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QUANTUM MECHANICS OF GUIDING CENTER MOTION IN STRONG MAGNETIC FIELD (COHERENT STATE PATH INTEGRAL APPROACH)

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Abstract. A new approach is proposed for the quantum mechanics of guiding center motion in strong magnetic field. This is achieved by use of the coherent state path integral for the coupled systems of the cyclotron and the guiding center motion. We are specifically concerned with the effective action for the guiding center degree, which can be used to get the Bohr-Sommerfeld quantization scheme. The quantization rule is similar to the one for the vortex motion as a dynamics of point particles.

1. Introduction. The physics of charged particles in strong magnetic field has been one of the central problems in quantum mechanics inspired by condensed matter physics [2]. The unique feature is that the system is described by two degrees of freedom: the cyclotron(CT) and guiding center (GC) motion, if one chooses the special type of gauge for writing the vector potential. In the case that we have the uniform magnetic field only, the GC degree is not active; this is manifested by the infinite degeneracy of Landau levels. The degeneracy

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implies some symmetry which is governed by nothing but the GC coordinates. Here it is notable that the GC coordinates, say, (X, Y) form the canonical pair which reminds us of the similarity with the motion of vortices as point particles [3]. Now if a potential of non-magnetic origin is added, the GC degree begins to be active. The energy exchange occurs between the CT and GC motion, namely, we expect the mixing between inter-Landau levels. Thus the problem is reduced to the study of the coupled system of two types degree.

In this note we focus our attention on the motion of the guiding center. We are particularly interested in the effective action to get the semiclassical quantization rule. In order to achieve this, we use the coherent state path integral. It is well known that both the CT and GC motion can be described by the raising and lowering operators for harmonic oscillator and hence the coherent state provides with a natural means for this specific problem. Concerning the semiclassical quantization, Levit etal studied it in the framework of the Born-Oppenheimer approximation: [4] the GC motion is treated as the slow degree, whereas the CT motion the fast one which is firstly handled to lead to the adiabatic effective action for the slow guiding center motion that includes the adiabatic quantum phase factor [5]. On the contrary, our method does not rely upon the use of the specific assumption of adiabaticity. This paper is a preliminary account of the basic idea and the more detailed argument will be given in the forthcoming paper.

2. Coherent state. We shall start with a brief account for the coherent state representing the quantum state of the charged particle in strong magnetic field. Let us introduce the guiding center coordinates:

$$X = x - \Pi_y/\mu\omega, \quad Y = y + \Pi_x/\mu\omega \tag{1}$$

with $\vec{\Pi} = \vec{p} + \frac{e}{c}\vec{A}$, which satisfies

$$[X, Y] = \frac{i\hbar}{\mu\omega} \tag{2}$$

and the vector potential is given by $\vec{A} = \left(-\frac{1}{2}By, \frac{1}{2}Bx, 0\right)$.

The Hamiltonian in the uniform magnetic field H is written by use of the vector potential and the additional potential possibly coming from the impurities:

$$H = \left(a^\dagger a + \frac{1}{2}\right)\hbar\omega + \phi(x, y) \tag{3}$$

The angular momentum operator $L_z = xp_y - yp_x$ is expressed in terms of a, b operators

$$L_z = (a^\dagger a - b^\dagger b)\hbar \tag{4}$$

where $(a, a^\dagger)$ and $(b, b^\dagger)$ are defined as

$$\left\{\begin{array}{lcl} a & = & \frac{1}{\sqrt{2\mu\omega}}(\Pi_x - i\Pi_y) \\ a^\dagger & = & \frac{1}{\sqrt{2\mu\omega}}(\Pi_x + i\Pi_y) \end{array}\right. \qquad \left\{\begin{array}{lcl} b & = & \frac{1}{\sqrt{2}\ell}(X + iY) \\ b^\dagger & = & \frac{1}{\sqrt{2}\ell}(X - iY) \end{array}\right. \tag{5}$$

which satisfy the commutation relation: $[a, a^\dagger] = 1$ and $[b, b^\dagger] = 1$ and all the others vanish. ℓ is the so-called the magnetic length; $\ell \equiv \sqrt{\frac{\hbar}{\mu\omega}}$.

Now let us define $|N, m\rangle$ as the simultaneous eigenstate of $H_0 = \hbar\omega(a^\dagger a + \frac{1}{2})$ and L_z,

$$\begin{array}{lcl} H_0\,|N, m\rangle & = & \left(N + \frac{1}{2}\right)\hbar\omega\,|N, m\rangle \\ L_z\,|N, m\rangle & = & m\hbar\,|N, m\rangle \end{array} \tag{6}$$

which satisfy the following relations

$$\left\{\begin{array}{lcl} a^\dagger|N, m> & = & (N+1)^{\frac{1}{2}}|N+1, m+1> \\ a|N, m> & = & N^{\frac{1}{2}}|N-1, m-1> \\ b^\dagger|N, m> & = & (N-m+1)^{\frac{1}{2}}|N, m-1> \\ b|N, m> & = & (N-m)^{\frac{1}{2}}|N, m+1> \end{array}\right. . \tag{7}$$

Here N, m take the value such that $N = 0, 1, 2, \cdots, \infty$, $m = -\infty, \cdots, N-1, N$. This feature indicates that the Landau level labelled by the quantum number N has a degeneracy of infinite dimension. Thus $|N, m>$ can be constructed as follows:

$$|N, m\rangle = \frac{(b^\dagger)^{N-m}(a^\dagger)^N}{N!(N-m)!}\,|0, 0\rangle\,. \tag{8}$$

Now the cohrent state $|z, \xi>$ is introduced such that

(9) $$a|z, \xi >= z|z, \xi >, b|z, \xi >= \xi|z, \xi >$$

with z, ξ represent the complex number. $|z, \xi>$ is constructed as

(10) $$|z, \xi >= \exp\left(za^\dagger + \xi b^\dagger\right)|0, 0> .$$

Note that this forms the so-called "overcomplete set of states". The overlap between two CS's is calculated as

(11) $$< z', \xi'|z, \xi >= \exp\left[-\frac{1}{2}\left(|z|^2 + |z'|^2 + |\xi|^2 + |\xi'|^2\right) + (z')^* z + (\xi')^*\xi\right].$$

The most important property of the coherent state is the resolution of unity,

(12) $$\int |z, \xi > d\mu[z, \xi] < z, \xi| = 1$$

and $d\mu(z, \xi)$ means the measure of for the complex parameter space given as

(13) $$d\mu[z, \xi] = \frac{dz^r dz^i}{\pi} \cdot \frac{d\xi^r d\xi^i}{\pi}.$$

The wave function representing the coherent state is written as

(14) $$\begin{aligned} \psi_{z,\xi}(x, y) &\equiv < x, y|z, \xi > \\ &= \frac{1}{\sqrt{2\pi}\ell}\exp\left\{-\frac{1}{4\ell^2}\left[x - \sqrt{2}\ell(iz + \xi)\right]^2 \right. \\ &\qquad \left. -\frac{1}{4\ell^2}\left[y + \sqrt{2}\ell(z + i\xi)\right]^2 + iz\xi\right\} \end{aligned}$$

which is used to evaluate the matrix elements of the Hamiltonian.

3. Coherent state path integral. We now consider the quantum transition in terms of the path integral using the coherent state constructed above. In the following argument we are concerned with the one-body problem for the charged particle in strong magnetic field as well as the force of non-magnetic origin: $H = H_0 + \phi(x, y)$. H_0 is the one for the pure cyclotron motion, while $\phi(x, y)$ involves the coupling between the guiding center and cyclotron motion. Here we make the following remark: In order to deal with the path integral for two systems under the mutual interaction, the two step procedure is adopted, namely, one first considers the transition amplitude as if one degree were fixed, and after having accomplished this, the transition amplitude for the remaining degree is considered. Now, we consider the transition amplitude

$$K = \left\langle \xi_f, z_f \left| \exp[-\frac{i}{\hbar}\hat{H}t] \right| \xi_i, z_i \right\rangle . \tag{15}$$

Following the usual procedure of the coherent state path integral, we first divide the time t into N small interval (and take the limit $N \to \infty$) and next insert the resolution of unity relation for the coherent state at each time division point, then K is cast into the infinite dimensional integral [6]

$$K(\xi_f, z_f t|\xi_i, z_i, 0) = \lim_{N\to\infty} \int \cdots \int \prod_{k=1}^{N-1} d\mu(\xi_k) \prod_{k=1}^{N} \left\langle \xi_k \left| \exp[-\frac{i}{\hbar}\hat{H}\epsilon] \right| \xi_{k-1} \right\rangle . \tag{16}$$

In the limit of $\epsilon \to 0$, the infinitesimal factor can be approximated in the case of the limit $\epsilon \to 0$;

$$\begin{aligned} \left\langle \xi_k \left| \exp[-\tfrac{i}{\hbar}\hat{H}\epsilon] \right| \xi_{k-1} \right\rangle &\simeq \left\langle \xi_k \left| (1 - \frac{i}{\hbar}\hat{H}\epsilon) \right| \xi_{k-1} \right\rangle \\ &= \langle \xi_k | \, \xi_{k-1} \rangle \left(1 - \frac{i}{\hbar}\epsilon \frac{\left\langle \xi_k \left| \hat{H} \right| \xi_{k-1} \right\rangle}{\langle \xi_k | \, \xi_{k-1} \rangle} \right) . \end{aligned} \tag{17}$$

The first factor is just the overlap between the infinite small interval, which is explicitly written as

$$\langle \xi_k | \, \xi_{k-1} \rangle = \exp \left[\xi_k^* \xi_{k-1} - \frac{|\xi_k|^2}{2} - \frac{|\xi_{k-1}|^2}{2} \right] . \tag{18}$$

By using this form, the propagator turns out to be

$$K(\xi_f, z_f, t : \xi_i, z_i, 0) = \int \exp[\frac{i}{\hbar} \int \frac{i\hbar}{2} (\xi^* \dot{\xi} - c.c) dt] T(z_f, z_i) \mathcal{D}\mu[\xi(t), \xi^*(t)]. \tag{19}$$

$T(z_f, z_i)$ means the amplitude for the transition from the initial state $|z_i\rangle$ to the final state $|z_f\rangle$ under the motion of the guiding center $\xi(t)$, namely,

$$T(z_f, z_i) = \left\langle z_f \left| T \exp[-\frac{i}{\hbar} \int_0^T h(a^\dagger, a; \xi(t), \xi^*) dt] \right| z_i \right\rangle . \tag{20}$$

Here $h(a^\dagger, a; \xi, \xi^*) \equiv \left\langle \xi \left| H(a^\dagger, a, b^\dagger, b) \right| \xi \right\rangle$, which implies that in calculating the transition amplitude for the cyclotron motion the guiding center degree is replaced as if it were the c-number coordinate. Thus, if using the relation of the resolution of unity holding for $|z\rangle$, $T(z_f, z_i)$ is written in terms of the path integral over the $z(t)$ space:

$$(21)\; T(z_f, z_i) = \int \exp\left[\frac{i}{\hbar}\int_0^T \frac{i\hbar}{2}\{(z^*\dot{z} - c.c) - (z^*z + \frac{1}{2})\hbar\omega - \langle z\,|h|\,z\rangle\}dt\right]\mathcal{D}\mu(z(t))$$

where the expectation value $\langle z\,|h|\,z\rangle$ is given

$$\Phi(z, z^*, \xi, \xi^*) = \langle z\,|h|\,z\rangle = \frac{\langle z, \xi\,|\phi(x, y)|\,z, \xi\rangle}{\langle z, \xi|\; z, \xi\rangle}. \tag{22}$$

4. Semiclassical quantization formula. We shall consider the semiclassical limit of the propagator given by the form: (19). The semiclassical limit means that $\hbar \to 0$. Noting that the propagator (18) consists of integral over two different degrees, we take the semiclassical limit in two step process, namely, in the first step, the semiclassical limit is taken for the integration for the cyclotron degree, thus

$$T(z_f, z_i) = \exp\left[\frac{i}{\hbar}\Gamma(\tilde{C})]\exp[-\frac{i}{\hbar}\int_0^T \left\langle z \left| \hat{h}(\xi(t), \xi^*(t)) \right| z \right\rangle dt\right] \tag{23}$$

with Γ

$$\Gamma = \int \left\langle z \left| i\hbar \frac{\partial}{\partial t} \right| z \right\rangle dt \tag{24}$$

where $\tilde{C}$ represents the orbit satisfying the action principle

$$\delta\int_0^T \left\langle z \left| i\hbar\frac{\partial}{\partial t} - \hat{h}(\xi(t), \xi^*(t)) \right| z \right\rangle dt = 0. \tag{25}$$

The variation equation yields the equation of motion for the complex variables z, z^*, which leads to the classical equation of motion in the coupled form

$$\begin{cases} i\hbar\dot{z}^* + \hbar\omega z^*[12pt] + \dfrac{\partial\Phi}{\partial z} &= 0 \\ -i\hbar\dot{z} + \hbar\omega z + \dfrac{\partial\Phi}{\partial z^*} &= 0 \end{cases}. \tag{26}$$

Note that the solution of these equations are given in terms of the functions of the orbit $\xi(t)$. Thus the evolution operator K is given by

$$K_{eff}(T) = \int \exp\left[\frac{i}{\hbar} S_{eff}\right] \prod d\mu[C] \tag{27}$$

where the effective action is given by

$$S_{eff} = S_0 + \hbar\Gamma - \int_0^T \left\langle z \middle| \hat{h} \middle| z \right\rangle dt \tag{28}$$

where $S_0 = \int \frac{i\hbar}{2}(\xi^\dagger \dot{\xi} - c.c)dt$. If we note that z and z^* are given by the functions of the orbit ξ, the canonical term is written as

$$\left\langle z \middle| i\hbar \frac{\partial}{\partial t} \middle| z \right\rangle = (\left\langle z \middle| i\hbar \frac{\partial}{\partial z} \middle| z \right\rangle \frac{\partial z}{\partial \xi} + \left\langle z \middle| i\hbar \frac{\partial}{\partial z^*} \middle| z \right\rangle \frac{\partial z^*}{\partial \xi}) \frac{d\xi}{dt} \tag{29}$$

The canonical term in the effective action for the guiding center motion is thus changed by an amount of ω, where

$$\omega = (A^i_z + A^i_{\bar{z}}) d\xi_i. \tag{30}$$

A represents the so-called “connection field” that is defined as

$$\begin{aligned} A^i_z &= \left\langle z \middle| i\hbar \frac{\partial}{\partial z} \middle| z \right\rangle \frac{\partial z}{\partial x_i} \\ A^i_{\bar{z}} &= \left\langle z \middle| i\hbar \frac{\partial}{\partial z^*} \middle| z \right\rangle \frac{\partial z^*}{\partial x_i} \end{aligned} \tag{31}$$

where x_i denotes the abbreviation of the guiding center coordinate (X, Y). The appearance of the phase Γ arises from taking the semiclassical limit of the expression of the time-ordered product in (20). This process is a counterpart of the adiabatic process which picks up a specific path from an infinite sum given by the time-ordered product [5].

Now if we use the effective action of the form we can obtain the semiclassical quantization formula for the guiding center motion. We first consider the semiclassical approximation for the effective propagator; $K^{sc}_{eff}(T)$. We next take the Fourier transform of $K^{sc}_{eff}(T)$ to apply the stationary phase approximation over the T-integral, and finally take into account of the contribution of multiple traversals of the basic periodic orbit, thus we have

$$K_{sc}(E) = \sum \frac{\exp[iW(E)]}{1 - \exp[iW(E)]} \tag{32}$$

with $W(E) = \oint \frac{i\hbar}{2}(\xi^*\dot{\xi} - c.c)dt + \Gamma(C)$ Then, one can get the quantization rule from picking up the pole of $K_{sc}(E)$.

$$\oint \frac{i\hbar}{2}(\xi^*\dot{\xi} - c.c)dt = \left(n - \frac{\Gamma(C)}{2\pi}\right)2\pi\hbar \tag{33}$$

which just gives the corrected Bohr-Sommerfeld formula including the geometric phase Γ. C means the periodic orbits determined by the action principle $\delta S_{eff} = 0$, which turns out to be

$$\begin{cases} i\hbar\dot{\xi}^* + \dfrac{\partial\Phi}{\partial\xi} &= 0 \\ -i\hbar\dot{\xi} + \dfrac{\partial\Phi}{\partial\xi^*} &= 0 \end{cases}. \tag{34}$$

The formula (33) is just a generalization of the previously obtained that is the one including the effect of the adiabatic phase [4].

5. Simple example. We consider the case of the central force; $\phi(x, y) = \phi(r), r = \sqrt{x^2 + y^2}$ for which we get $\Phi(z, z^*, \xi, \xi^*) = \Phi(|\chi|^2)$ where χ and hence the equation of motion yields

$$\begin{cases} i\hbar\dot{z}^* + \hbar\omega z^* + \dfrac{i}{2}(-iz^* + \xi)\dfrac{d\Phi}{d|\chi|^2} &= 0 \\ -i\hbar\dot{z} + \hbar\omega z - \dfrac{i}{2}(iz + \xi^*)\dfrac{d\Phi}{d|\chi|^2} &= 0 \\ i\hbar\dot{\xi}^* + \dfrac{1}{2}(iz + \xi^*)\dfrac{d\Phi}{d|\chi|^2} &= 0 \\ -i\hbar\dot{\xi} + \dfrac{1}{2}(-iz^* + \xi)\dfrac{d\Phi}{d|\chi|^2} &= 0 \end{cases}. \tag{35}$$

As a special case of the lowest Landau Level : $N = 0$ namely, $z = z^* = 0$, the Lagrangian turns out to be

$$L = \frac{i\hbar}{2}(\xi^*\dot{\xi} - \dot{\xi}^*\xi) - \frac{1}{2}\hbar\omega - \Phi(|\xi|^2) \tag{36}$$

which yields the equation of motion

$$\begin{cases} i\hbar\dot{\xi}^* + \dfrac{1}{2}\xi^*\dfrac{d\Phi}{d|\xi|^2} &= 0 \\ -i\hbar\dot{\xi} + \dfrac{1}{2}\xi\dfrac{d\Phi}{d|\xi|^2} &= 0 \end{cases} \tag{37}$$

from which we get $\dfrac{d(|\xi|^2)}{dt} = 0$ which gives $|\xi|^2 = \text{const}$.

Physically speaking, the restriction to the LLL means the strong magnetic field limit: $B \longrightarrow \infty$ and in terms of the cyclotron frequency, $\omega \longrightarrow \infty$ $(\omega \propto B)$. Now we shall consider the Bohr-Sommerfeld quantization, which is obtained as special case of (33), namely, by putting $z = z^* = 0$,

$$\frac{i\hbar}{2}\oint_C (\xi^* d\xi - \xi d\xi^*) = 2\pi m\hbar. \tag{38}$$

If we put $\xi = \dfrac{1}{\sqrt{2}\ell}(X + iY)$, $\xi^* = \dfrac{1}{\sqrt{2}\ell}(X - iY)$, this can be rewritten as the form

$$\begin{aligned} \frac{i\hbar}{2}\oint_C \frac{i}{l^2}(XdY - YdX) &= -\frac{\hbar}{2\ell^2}\int_S 2dX \wedge dY \\ &= -2\pi\hbar|\xi|^2 \end{aligned} \tag{39}$$

Thus, we have

$$|\xi|^2 = -m$$

and hence the energy Spectrum is given by

$$E_m = \frac{1}{2}\hbar\omega - \Phi(-m). \tag{40}$$

Note that the above quantization rule gives nothing but the angular momentum quantization, since $|\xi|^2$ can be given as the expectation value of L_z with respect to the coherent state.

As a special case we consider the Coulomb potential for the interaction $\phi(r) = -\dfrac{e^2}{r}$, for which $\Phi(|\xi|)$ becomes

$$\Phi(|\chi|) = \frac{e^2\pi}{\sqrt{2\pi}l}\mathrm{e}^{-|\chi|^2/2} I_0(|\chi|^2/2) \tag{41}$$

where $I_0(|\xi|)$ stands for the modified Bessel function. The energy spectrum is obtained by substituting the relation $|\xi|^2 = -m$.

6. Final remarks. In this note we have developed a basic idea for the effective action for the guiding center motion to yield the semiclassical quantization rule. The formula presented here is mainly concerned with the single particle in strong magnetic field together with the potential of non-magnetic origin. Here an outline is given of the possible problems that are expected to be developed on basis of the present general formalism. (i): To take into account of the motion of the cyclotrons motion in the quantization rule, which is to consider the effect of the geometric phase Γ after all. (ii): To extend the case of the lowest Landau level to the many particle case, which may be regarded as a similarity with the vortex gas for which the force is given by the logarithmic potential. (iii): In relation to the many-particle extension, it may be interesting to consider the charged particle with two kind of charges: namely, with opposite charges each other, which may be analogous to the two kind of vortices with opposite vortex strength. These will be left as subjects for the future study.

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