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NADEL'S SUBSCHEMES OF FANO MANIFOLDS X WITH A PICARD GROUP Pic(X) ISOMORPHIC TO Z

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Abstract. In this paper we find a global sufficient condition for suitable subschemes of Fano manifolds to be Nadel's subschemes. We apply this condition to one-dimensional subschemes of a projective space.

1. Introduction and basic notations. In this paper we consider Fano manifolds, i.e. compact complex manifolds X with positive First Chern class: $c_1(X) > 1$. A subsceme Y of such manifold is called Nadel's if for any Nakano semi-positive holomorphic vector bundle E on X all the higher cohomology groups

$$H^q(X, \mathcal{E}\otimes\mathcal{I}_Y)$$

vanish. (The definition of Nakano semi-positivity is given bellow). Here $\mathcal{E}$ denotes the sheaf of germs of holomorphic sections of E, and $\mathcal{I}_Y$ denotes the ideal sheaf on X which defines the subsceme Y. As was shown by A. M. Nadel [5] and by the author [9] the existence of Nadel's subschemes is related to the presence of Kähler-Einstein metrics on X:

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If there are no Nadel's subscemes on X,
then X is Kähler-Einstein.

Unfortunately, every reduced point on X is a Nadel's subscheme of X. But often it is very useful to know that on a Fano manifold there are no any Nadel's subschemes of a certain type. For example, if the twisted cubic C_3 in P^3 is not a Nadel's subscheme of P^3, then the blow-up of P^3 in C_3 is Kähler-Einstein.

Our main purpose in this paper is to investigate what global conditions on a subscheme Y of X are sufficient for Y to be a Nadel's subscheme of X. We find such conditions in case of reduced Y with smooth irreducible components which intersect each-other transversally, and with $\mathrm{Pic}(X)$ isomorphic to Z (Theorem 3.1 and Corollary 3.6). For proving this result we construct a fine resolution $\mathcal{F}^{\cdot}$ for the sheaf $\mathcal{E}\otimes\mathcal{I}_Y$, whose complex of global sections is acyclic in positive dimensions. The construction of $\mathcal{F}^{\cdot}$ uses classic methods of Andreotti-Vesentini and Hörmander about the existence of solutions of a $\overline{\partial}$ -problem on a complex manifold. We conclude with some examples to which the results of this paper are applicable: Grassman manifolds and their smooth divisors. As a consequence we give a sufficient-and-necessary condition for a reduced curve of degree 3 in P^n to be a Nadel's subscheme. For example, C_3 is Nadel's in P^3.

The approach to the problem in this paper, in its final variant, was influenced by Demailly's papers [1] and [2]. The author would like to acknowledge K. Ranestad for very helpful discussions during the conference "Geometry and Mathematical Physics" in Zlatograd, 1995.

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The following basic notations we shall persistently use in the paper.

Let X be a compact complex manifold and (E, h) be an Hermitian holomorphic vector bundle over X.

Definition 1.1. *With $\Omega(h)$ we denote the following section of the vector bundle $(E\otimes TX)^{*}\otimes(\overline{E\otimes TX})^{*}$, which in local coordinates has the form*

$$\Omega(h) = -\partial\overline{\partial}h + \partial h.h^{-1} \wedge \overline{\partial}h$$

Remark 1.1. If ∇ is the Hermitian connection in E, and $\Theta = \nabla^2$ its curvature, then in local coordinates

$$\Omega(h) =^{t} \Theta.h.$$

It's well known that $\Omega(h)$ is a Hermitian form on $E \otimes TX$.

Definition 1.2. *A holomorphic vector bundle E over X is called Nakano (semi-)positive if there exists an Hermitian metric h on E, the corresponding Hermitian form $\Omega(h)$ of which is (semi-)positive definite on $E \otimes TX$.*

In particular, X is a Fano manifold iff its anticanonical line bundle K_X^* is Nakano positive.

Remark 1.2. (i) Suppose L is a holomorphic line bundle over X, and l its Hermitian metric. Then locally $\frac{\sqrt{-1}}{2\pi}.\frac{\Omega(l)}{l}$ represents the First Chern class of L : $c_1(L)$. Consequently, L is Nakano positive iff $c_1(L) > 0$. By the famous Kodaira theorem the latter is satisfied iff L is an ample line bundle.

(ii) For the connections between various notions of positivity of a vector bundle and the Nakano (semi-)positivity we refer to the book of Shiffman and Sommese [7].

Definition 1.3. 1) *Suppose $\varphi \in L^1_{loc}(X)$, and (E, h) is an Hermitian vector bundle over X. The almost everywhere defined bilinear form*

$$\tilde{h} = h.\exp(-\varphi)$$

is called a singular metric *on E. By definition*

$$\Omega(\tilde{h}) = \exp(-\varphi).(\Omega(h) + \partial\overline{\partial}\varphi.h.$$

The coefficients of $\Omega(\tilde{h})$ are $(1,1)$-currents.

2) *For $\varphi \in L^1_{loc}(X)$ we define $\mathcal{I}(\varphi)$ to be the sheaf of germs of holomorphic functions f on X for which*

$$|f|^2.\exp(-\varphi)$$

is locally integrable (with respect to any smooth volume form on X).

As usual, $\mathcal{O}_X$ denotes the structure sheaf of X as a complex manifold, and $\mathcal{A}_X$ – the structure sheaf of X as a smooth manifold. If Y is a subscheme of (the complex manifold) X, then $\mathcal{I}_Y$ denotes the ideal sheaf which defines Y. We have

$$(Y, \mathcal{O}_Y) = (Spec(\mathcal{O}_X/\mathcal{I}_Y), \mathcal{O}_X/\mathcal{I}_Y).$$

For any holomorphic vector bundle E over X, $\mathcal{E}$ denotes the sheaf of germs of holomorphic sections of E.

2. Singular metrics defining submanifolds of a complex manifold. Let X be a compact complex manifold, $L_1, L_2, \dots, L_s$ be holomorphic line bundles over X, and Y – a subscheme of X.

Definition 2.1. *(i) We shall say that Y is* scheme-theoretically determined by $L_1, \dots, L_s$, *if there exist sections $\sigma_i \in H^0(X, L_i)$, $i = 1, \dots, s$, that satisfy the following property*

For any $x \in X$ there exists an open neighbourhood U of x such that

$$\sigma_i|_U = f_i^U.e_i, \quad (i = 1, \dots, s), \quad f_i^U \in O_X(U)$$

and

$$\mathcal{I}_Y(U) = (f_1^U, \dots, f_s^U).\mathcal{O}_X(U).$$

(ii) Y is called a globally complete intersection, if there exist $L_1, \dots, L_s$ that determine Y scheme-theoretically, and $s = \operatorname{codim}_X Y$.

Remark 2.1. If X is a projective manifold, then every subscheme of X is scheme-theoretically determined by some line bundles $L_1, \dots, L_s$.

Let's consider the following situation. Suppose Y is a smooth submanifold of X scheme-theoretically determined by $L_1, \dots, L_s$ and

$$L_1 = L^{\otimes n_i} \qquad i = 1, \dots, s$$

for a very ample line bundle L over X. We shall construct a function $\varphi \in L^1_{loc}(X)$, for which

$$\mathcal{I}_Y = \mathcal{I}(\varphi).$$

Remark 2.2. In fact, for the construction of φ below L suffices to be ample, not very ample. Hence this construction is valid for any projective X with $\mathrm{Pic}(X) \cong \mathrm{Z}$.

The construction of φ.

L is very ample, so there exists an embedding

$$\Phi_L \quad : X \longrightarrow \mathrm{P}(H^0(X, L)^*) =: \mathrm{P}^N,$$

such that $\mathcal{L} \cong \Phi_L^* \mathcal{O}_{\mathrm{P}^N}(1)$.

Let $(x_0, \dots, x_N)$ be homogeneous coordinates of P^N, and $U_j = \{x_j \neq 0\}$, $j = 1, \dots, N$, be the standard affine open subsets of P^N. Denote by V_j the preimage of U_j under Φ_L. Then L_j is trivial over V_j for all possible i and j. By our assumption Y is scheme-theoretically determined by $L_1, \dots, L_s$. Hence there exist sections $\sigma_i \in H^0(X, L_i)$, $(i = 1, \dots, s)$, for which

$$\sigma_i|_{U_j} = f_i^j.e_j^{\otimes n_i}$$

and

$$\mathcal{I}_Y(V_j) = (f_1^j \ldots, f_s^j).\mathcal{O}_X(V_j),$$

where e_j is a local frame for $L_i|_{V_j}$.

Let l be the standard metric on $\mathcal{O}_{\mathbf{P}^N}(1)$ lifted to L via the map Φ_L, and l_j – its local representation over V_j $(j = 1, \ldots, N)$. We set

$$\varphi_j := r.\log\left(\sum_{i=1}^{s} \frac{|f_i^j|^2}{l_j^{n_i}}\right) \quad j = 1, \ldots, N,$$

where $r = \operatorname{codim}_X Y$.

Obviously,

$$\varphi_j|_{V_j \cap V_i} = \varphi_i|_{V_j \cap V_i},$$

and φ_i is an almost pluri-subharmonic function, i.e. it can be represented locally as a sum of a smooth and a pluri-subharmonic function. Consequently, $\varphi = \{\varphi_j\}_j$ as a function belongs to $L^1_{loc}(X)$.

Theorem 2.3. *For the function just defined we have*

$$\mathcal{I}_Y = \mathcal{I}(\varphi).$$

First we shall prove the following general result about almost pluri-subharmonic functions.

Proposition 2.4. *For every almost pluri-subharmonic function φ the sheaf $\mathcal{I}(\varphi)$ is a coherent ideal sheaf in $\mathcal{O}_X$.*

Proof. (We follow Demailly, [1]) The assertion is local, so we may assume that X is a Stein manifold.

Let S denote the set of all holomorphic functions f on X for which

$$|f|^2.\exp(-\varphi)$$

is locally integrable (with respect to some smooth and bounded measure). Since S is an ideal in $\mathcal{O}_X$, which is a Noetherian sheaf, then S defines a coherent ideal sheaf $\mathcal{S}$ on X by a standard way. We have

$$\mathcal{S} \subseteq \mathcal{I}(\varphi).$$

We shall prove that $\mathcal{S}_x = \mathcal{I}(\varphi)_x$ for each $x \in X$ from which the Proposition 2.4 will follow immediately.

Let m_x be the maximal ideal in the local ring $\mathcal{O}_{X,x}$. By the Krull Intersection Theorem we have

$$\bigcap_{k\geq 0} (\mathcal{S}_x + m_x^k.\mathcal{I}(\varphi)_x) = \mathcal{S}_x,$$

hence it suffices to prove that for each $k \in \mathrm{N}$

$$\mathcal{I}(\varphi)_x \subseteq \mathcal{S}_x + m_x^k.\mathcal{I}(\varphi)_x.$$

Let $f_x \in \mathcal{I}(\varphi)_x$ be the representative of a holomorphic function $f \in \mathcal{I}(\varphi)(V)$, i.e. $|f|^2.\exp(-\varphi)$ is integrable over V. Without loss of generality we may assume that over V

$$\varphi = \psi + \psi',$$

where ψ is pluri-subharmonic, and ψ' is smooth over V. Set

$$\tilde{\varphi}(z) = \psi(z) + 2.(n+k).\log(|z-x|) + |z|^2,$$

where $z \in V$, $n = \dim X$. Let $W \subset V$ be an open neighbourhood of x, and let ρ be a cut off function in V for which

$$\rho(z) = 1 \quad \text{for} \quad z \in W, \quad \rho(z) = 0 \quad \text{for} \quad z \notin W.$$

Since the $\overline{\partial}$-closed form $\alpha = \overline{\partial}(\rho.f)$ is integrable over V, and since

$$\partial\overline{\partial}(\tilde{\varphi}) \geq \sum_{i=1}^{n} dz^i \wedge d\overline{z}^i,$$

we can apply the Hörmander Existence Theorem (see Nadel [5], Proposition 1.1):

There exists a smooth function g over V such that $\overline{\partial} g = \alpha$, and

$$\int_V |g|^2.\exp(-\tilde{\varphi})dV \leq \int_V |\alpha|^2 \exp(-\tilde{\varphi})dV.$$

Here $dV = \left(\frac{\sqrt{-1}}{2}\right)^n . \frac{\left(\sum dz^i \wedge \overline{z^i}\right)^n}{n!}$.

Obviously,

$$u := \rho.f - g \in \mathcal{S}(V), \quad g \in \mathcal{O}_X(W)$$

and

$$f_x = u_x + g_x \in \mathcal{S}_x + \mathcal{I}(\varphi)_x \bigcap m_x^{k+1}.$$

Hence $\mathcal{I}(\varphi)_x \subset \mathcal{S}_x + \mathcal{I}(\varphi)_x \bigcap m_x^k$ for every $k \in \mathrm{N}$. $\square$

Corollary 2.5 *Let (E, h) be an Hermitian vector bundle over X, and φ be an almost pluri-subharmonic function on E. Denote by $\mathcal{S}$ the sheaf of germs of holomorphic sections of E, which are locally $\tilde{h} = h.\exp(-\varphi)$-integrable. Then $\mathcal{S}$ is a coherent sheaf and*

$$\mathcal{S} = \mathcal{E} \otimes \mathcal{I}(\varphi).$$

$\lhd$ The assertion is local and, since X is locally compact, easily reduces to the case of a trivial E with the standard Hermitian metric. Corollary 2.5 then follows from Proposition 2.4. $\rhd$

Proof of Theorem 2.3. It follows from Proposition 2.4 that $\mathcal{I}(\varphi)$ defines a subscheme of X. Obviously,

$$Supp(\mathcal{O}_X/\mathcal{I}(\varphi)) \bigcap V_j = \{f_1^j = 0, \ldots, f_s^j = 0\} = Y \bigcap V_j.$$

Since Y is reduced we have

$$\mathcal{I}(\varphi) \subset \mathcal{I}_Y.$$

It remains to prove that

Lemma 2.6. *For every $x \in X$ $\quad (\mathcal{I}_Y)_x \subset \mathcal{I}(\varphi)_x$.*

$\lhd$ If $x \notin Y$, then $(\mathcal{I}_Y)_x = \mathcal{I}(\varphi)_x = \mathcal{O}_{X,x}$.

Let $x \in Y \bigcap V_j$. Since Y is smooth at x, within the set $\{f_1^j, \ldots, f_s^j\}$ there exist $r (= \operatorname{codim}_X Y)$ functions which are a part of a parameter system of $\mathcal{O}_{X,x}$. Suppose these functions are $f_1^j, \ldots, f_r^j$. Moreover, we have

$$\{(f_{r+1}^j)_x, \ldots, (f_s^j)_x\} \subset ((f_1^j)_x, \ldots, (f_r^j)_x).\mathcal{O}_{X,x}.$$

Hence we must prove that $(f_i^j)_x \in \mathcal{I}(\varphi)_x$ for $i = 1, \ldots, r$, or, in other words, that $f_1^j, \ldots, f_r^j$ are integrable at x with respect to the weight function φ.

After an appropriate change of the local coordinates at x, the last assertion is equivalent to the fact that the functions

$$\frac{|z_i|^2}{\left(\sum\limits_{j=1}^{r} \frac{|z_j|^2}{g_j} + G\right)^r} \quad i = 1, \ldots, r$$

are integrable at $z = 0$, where g_i $(i = 1, \ldots, r)$ is a smooth and positive at x function, and $G = \sum |G_j|^2$ for $G_j \in (z_1, \ldots, z_r).\mathcal{A}_{X,x}$. $\rhd$

This completes the proof of Theorem 2.3. □

From now on we shall work with projective X for which $\mathrm{Pic}(X) \cong \mathrm{Z}$. Suppose that Y is a subscheme of X and

$$Y = Y_1 \bigcup \ldots \bigcup Y_m$$

is its decomposition into irreducible components. We shall say that Y is *suitable* for our considerations if the following holds:

i) each Y_i $(i = 1, \ldots, m)$ is smooth,

ii) if $x \in X$ and $Y_{i_1}, \ldots, Y_{i_k}$ are all the components of Y passing through x, then

$$Y_{i_1} \bigcap \ldots \bigcap Y_{i_j} \quad \text{and} \quad Y_{i_{j+1}} \quad j = 1, \ldots, k-1,$$

intersect each other transversally.

By Theorem 2.3 we can construct almost pluri-subharmonic functions $\{\varphi\}_{i=1}^m$ with the property

$$\mathcal{I}(\varphi_i) = \mathcal{I}_{Y_i} \quad i = 1, \ldots, m.$$

Denote by φ the sum $\varphi_1 + \ldots + \varphi_m$. Obviously, this function is almost pluri-subharmonic.

Theorem 2.7. *If Y is a suitable subscheme of X, and φ is the almost pluri-subharmonic function defined above, then*

$$\mathcal{I}(\varphi) = \mathcal{I}_Y.$$

P r o o f. We shall prove this theorem for $m = 2$. After that the general case will be clear.

By Theorem 2.3 we have that $\mathcal{I}(\varphi)$ defines $Y \backslash Y_1 \bigcap Y_2$ on $X \backslash Y_1 \bigcap Y_2$. Suppose that $x \in Y_1 \bigcap Y_2$ and at x the functions φ_1 and φ_2 are represented by

$$\varphi_1 = r_1 . \log \left(\sum_{i=1}^{n_1} \frac{|f_i|^2}{l^{\nu_i}} \right) \quad \text{and} \quad \varphi_2 = r_2 . \log \left(\sum_{j=1}^{n_2} \frac{|g_j|^2}{l^{\mu_j}} \right),$$

where $r_i = \mathrm{codim}\,_X Y_i$, $i = 1, 2$. (See the construction of φ before Theorem 2.3).

We must prove that

$$f_i . g_j \quad \text{is} \quad \varphi - \text{integrable} \quad \text{for all} \quad i = 1, \ldots, n_1, \quad j = 1, \ldots, n_2.$$

Similar to the proof of Lemma 2.6. we can choose

$$f_1, \dots, f_{r_1}, g_1, \dots, g_{r_2}$$

to be a part of a parameter system of $\mathcal{O}_{X,x}$ with

$$(\mathcal{I}_{Y_1})_x = ((f_1)_x, \dots, (f_{r_1})_x).\mathcal{O}_{X,x} \text{ and } (\mathcal{I}_{Y_2})_x = ((g_1)_x, \dots, (g_{r_2})_x).\mathcal{O}_{X,x}.$$

By the same arguments as in the proof of Theorem 2.3 we must prove that

$$(f_i.g_j)_x \in \mathcal{I}(\varphi)_x \text{ for all possible } i, j$$

This last, by an appropriate change of the coordinates at x, is equivalent to the fact that for $k = 1, \dots, r_1$, $l = 1, \dots, r_2$ the function

$$\frac{|z'_k|^2.|z''_l|^2}{\left(\sum\limits_{i=1}^{r_1} \frac{|z'_i|^2}{g'_i} + G'\right)^{r_1} . \left(\sum\limits_{j=1}^{r_2} \frac{|z''_j|^2}{g''_j} + G''\right)^{r_2}}$$

is integrable at $z = 0$ with respect to the standard volume form in $\mathbf{C}^n$, where

$$g'_i > 0, \quad i = 1, \dots, r_1, \quad G' = \sum |G'_i|^2 \quad G'_i \in (z'_1, \dots, z'_{r_1}).\mathcal{A}_{X,x},$$

$$g''_j > 0, \quad j = 1, \dots, r_2, \quad G'' = \sum |G''_j|^2, \quad G''_j \in (z''_1, \dots, z''_{r_2}).\mathcal{A}_{X,x}.$$

This completes the proof of Theorem 2.7. □

3. Vanishing theorems and Nadel's subschemes. Suppose we are given the following data:

- X is a complex manifold with $\mathrm{Pic}(X) \cong \mathrm{Z}$,
- $Y \hookrightarrow X$ is a suitable subscheme,
- φ is the almost pluri-subharmonic function corresponding to Y by means of Theorem 2.7,
- g is a Kähler metric on X with k - the induced metric on the anticanonical line bundle K^*_X,
- (E, h) is an Hermitian vector bundle over X.

Theorem 3.1. *Let $\tilde{h}$ be the singular metric on E, defined by*

$$\tilde{h} = h. \exp(-\varphi).$$

Suppose that there exists a positive number ϵ and a Kähler form ω on X, for which

$$k. \exp(\varphi).\Omega(\tilde{h}) + \Omega(k).h \geq \epsilon.\omega.h,$$

(as an inequality between Hermitian forms with currents as coefficients). Then for each positive q

$$H^q(X, \mathcal{E} \otimes \mathcal{I}_Y) = 0.$$

P r o o f. The idea of the proof is in constructing of a fine resolution $\mathcal{F}^{\cdot}$ for $\mathcal{E} \otimes \mathcal{I}(\varphi)$, the corresponding complex $\{\Gamma(\mathcal{F}^{\cdot})\}$ of which is acyclic in positive dimensions. Theorem 3.1 follows then from the fact that $\mathcal{I}(\varphi) = \mathcal{I}_Y$ (Theorem 2.7).

Let $\mathcal{F}^q \subset \mathcal{A}^{0,q}(E)$ denote the sheaf of germs of those smooth $(0,q)$-forms α with coefficients in E, for which both α and $\overline{\partial}\alpha$ are locally integrable with respect to the metric induced by $\tilde{h}$ and g. Then $\{\mathcal{F}^q, d^q = \overline{\partial}\}_{q \geq 0}$ is a differential complex with $\mathrm{ker} d^0 = \mathcal{E} \otimes \mathcal{I}(\varphi)$ (by Corollary 2.5).

Lemma 3.2. *The complex $\{F^q = \Gamma(X, \mathcal{F}^q), d^q = \overline{\partial}\}_{q \geq 0}$ is acyclic in positive dimensions, i.e.*

$$\mathrm{Ker} d^q = \mathrm{Im} d^{q-1}$$

for $q \geq 1$.

$\triangleleft$ In the notations of the proof of Theorem 2.7 let

$$\varphi_i = r_i . \log \left(\sum_{j=1}^{n_1} \frac{|f_{ij}|^2}{l^{\nu_{ij}}} \right).$$

We have (a global) regularization of φ_i:

$$\varphi_{i,n} = r_i . \log \left(\frac{1}{n} + \sum_{j=1}^{n_1} \frac{|f_{ij}|^2}{l^{\nu_{ij}}} \right).$$

Hence the following pluri-subharmonic functions

$$\varphi_n = \varphi_{1,n} + \cdots + \varphi_{m,n}$$

form a regularization of a φ, and $\varphi_n \searrow_n \varphi$.

Denote by h_n the Hermitian metric $h.\exp(-\varphi_n)$. It follows that there exists an integer n_0 and a positive number ϵ', less then ϵ such that

$$k.\exp(\varphi_n).\Omega(h_n) + \Omega(k).h \geq \epsilon'.\omega.h.$$

for each $n \geq n_0$.

Denote by $\|.\|_{\tilde{h}}$ $(\|.\|_{h_n})$ the norm in $\mathcal{A}^{0,q}(E)$ induced by the metrics $\tilde{h}$ and g (respectively h_n and g).

Suppose now that $\alpha \in F^q$ and $d^q\alpha = 0$ for some $q \geq 1$. By definition the number

$$C = \|\alpha\|^2_{\tilde{h}}$$

is finite. Since $\|\alpha\|^2_{h_n} \leq \|\alpha\|^2_{\tilde{h}}$ for each n, then the sequence $\{\|\alpha\|^2_{h_n}\}_n$ is bounded from above by C. On the other hand, by using the classic methods of Andreotti-Vesentini and Hörmander, for each $n \geq n_0$ there exists a smooth $(0,q)$-form β_n such that

$$\tilde{\partial}\beta_n = \alpha \;\text{ and }\; \|\beta_n\|^2_{h_n} \leq \frac{1}{\epsilon'.q}\|\alpha\|^2_{h_n}.$$

Hence $\{\beta_n\}_{n\geq n_0}$ is uniformly bounded on the compact subsets of X. We can choose a subsequence $\{\beta_{n_k}\}_{k\geq 1}$ which has a limit in the weak topology of $L^{0,q-1}(E)$:

$$\beta_{n_k} \to_k \beta.$$

Since $\overline{\partial}$ is a continuous and regular operator we have

$$\alpha = \overline{\partial}\beta = d^{q-1}\beta$$

and β is smooth. Finally

$$\frac{C}{\epsilon'.q} \geq \|\beta_{n_k}\|^2_{h_{n_k}} \longrightarrow_k \|\beta\|^2_{\tilde{h}},$$

and we get that

$$\beta \in F^{q-1} \;\text{ and }\; d^{q-1}\beta = \alpha,$$

which proves our lemma. ▹

Lemma 3.3. *The complex of sheaves $\{\mathcal{F}^q, d^q\}_{q\geq 0}$ is a resolution for $\mathcal{E}\otimes\mathcal{I}(\varphi)$.*

◃ The proof is identical with that of Lemma 3.2 but for Stain open subsets of X instead of X.▹

Obviously $\{\mathcal{F}^q, d^q\}_{q\geq 0}$ is a fine resolution for $\mathcal{E}\otimes\mathcal{I}(\varphi)$ and so

$$H^q(X, \mathcal{E}\otimes\mathcal{I}(\varphi)) \cong H^q(F^\cdot, d^\cdot).$$

Lemma 3.2 gives us that $H^q(X, \mathcal{E}\otimes\mathcal{I}(\varphi)) = 0$ for $q \geq 1$ ▹

Remark 3.4. The assertion in Theorem 3.1 is a special case of the following more general result

Theorem 3.5. *Let X be a compact complex manifold with a Kähler metric g; let k be the induced metric on K^*_X. Suppose (E,h) is an Hermitian* *vector bundle over X, and φ is an almost pluri-subharmonic function on X with Y the corresponding to φ subscheme of X.If*

$$k.\exp(\varphi).\Omega(\tilde{h}) + \Omega(k).h \geq \epsilon.\omega.h$$

where $\tilde{h} = h.\exp(-\varphi), \epsilon > 0$, and ω is a Kähler form on X, then for all $q > 0$

$$H^q(X, \mathcal{E} \otimes \mathcal{I}_Y) = 0.$$

In this paper we don't need this general result.

Now we want to apply Theorem 3.1 to Fano manifolds.

Definition. *A subscheme Y of a Fano manifold X is called* Nadel's subscheme *of X if for every Nakano semi-positive vector bundle E*

$$H^q(X, \mathcal{E} \otimes \mathcal{I}_Y) = 0$$

for each positive q.

We refer to Nadel [5] and Yotov [9] for the properties of Nadel's subschemes.

Let $Y = Y_1 \cup \ldots \cup Y_m$ be the decomposition of a suitable subscheme of a Fano manifold X into irreducible components. Let $\mathrm{Pic}(X) = \mathrm{Z}.L$, where L is ample. Then each Y_i is scheme-theoretically determined by $L_{ij} \in \mathrm{Pic}(X)$, $(j = 1, \ldots, m_i)$, for which

$$L_{ij} = L^{\otimes n_{ij}} \quad \text{where} \quad n_{ij} \quad \text{are} \quad \text{positive} \quad \text{integers.}$$

Denote by n_i the maximum of $n_{i1}, \ldots, n_{im_i}$.

Corollary 3.6. *Let $r_i = \mathrm{codim}\,_X Y_i$, $i = 1, \ldots, m$, and $K_X^* = L^{\otimes s}$. If*

$$\sum_{i=1}^{m} r_i.n_i + 1 \leq s,$$

then Y is a Nadel's subscheme of X.

P r o o f. Since L is ample, then there exists a metric l on L with $\Omega(l) > 0$. Without loss of generality we may assume that l is induced by a Kähler metric on X.

Let (E, h) be an Hermitian vector bundle over X for which $\Omega(h) \geq 0$, and let $\tilde{h} = h.\exp(-\varphi)$, where φ is the almost pluri-subharmonic function corresponding to Y via Theorem 2.7. We have

$$l^s.\exp(\varphi).\Omega(\tilde{h}) + \Omega(l^s).h \geq \Omega(h) + \left(s - \sum_{i=1}^{m} r_i n_i\right)\Omega(l).h \geq \Omega(l).h.$$

Now we can apply Theorem 3.1 to deduce that

$$H^q(X, \mathcal{E} \otimes \mathcal{I}_Y) = 0 \quad \text{for} \quad q \geq 1.$$

This proves the Corollary 3.6. □

4. Some examples. Let X be the Grassman manifold $\mathbf{G}(k, n)$ of k-planes in P^n. It's well known that $\mathrm{Pic}(X) = \mathrm{Z}.\mathcal{L}$, where $\mathcal{L}$ is the pull-back of $\mathcal{O}_{\mathrm{P}^N}(1)$ via the Plücker map

$$Pl : X \longrightarrow \mathrm{P}^N, \quad N = \binom{n+1}{k+1} - 1.$$

Here $K_X^* \cong \mathcal{L}^{\otimes(n+1)}$. The Theorem 3.1 is applicable to X. The special case of $k = 0$ is very interesting.

1. Let Y be an equidimensional *suitable* subscheme of P^n of codimension 1. In this case Theorem 3.1 doesn't give anything new:

If deg$Y \leq n$, then Y is a Nadel's subscheme of P^n .

In fact, deg$Y \leq n$ is sufficient-and-necessary condition for a divisor on P^n to be a Nadel's subscheme.

2. Another interesting case is when Y is a (suitable) complete intersection of codimension 2. Now Y is determined by $\mathcal{O}(d_1)$ and $\mathcal{O}(d_2)$, and deg$Y = d_1.d_2$. If Y is nondegenerate, which is the only interesting case (as we shall see later on), we get

If deg$Y \leq n$, then Y is a Nadel's subscheme of P^n .

3. The third case we want to apply Theorem 3.1 to is of one-dimensional subscheme Y, and $n \geq 3$. Here Y is suitable iff Y is smooth, i.e. Y is a disjoint union of its smooth components.

These are some well known facts about Nadel's subschemes we shall use in what follows (see Nadel [5]):

Fact 1. Every Nadel's subscheme is connected as a topological space.

Fact 2. If Y is 1-dimensional Nadel's subscheme, then Y_{red} consists of smooth rational curves which intersect each-other at most once. Moreover, there must not be any circles of lines in Y_{red}.

It follows from Fact 2 that if Y is smooth, then it is isomorphic to P^1. Suppose that Y is smooth and deg$Y = d$.

3.1. Let $d \geq n+1$. It is easy to see that Y is not Nadel's. Indeed, let E be the line bundle $[H]$, where H is a hyperplane in P^n. Since E is ample, it is Nakano positive. The short exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_Y \otimes \mathcal{O}_{\mathrm{P}^n}(1) \longrightarrow \mathcal{O}_{\mathrm{P}^n}(1) \longrightarrow \mathcal{O}_{\mathrm{P}^n}(1) \otimes \mathcal{O}_Y \longrightarrow 0$$

gives that

$$h^0(\mathrm{P}^n, \mathcal{O}_{\mathrm{P}^n}(1)) = n+1, \quad h^0(Y, \mathcal{O}_{\mathrm{P}^n}(1) \otimes \mathcal{O}_Y) = d+1 \geq n+2.$$

Hence,

$$h^1(\mathrm{P}^n, \mathcal{I}_Y \otimes \mathcal{O}_{\mathrm{P}^n}(1)) \neq 0,$$

and Y is not a Nadel's subscheme of P^n.

3.2. Let $d \leq n-1$. In this case Y is degenerate (i.e., Y lies in a proper linear subspace of P^n. Let P^m be a subspace of minimal dimension in P^n containing Y. Hence, $d \geq m$. The short exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_{\mathrm{P}^m}(1) \longrightarrow \mathcal{O}_{\mathrm{P}^n}(1) \longrightarrow \mathcal{O}_{\mathrm{P}^m}(1) \longrightarrow 0,$$

combined with the Bott formula about the cohomology groups of a projective space, gives us that

$$h^i(\mathrm{P}^n, \mathcal{I}_{\mathrm{P}^m}(1)) \leq h^i(\mathrm{P}^n, \mathcal{O}_{\mathrm{P}^n}(1)) = 0, \quad i = 1, 2.$$

On the other hand, from the exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_{\mathrm{P}^m}(1) \longrightarrow \mathcal{I}_Y(1) \longrightarrow \mathcal{I}_Y \otimes \mathcal{O}_{\mathrm{P}^m}(1) \longrightarrow 0$$

we get that

$$h^1(\mathrm{P}^n, \mathcal{I}_Y(1)) = h^1(\mathrm{P}^m, \mathcal{I}_Y(1) \otimes \mathcal{O}_{\mathrm{P}^m}).$$

Hence, if $d \geq m+1$, then Y is not a Nadel's subscheme of P^n.

3.3. The only essential case is when $Y = C_n$ is a rational normal curve of degree n in P^n.

Claim 1. *There exists one-dimensional smooth deformation of a non-degenerate $Y = C_{n-1} \bigcup l \subset \mathrm{P}^n$ with rational normal curves C_n outside the central fibre.*

Indeed, the corresponding deformation is given in $\mathrm{P}^n \times \mathbf{C}^1$ by the equations

$$rk \begin{pmatrix} z_0 & \dots & z_{n-2} & t.z_{n-1} \\ z_1 & \dots & z_{n-1} & z_n \end{pmatrix} \leq 1.$$

Here C_{n-1} is a rational normal curve in $\{z_n = 0\}$, and l is the line $\{z_0 = z_1 = \cdots = z_{n-2} = 0\}$. Let Y_t denote the fiber of this deformation over t. Hence, Y is isomorphic to Y_0.

Claim 2. *For each Nakano semi-positive E* $h^1(\mathrm{P}^n, \mathcal{I}_{Y_0} \otimes \mathcal{E}) = 0$.

Let H_n be the hyperplane $\{z_n = 0\}$. Obviously, $H_n \cup l$ is a suitable subscheme of P^n to which we can apply Corollary 3.6. We get

$$H^q(\mathrm{P}^n, \mathcal{I}_{H_n \cup l} \otimes \mathcal{E}) = 0, \quad \text{for} \quad q > 0.$$

On the other hand, we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_{H_n \cup l} \longrightarrow \mathcal{I}_{Y_0} \longrightarrow \mathcal{I}_{C_{n-1}} \otimes \mathcal{O}_{H_n} \longrightarrow 0.$$

Tensoring this sequence by E, the corresponding long exact sequence

$$\ldots \longrightarrow H^1(\mathrm{P}^n, \mathcal{I}_{H_n \cup l} \otimes \mathcal{E}) \longrightarrow H^1(\mathrm{P}^n, \mathcal{I}_{Y_0} \otimes \mathcal{E}) \longrightarrow H^1(H_n, \mathcal{I}_{C_{n-1}} \otimes \mathcal{E})$$
$$\longrightarrow H^2(\mathrm{P}^n, \mathcal{I}_{H_n \cup l} \otimes \mathcal{E}) \longrightarrow \ldots$$

gives us that

$$H^1(\mathrm{P}^n, \mathcal{I}_{Y_0} \otimes \mathcal{E}) \cong H^1(H_n, \mathcal{I}_{C_{n-1}} \otimes \mathcal{E}).$$

It is a well known fact that Nakano semi-positivity of a vector bundle remains valid when restricting on submanifolds. So, we can proceed by induction. The fact that C_2 is a Nadel's subscheme of P^2 completes the proof of our claim.

Since the deformation of Y_0 in **Claim 1.** is flat and proper we can apply the theorem of semicontinuity of cohomology groups

$$h^1(\mathrm{P}^n, \mathcal{I}_{Y_t} \otimes \mathcal{E}) \leq h^1(\mathrm{P}^n, \mathcal{I}_{Y_0} \otimes \mathcal{E}).$$

But Y_t $(t \neq 0)$ is isomorphic to C_n, and we conclude that

Proposition 4.1. *The rational normal curve* C_n *is a Nadel's subscheme of* P^n.

By using the method of the proof of **Claim 2.** one easily can prove the following

Proposition 4.2. *Suppose that* Y *is a reduced curve in* P^n *of degree* 3 $(n \geq 3)$. *If* Y *is a Nadel's subscheme of* P^n, *then*
either
1) Y *is a rational normal curve in some three-dimensional projective subsspace*
or

2) *Y is a noncomplanar connected union of a conic with a line $q \cup l$*
or
3) *Y is a noncomplanar connected union of three lines $l_1 \cup l_2 \cup l_3$.*

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