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ON TYPICAL COMPACT CONVEX SETS IN HILBERT SPACES

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Abstract. Let $\mathbb{E}$ be an infinite dimensional separable space and for $e \in \mathbb{E}$ and X a nonempty compact convex subset of $\mathbb{E}$, let $q_X(e)$ be the metric antiprojection of e on X. Let $n \geq 2$ be an arbitrary integer. It is shown that for a typical (in the sence of the Baire category) compact convex set $X \subset \mathbb{E}$ the metric antiprojection $q_X(e)$ has cardinality at least n for every e in a dense subset of $\mathbb{E}$.

1. Introduction. It is well known that Baire category techniques are a powerful tool in order to prove the existence of mathematical objects with elusive and, sometimes, unexpected properties. While this was soon realized in Analysis applications to Geometry have been found much later.

The first significant applications of the Baire category to Convex Geometry are contained in a classical paper by Klee [8], published in 1959. Further contributions were given independently by Gruber [6] in a paper appeared in 1977, in which some of Klee's results are proved again and several new ones established. Since then Baire category techniques have been used by many mathematicians

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in order to discover geometric objects whose existence was not easy to prove or was unknown at all. For a survey and a comprehensive bibliography about Baire category in Geometry see Gruber [7] and a Zamfirescu [14].

Let $\mathbb{E}$ be a real Hilbert space and let $\mathcal{C}(\mathbb{E})$ be the space of all nonempty compact convex subsets of $\mathbb{E}$. For $X \in \mathcal{C}(\mathbb{E})$ denote by q_X the *metric antiprojection* mapping that is the multifunction which associates to each $e \in \mathbb{E}$ the set $q_X(e)$ of all $x \in X$ whose distance from e attains the maximum value. For $X \in \mathcal{C}(X)$ and $n \in \mathbb{N}$ we denote by $M^{n+1}(X)$ and $S(X)$ respectively, the *multivalued locus* of q_X of order $n+1$ (i.e. the set of all $e \in \mathbb{E}$ with $\operatorname{card} q_X(e) \geq n+1$) and the *singlevalued locus* (i.e. the set of all $e \in \mathbb{E}$ with card $q_X(e) = 1$).

If M is a complete metric space, the elements of any residual subset of M will be also called *typical* elements of M.

In the present paper we study the multivalued locus of the metric antiprojection mapping q_X, when X is a typical element of $\mathcal{C}(\mathbb{E})$.

Suppose that $\mathbb{E}$ is infinite dimensional and separable. Then we shall prove that for a typical compact convex set $X \subset \mathbb{E}$ the multivalued locus $M^{n+1}(X)$ of q_X of order $n+1$, $n \in \mathbb{N}$ arbitrary, is dense in $\mathbb{E}$. Consequently, for a typical $X \in \mathcal{C}(\mathbb{E})$, each point $e \in \mathbb{E}$ is limit of a sequence of points $e_n \in \mathbb{E}$ with $\operatorname{card} q_X(e_n) \geq n+1$, hence tending to infinity with n. So far no example of a compact convex set $X \subset \mathbb{E}$ with this property seems to be known.

Our approach is based upon Baire category techniques, following some ideas of Klee, Gruber and Zamfirescu. Furthermore, a key role is played by a topological theorem, due to Brouwer [2] and Miranda [10], which turns out to be equivalent to Brouwer's fixed point theorem (see [10]).

2. Notation and preliminaries. Throughout the present paper $\mathbb{E}$ denotes a real infinite dimensional Hilbert space with inner product $\langle x, y\rangle$ and induced norm $\|x\|$, $x, y \in \mathbb{E}$, and $\mathcal{C}(\mathbb{E})$ (resp. $\mathcal{K}(\mathbb{E})$) the space of all nonempty compact convex (resp. nonempty compact) subsets of $\mathbb{E}$ endowed with the Hausdorff metric h. As is well known, under this metric $\mathcal{C}(\mathbb{E})$ and $\mathcal{K}(\mathbb{E})$ are complete metric spaces. For any $X \subset \mathbb{E}$ we denote by $\overline{co}X$ the closed convex hull of X. For $x, y \in \mathbb{E}$, $[x, y]$ stands for the closed line interval contained in $\mathbb{E}$ with end points x and y.

Let M be a metric space. By $U_M(x, r)$ (resp. $\hat{U}_M(x, r)$) we mean on open (resp. closed) ball in M with center x and radius r. In $\mathbb{E}$ we put, for brevity,

$U = U_{\mathbb{E}}(0,1)$. If $X \subset M$, $\operatorname{diam} X$, $\operatorname{card} X$ $(X \neq \emptyset)$ we denote respectively, the diameter, and the cardinality of X.

A set $X \subset M$, M a complete metric space, is called *residual* in M, if $M \setminus X$ is a set of the Baire first category in M. As is well known, X is a residual subset of M if and only if X contains a dense G_δ – subset of M. The elements of a residual subset of M are also called *typical* elements of M.

A map $F : M \to \mathcal{K}(\mathbb{E})$ is called *upper semicontinuous* if for every $x \in M$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that $F(y) \subset F(x) + \varepsilon U$ for every $y \in U_M(x, \delta)$.

For X a nonempty bounded subset of $\mathbb{E}$ and $e \in \mathbb{E}$, we put

$$\delta(X,e) = \sup\{\|x-e\| \mid x \in X\} \quad \gamma(X,e) = \inf\{\|x-e\| \mid x \in X\}.$$

Let $X \in \mathcal{K}(\mathbb{E})$ and $e \in \mathbb{E}$ be any. The set $q_X(e)$ given by

$$q_X(e) = \{x \in X \mid \|x-e\| = \delta(X,e)\} \tag{2.1}$$

is called *metric antiprojection* from e to X.

Clearly $q_X(e) \in \mathcal{K}(\mathbb{E})$, thus for a fixed $X \in \mathcal{K}(\mathbb{E})$, (2.1) defines a map $q_X : \mathbb{E} \to \mathcal{K}(\mathbb{E})$, called *metric antiprojection* mapping from $\mathbb{E}$ to X. Observe also that the map $(X, e) \mapsto q_X(e)$, from $\mathcal{K}(\mathbb{E}) \times \mathbb{E}$ to $\mathcal{K}(\mathbb{E})$, is upper semicontinuous.

Let $X \in \mathcal{K}(\mathbb{E})$ and $\varepsilon > 0$ be any. The sets $M^{n+1}(X)$ and $M^{n+1,\varepsilon}(X)$ given by

$$M^{n+1}(X) = \{e \in \mathbb{E} \mid \operatorname{card} q_X(e) \geq n+1\},$$

$$M^{n+1,\varepsilon}(X) = \{e \in \mathbb{E} \mid \operatorname{card} q_X(e) \geq n+1 \;\text{ and }\; \operatorname{diam} q_X(e) \leq \varepsilon\}$$

are called respectively, *multivalued locus of* q_X of *order* $n+1$, and ε*–multivalued locus of* q_X of *order* $n+1$. Moreover, the set

$$S(X) = \{e \in \mathbb{E} \mid \operatorname{card} q_X(e) = 1\}$$

is called *singlevalued locus* of q_X.

In the sequel we will use the following topological result, contained in an implicit form in Brouwer [2], which, as shown by Miranda [10], is equivalent to Brouwer's fixed point theorem.

Brouwer–Miranda Theorem. *Let $f_k : Q_n^\theta \to \mathbb{R}$, $k = 1, \ldots, n$ be n continuous functions defined in the hypercube $Q_n^\theta = [-\theta, \theta] \times \cdots \times [-\theta, \theta]$ (n* *times), $\theta > 0$, and for $k = 1, \dots, n$, set $L_k^{\pm\theta} = \{(t_1, \dots, t_n) \in Q_n^\theta | t_k = \pm\theta\}$. If for $k = 1, \dots, n$ we have*

$$f_k(t) < 0 \quad \textit{for every} \quad t \in L_k^{-\theta}, \qquad f_k(t) > 0 \quad \textit{for every} \quad t \in L_k^{+\theta},$$

where $t = (t_1, \dots, t_n)$, then there exists a point $\hat{t} \in Q_n^\theta$ such that $f_k(\hat{t}) = 0$ for $k = 1, \dots, n$.

3. Multivalued loci. In this section we show that for a typical compact convex set $X \subset \mathbb{E}$ the multivalued locus of q_X of order $n+1$ is dense in $\mathbb{E}$. To this end we need the following technical lemma.

Lemma. *Let $\mathbb{E}$ be a real infinite dimensional Hilbert space. Let $A_0 \in \mathcal{C}(\mathbb{E})$ and $e_0 \in \mathbb{E}$ be such that $\delta(A_0, e_0) > 0$. Let $n \in \mathbb{N}$, $\varepsilon > 0$, $\lambda > 0$ and $r > 0$ be arbitrary. Then there exist $B \in \mathcal{C}(\mathbb{E})$ and $\sigma > 0$, with $U_{\mathcal{C}(\mathbb{E})}(B, \sigma) \subset U_{\mathcal{C}(\mathbb{E})}(A_0, \lambda)$, such that for every $X \in U_{\mathcal{C}(\mathbb{E})}(B, \sigma)$ we have*

$$M^{n+1,\varepsilon}(X) \cap U_{\mathbb{E}}(e_0, r) \neq \emptyset.$$

P r o o f. The proof, rather long, will be divided into five steps.

Step 1. Construction of B.

Take $a_0 \in A_0$ satisfying $\|a_0 - e_0\| = \delta(A_0, e_0)$. Fix γ and β so that

$$1 < \gamma < \min\left\{2, 1 + \frac{\omega}{4\|a_0 - e_0\|}\right\} \tag{3.1}$$

$$\gamma > \beta > \max\left\{1, \gamma - \frac{\omega^2}{64\|a_0 - e_0\|^2}, \frac{n-1}{n+1}\gamma\right\}, \tag{3.2}$$

where

$$0 < \omega < \min\{\varepsilon, \lambda\}. \tag{3.3}$$

Let $\{u_k\}_{k=1}^n$ be a set of n mutually orthogonal vectors u_k of norm one contained in the hyperplane $\{x \in \mathbb{E} \mid \langle x, a_0 - e_0 \rangle = 0\}$. This set certainly exists for $\dim \mathbb{E} = +\infty$. Now, construct a set $\{b_k\}_{k=0}^n$ of $n+1$ vectors b_k by

$$\begin{aligned} b_0 &= e_0 + \gamma(a_0 - e_0) \\ b_k &= e_0 + \beta(a_0 - e_0) + v_k \quad \text{where} \quad v_k = \sqrt{\gamma^2 - \beta^2}\|a_0 - e_0\| u_k \quad k = 1, \dots, n. \end{aligned}$$

Clearly,

$$\|b_k - b_0\| = \sqrt{2\gamma(\gamma-\beta)}\|a_0 - e_0\|, \quad k = 1,\dots,n \tag{3.4}$$

$$\|b_k - b_h\| = \sqrt{2(\gamma^2-\beta^2)}\|a_0 - e_0\|, \quad k,h = 1,\dots,n, \ k \neq h \tag{3.5}$$

$$\|b_k - e_0\| = \gamma\|a_0 - e_0\|, \quad k = 0,1,\dots,n. \tag{3.6}$$

Set

$$B = \overline{co}\{b_0, b_1, \dots, b_n, A_0\},$$

and observe that $B \in \mathcal{C}(\mathbb{E})$, by Mazur's theorem. From (3.4), (3.5) and (3.6), it follows that $\{b_k\}_{k=0}^n$ is a set of $n+1$ different points lying on the boundary of the open ball $U_{\mathbb{E}}(e_0, \gamma\|a_0 - e_0\|)$. Hence as A_0 is contained in this ball, we have

$$q_B(e_0) = \{b_0, b_1, \dots, b_n\} \text{ and } \operatorname{card} q_B(e_0) = n+1.$$

From (3.4) as $\gamma < 2$ and $\gamma - \beta < \omega^2/(64\|a_0 - e_0\|^2)$ we have $\|b_k - b_0\| < \omega/4$, $k = 1,\dots,n$. Furthermore $\|b_0 - a_0\| < \omega/4$, as $\|b_0 - a_0\| = (\gamma-1)\|a_0 - e_0\|$ and $\gamma - 1 < \omega/(4\|a_0 - e_0\|)$ hence, by the triangle inequality,

$$\|b_k - b_h\| < \frac{\omega}{2}, \quad h,k = 0,1,\dots,n \tag{3.7}$$

$$\|b_k - a_0\| < \frac{\omega}{2}, \quad k = 0,1,\dots,n. \tag{3.8}$$

As $\omega < \min\{\varepsilon, \lambda\}$, (3.7) and (3.8) imply, respectively

$$\operatorname{diam} q_B(e_0) < \frac{\varepsilon}{2} \tag{3.9}$$

$$h(B, A_0) < \frac{\lambda}{2}. \tag{3.10}$$

In view of Step 2, we introduce some further notation. For $k = 0,1,\dots,n$ put

$$\tilde{A}_k = \overline{co}\{b_0, \dots, b_{k-1}, b_{k+1}, \dots, b_n, A_0\}.$$

It is routine to verify that $\gamma(\tilde{A}_k, b_k) > 0$. Next, fix $\tilde{\eta} > 0$ satisfying

$$\tilde{\eta} < \min\Big(\{\gamma(\tilde{A}_k, b_k) | k=0,1,\dots,n\} \cup \{\|b_k - b_h\|/4 \mid k,h=0,1,\dots,n, \ k\neq h\}\Big). \tag{3.11}$$

For $k = 0, 1, \dots, n$ and $0 < \eta < \tilde{\eta}$ set

$$B_k = B \cap \tilde{U}_{\mathbb{E}}(b_k, \eta) \quad D_k = B \cap \tilde{U}_{\mathbb{E}}(b_k, \tilde{\eta}) \quad \tilde{D} = B \setminus \bigcup_{k=0}^{n} U_{\mathbb{E}}(b_k, \tilde{\eta})$$

and observe that B_k, D_k and $\tilde{D}$ are compact nonempty.

Step 2. Let $\tilde{\eta}$ satisfy (3.11). *Then there exists $0 < \eta < \min\{\tilde{\eta}, \omega/4\}$ such that for every $e \in U_{\mathbb{E}}(e_0, \eta)$ we have*

$$q_{B_k}(e) = b_k \quad k = 0, 1, \dots, n. \tag{3.12}$$

It is easy to see that for $k = 0, 1, \dots, n$

$$D_k = \bigcup_{b \in \tilde{A}_k} [b_k, r(b)] \text{ where } r(b) = b_k + \frac{b - b_k}{\|b - b_k\|}\tilde{\eta}. \tag{3.13}$$

Set $\delta_0 = \delta(B, e_0)$. Since $\tilde{D}$ is a compact set contained in the open ball $U_{\mathbb{E}}(e_0, \delta_0)$, for some $0 < \tilde{\delta} < \delta_0$ we have $\tilde{D} \subset U_{\mathbb{E}}(e_0, \tilde{\delta})$. Now fix η satisfying

$$0 < \eta < \min\left\{\tilde{\eta}, \frac{\delta_0(\delta_0 - \tilde{\delta})}{2\delta_0 + \tilde{\eta}}, \frac{\omega}{4}\right\}. \tag{3.14}$$

With this choice of η, the statement of Step 2 is verified. In fact, let $0 \le k \le n$ be any. It suffices to show that if $a \in B_k$, $a \neq b_k$, and $e \in U_{\mathbb{E}}(e_0, \eta)$ are arbitrary, we have

$$\|a - e\| < \|b_k - e\|. \tag{3.15}$$

Clearly $a \in D_k$, for $\eta < \tilde{\eta}$, thus by (3.13) there exist $b \in \tilde{A}_k$ and $0 < t \le \eta/\tilde{\eta}$ such that $a = (1 - t)b_k + tr(b)$. Further, $r(b) \in \tilde{D}$ because $\|r(b) - b_k\| = \tilde{\eta}$ and, if $h \neq k$, $\|r(b) - b_h\| \ge \|b_h - b_k\| - \|b_k - r(b)\| > \tilde{\eta}$, by (3.11).

We have

$$\begin{aligned}
\|b_k - e\|^2 - \|a - e\|^2 &= \|b_k - e\|^2 - \|(1 - t)(b_k - e) + t(r(b) - e)\|^2 \\
&= t[(2 - t)\|b_k - e\|^2 - t\|r(b) - e\|^2 - 2(1 - t)\langle b_k - e, r(b) - e\rangle] \\
&= t[-t(\|b_k - e\|^2 + \|r(b) - e\|^2 - 2\langle b_k - e, r(b) - e\rangle) + 2\|b_k - e\|^2
\end{aligned}$$

$$
\begin{aligned}
&\qquad -2\langle b_k - e, r(b) - e\rangle] \\
&= t[-t\|b_k - r(b)\|^2 + 2\|b_k - e\|^2 - 2\langle b_k - e, r(b) - e\rangle] \\
&\geq t[-\eta\tilde{\eta} + 2\|b_k - e\|(\|b_k - e\| - \|r(b) - e\|)].
\end{aligned}
$$

But, $\|b_k - e\| \geq \|b_k - e_0\| - \|e - e_0\| > \delta_0 - \eta$ and $\|r(b) - e\| \leq \|r(b) - e_0\| + \|e - e_0\| < \tilde{\delta} + \eta$, for $r(b) \in \tilde{D} \subset U_{\mathbb{E}}(e_0, \tilde{\delta})$. As $0 < \eta < \delta_0(\delta_0 - \tilde{\delta})/(2\delta_0 + \tilde{\eta}) < \delta_0/2$, we have

$$
\begin{aligned}
\|b_k - e\|^2 - \|a - e\|^2 &\geq t[-\eta\tilde{\eta} + 2(\delta_0 - \eta)(\delta_0 - \tilde{\delta} - 2\eta)] \\
&\geq t[-\eta\tilde{\eta} + \delta_0(\delta_0 - \tilde{\delta} - 2\eta)] = t(2\delta_0 + \tilde{\eta}) \left[\frac{\delta_0(\delta_0 - \tilde{\delta})}{2\delta_0 + \tilde{\eta}} - \eta \right],
\end{aligned}
$$

a strictly positive quantity. Hence (3.15) is verified. Thus (3.12) holds true, completing the proof of Step 2.

For $X \in \mathcal{C}(\mathbb{E})$ and $k = 0, 1, \ldots, n$, put

$$
X_k = X \cap \tilde{U}_{\mathbb{E}}(b_k, \eta) \quad \tilde{X} = X \setminus \bigcup_{k=0}^{n} U_{\mathbb{E}}(b_k, \eta), \tag{3.16}
$$

where η is as in Step 2. For every $X \in U_{\mathcal{C}(\mathbb{E})}(B, \varrho)$, where $0 < \varrho < \eta$, the set $\tilde{X}$ (as well as each X_k, $k = 0, 1, \ldots, n$) is compact nonempty, and

$$
X_0 \cup X_1 \cup \cdots \cup X_n \cup \tilde{X} = X.
$$

Step 3. Let η *satisfy* (3.14)*. Then there exists* ρ

$$
0 < \rho < \min\{\eta, r\}, \tag{3.17}
$$

such that, for every $X \in U_{\mathcal{C}(\mathbb{E})}(B, \rho)$ *and* $e \in U_{\mathbb{E}}(e_0, \rho)$ *we have*

$$
\delta(X_k, e) > \delta(\tilde{X}, e) \qquad k = 0, 1, \ldots, n,
$$

where the X_k*'s and* $\tilde{X}$ *are given by* (3.16).

In the contrary case, there are sequences $\{Y_p\} \subset \mathcal{C}(\mathbb{E})$ and $\{e_p\} \subset \mathbb{E}$ converging respectively, to B and e_0, and there is a k, $0 \leq k \leq n$, such that for every $p \in \mathbb{N}$ we have

$$
\delta(Y_{p,k}, e_p) \leq \delta(\tilde{Y}_p, e_p), \tag{3.18}
$$

where $Y_{p,k} = Y_p \cap \tilde{U}_{\mathbb{E}}(b_k,\eta)$ and $\tilde{Y}_p = Y_p \setminus \bigcup\limits_{i=0}^{n} U_{\mathbb{E}}(b_i,\eta)$. Let $\{y_p\}$, $y_p \in Y_p$, be a sequence converging to b_k. For all p large enough, $y_p \in Y_{p,k}$ thus

$$\delta(Y_{p,k}, e_p) \geq \|y_p - e_p\|. \tag{3.19}$$

On the other hand, for $p \in \mathbb{N}$, let $\tilde{y}_p \in \tilde{Y}_p$ satisfy

$$\delta(\tilde{Y}_p, e_p) = \|\tilde{y}_p - e_p\|. \tag{3.20}$$

By compactness a sequence, say $\{\tilde{y}_p\}$, converges to some $\tilde{y} \in \tilde{B}$, where $\tilde{B}$ is given by (3.16), with B in the place of X. From (3.18), (3.19) and (3.20) by letting $p \to +\infty$ we have $\|b_k - e_0\| \leq \|\tilde{y} - e_0\|$, which implies $\delta_0 \leq \delta(\tilde{B}, e_0)$. But $\tilde{B}$ is compact and satisfies $\tilde{B} \subset U_{\mathbb{E}}(e_o, \delta_0)$, thus $\delta(\tilde{B}, e_0) < \delta_0$, a contradiction. Hence Step 3 holds true.

Now fix θ so that

$$0 < \theta < \frac{\rho}{2n\|a_0 - e_0\|}, \tag{3.21}$$

where ρ is as in Step 3, and set $Q_n^\theta = [-\theta,\theta] \times \cdots \times [-\theta,\theta]$, n times, $L_k^{\pm\theta} = \{(t_1,\ldots,t_n) \in Q_n^\theta \mid t_k = \pm\theta\}$. Define $e : Q_n^\theta \to \mathbb{E}$ by

$$e(t) = e_0 + \sum_{k=1}^{n} t_k(b_k - b_0) \quad \text{where} \quad t = (t_1,\ldots,t_n).$$

Observe that, from (3.4), $\|b_k - b_0\| < 2\|a_0 - e_0\|$, $k = 1,\ldots,n$, thus by (3.21)

$$e(t) \in U_{\mathbb{E}}(e_0,\rho) \quad \text{for every} \quad t \in Q_n^\theta. \tag{3.22}$$

Step 4. Let η, ρ, θ satisfy (3.14), (3.17) *and* (3.21). *Then there is $\sigma > 0$ such that, for every $X \in U_{\mathcal{C}(\mathbb{E})}(B,\sigma)$ and $k = 1,\ldots,n$ we have*

$$\delta(X_0, e(t)) - \delta(X_k, e(t)) < 0 \qquad \textit{for every} \quad t \in L_k^{-\theta} \tag{3.23}$$

$$\delta(X_0, e(t)) - \delta(X_k, e(t)) > 0 \qquad \textit{for every} \quad t \in L_k^{+\theta}, \tag{3.24}$$

where $X_0, X_1, \ldots, X_n$ are given by (3.16).

Let $1 \leq k \leq n$ be arbitrary. We prove first that (3.23) is verified when $X = B$. Let $t \in L_k^{-\theta}$ be any. Since $\|e(t) - e_0\| < \rho < \eta$, by Step 2 we have

$$\delta(B_0, e(t)) - \delta(B_k, e(t)) = \|b_0 - e(t)\| - \|b_k - e(t)\|.$$

From the definition of $e(t)$, b_0 and b_k it follows

$$
\begin{aligned}
b_0 - e(t) &= \left[\gamma + (\beta-\gamma)\left(\theta - \sum_{\substack{i=1\\ i\neq k}}^{n} t_i\right)\right](a_0 - e_0) + \theta v_k \sum_{\substack{i=1\\ i\neq k}}^{n} t_i v_i \\
b_k - e(t) &= \left[\beta + (\beta-\gamma)\left(\theta - \sum_{\substack{i=1\\ i\neq k}}^{n} t_i\right)\right](a_0 - e_0) + (1+\theta) v_k - \sum_{\substack{i=1\\ i\neq k}}^{n} t_i v_i.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\|b_0 - e(t)\|^2 - \|b_k - e(t)\|^2 \\
&= \left\{\left[\gamma + (\beta-\gamma)\left(\theta - \sum_{\substack{i=1\\ i\neq k}}^{n} t_i\right)\right]^2 - \left[\beta + (\beta-\gamma)\left(\theta - \sum_{\substack{i=1\\ i\neq k}}^{n} t_i\right)\right]^2\right\}\|a_0 - e_0\|^2 \\
&\qquad\qquad -(1+2\theta)\|v_k\|^2 \\
&= \left[\gamma^2 - \beta^2 - 2(\gamma-\beta)^2\left(\theta - \sum_{\substack{i=1\\ i\neq k}}^{n} t_i\right)\right]\|a_0 - e_0\|^2 - (1+2\theta)(\gamma^2-\beta^2)\|a_0 - e_0\|^2 \\
&= -2(\gamma-\beta)\left[\theta(\gamma+\beta) + (\gamma-\beta)\left(\theta - \sum_{\substack{i=1\\ i\neq k}}^{n} t_i\right)\right]\|a_0 - e_0\|^2,
\end{aligned}
$$

which is strictly negative for, in view of (3.2),

$$
\theta(\gamma+\beta)+(\gamma-\beta)\left(\theta - \sum_{\substack{i=1\\ i\neq k}}^{n} t_i\right) \geq \theta(\gamma+\beta) - n\theta(\gamma-\beta) = (n+1)\theta\left[\beta - \frac{n-1}{n+1}\gamma\right] > 0.
$$

This shows that $\delta(B_0, e(t)) - \delta(B_k, e(t)) < 0$ for every $t \in L_k^{-\theta}$. Similarly one can prove that $\delta(B_0, e(t)) - \delta(B_k, e(t)) > 0$ for every $t \in L_k^{+\theta}$. Thus for $k = 1, \ldots, n$, (3.23) and (3.24) are satisfied with $X = B$.

Consider now the general case. Since $L_k^{-\theta}$ and $L_k^{+\theta}$ are compact, there is $\mu > 0$ so that, for $k = 1, \ldots, n$ we have:

$$
\delta(B_0, e(t)) - \delta(B_k, e(t)) < -\mu \qquad \text{for every} \quad t \in L_k^{-\theta} \tag{3.25}
$$

$$\delta(B_0, e(t)) - \delta(B_k, e(t)) > \mu \qquad \text{for every} \qquad t \in L_k^{+\theta}. \tag{3.26}$$

On the other hand, for $k = 0, 1, \ldots, n$, the map $X \mapsto X_k$, $X \in \mathcal{C}(\mathbb{E})$, is continuous at $X = B$. Hence there exists σ,

$$0 < \sigma < \min\left\{\rho, \frac{\lambda}{2}\right\}, \tag{3.27}$$

such that for every $X \in U_{\mathcal{C}(\mathbb{E})}(B, \sigma)$ we have $h(X_k, B_k) < \mu/2$, $k = 0, 1, \ldots, n$.

With this choice of σ the statement of Step 4 is verified. In fact, let $X \in U_{\mathcal{C}(\mathbb{E})}(B, \sigma)$ and $1 \leq k \leq n$ be arbitrary. For every $t \in L_k^{-\theta}$,we have

$$\begin{aligned} \delta(X_0, e(t)) &\leq \delta(B_0, e(t)) + h(X_0, B_0) < \delta(B_0, e(t)) + \frac{\mu}{2} \\ \delta(X_k, e(t)) &\geq \delta(B_k, e(t)) - h(X_k, B_k) > \delta(B_k, e(t)) - \frac{\mu}{2}, \end{aligned}$$

thus, in view of (3.25).

$$\delta(X_0, e(t)) - \delta(X_k, e(t)) < \delta(B_0, e(t)) - \delta(B_k, e(t)) + \mu < 0,$$

and (3.23) is proved. By a similar argument, using (3.26), one can show (3.24), completing the proof of Step 4.

Step 5. With B as in Step 1 and σ as in Step 4, verifying (3.27)*, the statement of Lemma* 1 *is satisfied.*

Clearly, $U_{\mathcal{C}(\mathbb{E})}(B, \sigma) \subset U_{\mathcal{C}(\mathbb{E})}(A_0, \lambda)$ as $h(B, A_0) < \lambda/2$, by (3.10), and $\sigma < \lambda/2$, by (3.27). Now, let $X \in U_{\mathcal{C}(\mathbb{E})}(B, \sigma)$ be arbitrary.

We claim that

$$M^{n+1,\varepsilon}(X) \cap U_{\mathbb{E}}(e_0, r) \neq \emptyset. \tag{3.28}$$

By Step 4, the n continuous functions $t \to \delta(X_0, e(t)) - \delta(X_k, e(t))$, $k = 1, \ldots, n$, defined for t in the hypercube Q_n^θ, satisfy (3.23) and (3.24). By Brouwer – Miranda theorem, there exists $\hat{t} \in Q_n^\theta$, such that, setting $\hat{e} = e(\hat{t})$, we have

$$\delta(X_k, \hat{e}) = \delta(X_0, \hat{e}) \qquad k = 1, \ldots, n. \tag{3.29}$$

We have $\hat{e} \in U_{\mathbb{E}}(e_0, r)$, for $\|\hat{e} - e_0\| < \rho$ by (3.22), and $\rho < r$ by (3.17). It remains to show that $\hat{e} \in M^{n+1,\varepsilon}(X)$. Clearly

$$\delta(X, \hat{e}) = \max\{\delta(X_0, \hat{e}), \ldots, \delta(X_n, \hat{e}), \delta(\tilde{X}, \hat{e})\}, \tag{3.30}$$

as $X = X_0 \cup \cdots \cup X_n \cup \tilde{X}$. Since $h(X, B) < \sigma < \rho$, by (3.27), and $\|\hat{e} - e_0\| < \rho$, by virtue of Step 3 we have $\delta(X_k, \hat{e}) > \delta(\tilde{X}, \hat{e})$, $k = 0, \ldots, n$. The latter inequality, (3.29) and (3.30) imply $\delta(X, \hat{e}) = \delta(X_k, \hat{e})$, $k = 0, 1, \ldots, n$. Consequently,

$$(3.31) \qquad q_X(\hat{e}) \cap X_k \neq \emptyset \qquad k = 0, 1, \ldots, n,$$

and so in each ball $\tilde{U}_{\mathbb{E}}(b_k, \eta)$, $k = 0, 1, \ldots, n$, there are points of $q_X(\hat{e})$. But all these balls are pairwise disjoint, since $\eta < \tilde{\eta}$ and $\tilde{\eta}$ satisfies (3.11), hence

$$(3.32) \qquad \operatorname{card} q_X(\hat{e}) \geq n + 1.$$

On the other hand, $q_X(\hat{e}) \subset \bigcup\limits_{k=0}^{n} X_k \subset \bigcup\limits_{k=0}^{n} \tilde{U}_{\mathbb{E}}(b_k, \eta)$. But $q_B(e_0) = \{b_0, b_1, \ldots, b_n\}$ and $\eta < \omega/4$, by (3.14), thus $q_X(\hat{e}) \subset q_B(e_0) + (\omega/4)U$. Hence

$$(3.33) \qquad \operatorname{diam} q_X(\hat{e}) < \varepsilon,$$

because $\operatorname{diam} q_B(e_0) < \varepsilon/2$, by (3.9), and $\omega < \varepsilon$, by (3.3). From (3.32) and (3.33) it follows that $\hat{e} \in M^{n+1,\varepsilon}(X)$. Thus $\hat{e} \in M^{n+1,\varepsilon}(X) \cap U_{\mathbb{E}}(e_0, r)$, and (3.28) is verified. Since $X \in U_{\mathcal{C}(\mathbb{E})}(B, \sigma)$ is arbitrary, the statement of the lemma holds true, completing the proof. □

Remark 1. The statement of the Lemma remains valid with $\delta(A_0, e_0) = 0$, (all other assumptions unchanged). In fact, take $\tilde{A} \in \mathcal{C}(\mathbb{E})$ such that $\delta(\tilde{A}, e_0) > 0$ and $h(\tilde{A}, A_0) < \lambda/2$. By the Lemma (with $\tilde{A}$, $\lambda/2$ in the place of A_0, λ) there exist $B \in \mathcal{C}(\mathbb{E})$ and $\sigma > 0$, with $U_{\mathcal{C}(\mathbb{E})}(B, \sigma) \subset U_{\mathcal{C}(\mathbb{E})}(\tilde{A}, \lambda/2)$, such that $X \in U_{\mathcal{C}(\mathbb{E})}(B, \sigma)$ implies $M^{n+1,\varepsilon}(X) \cap U_{\mathbb{E}}(e_0, r) \neq \emptyset$. As $U_{\mathcal{C}(\mathbb{E})}(B, \sigma) \subset U_{\mathcal{C}(\mathbb{E})}(A_0, \lambda)$, Remark 1 is proved.

Theorem. *Let $\mathbb{E}$ be a real infinite dimensional separable Hilbert space. Let $n \in \mathbb{N}$ be arbitrary. Then for a typical $X \in \mathcal{C}(\mathbb{E})$, the multivalued locus $M^{n+1}(X)$ of q_X of order $n+1$ is dense in $\mathbb{E}$.*

Proof. Let E_0 be a countable set dense in $\mathbb{E}$ and let Q^+ be the set of all rationals $r > 0$. For $e \in E_0$ and $r > 0$, put

$$\mathcal{C}_{e,r}^{n+1} = \{X \in \mathcal{C}(\mathbb{E}) \mid M^{n+1}(X) \cap U_{\mathbb{E}}(e, r) \neq \emptyset\}.$$

The set $\operatorname{int} \mathcal{C}_{e,r}^{n+1}$ is dense in $\mathcal{C}(\mathbb{E})$. In fact, let $A_0 \in \mathcal{C}(\mathbb{E})$ and $\lambda > 0$ be arbitrary. By the Lemma and Remark 1 (with e in the place of e_0 and $\varepsilon = 1$) there exist

$B \in \mathcal{C}(\mathbb{E})$ and $\sigma > 0$, with $U_{\mathcal{C}(\mathbb{E})}(B,\sigma) \subset U_{\mathcal{C}(\mathbb{E})}(A_0,\lambda)$, such that for every $X \in U_{\mathcal{C}(\mathbb{E})}(B,\sigma)$ we have

$$M^{n+1}(X) \cap U_{\mathbb{E}}(e,r) \neq \emptyset. \tag{3.35}$$

Consequently, $\operatorname{int}\mathcal{C}^{n+1}_{e,r} \supset U_{\mathcal{C}(\mathbb{E})}(B,\sigma)$ and so $\operatorname{int}\mathcal{C}^{n+1}_{e,r} \cap U_{\mathcal{C}(\mathbb{E})}(A_0,\lambda) \neq \emptyset$. As $A_0 \in \mathcal{C}(\mathbb{E})$ and $\lambda > 0$ are arbitrary, the set $\operatorname{int}\mathcal{C}^{n+1}_{e,r}$ is dense in $\mathcal{C}(\mathbb{E})$.

Now define

$$\mathcal{C}^{n+1} = \bigcap_{e\in E_0}\bigcap_{r\in Q^+} \mathcal{C}^{n+1}_{e,r}. \tag{3.36}$$

Let $X \in \mathcal{C}^{n+1}$, $u \in \mathbb{E}$ and $s > 0$ be arbitrary. Take $e \in E_0$ and $r \in Q^+$ so that $U_{\mathbb{E}}(e,r) \subset U_{\mathbb{E}}(u,s)$. Since $X \in \mathcal{C}^{n+1}_{e,r}$, (3.35) is satisfied and, a fortiori, $M^{n+1}(X) \cap U_{\mathbb{E}}(u,s) \neq \emptyset$. Hence $M^{n+1}(X)$ is dense in $\mathbb{E}$. As X is arbitrary in $\mathcal{C}^{n+1}$, a residual subset of $\mathcal{C}(\mathbb{E})$, the proof is complete. $\square$

Remark 2. The statement of the Theorem remains valid with $M^{n+1,\varepsilon}(X)$, $\varepsilon > 0$, in the place of $M^{n+1}(X)$.

Corollary. *Let $\mathbb{E}$ be as in Theorem 1. Then a typical $X \in \mathcal{C}(\mathbb{E})$ has the following property: for each $e \in \mathbb{E}$ there is a sequence $\{e_n\} \subset \mathbb{E}$, converging to e, satisfying*

$$\operatorname{card} q_X(e_n) \geq n+1 \quad \textit{and} \quad \operatorname{diam} q_X(e_n) \leq \frac{1}{n} \quad \textit{for every} \;\; n \in \mathbb{N}.$$

Remark 3. For each $X \in \mathcal{C}(\mathbb{E})$ the single valued locus $S(X)$ of q_X is a residual (hence dense) subset of $\mathbb{E}$. This follows from a theorem proved, in a much more general setting, by Asplund [1] and Edelstein [5] (see Lau [9] and Deville and Zizler [3] for generalizations), along a pattern developed by Stečkin [12] for the metric projection mapping. An account of the properties of the single valued loci for metric projection mappings and optimization problems can be found in Singer [11] and Dontchev and Zolezzi [4].

The first result concerning the existence of dense multivalued loci for the metric projection mapping on compact subsets of $\mathbb{R}^d$, $d \geq 2$, is due to Zamfirescu [13]. Infinite dimensional generalizations have been recently obtained by Zhivkov [15, 16], who proves also a sharp theorem about dense multivalued loci, with two-valued projections, in uniformly convex Banach spaces.

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