therefore the remaining case is $g = 9$ case. But in this case, if $\# \mathcal{W}^1_4(C) \geq 2$, then we have an embedding $C \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. In this case one easily proves that $h^0(C, \mathcal{O}(K_C - 3g_1^4)) = h^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-l + 2m)) = 0$, thus the invariants of $g_1^4$ are (2,2,2). Therefore the $e$-numbers of any $k_4^1 \in \mathcal{W}^1_4(C)$ are the same and equal to (2,2,2). This proves Corollary A. 


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