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ON THE STRUCTURE OF SPATIAL BRANCHING PROCESSES

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Abstract. The paper is a contribution to the theory of branching processes with discrete time and a general phase space in the sense of [2]. We characterize the class of *regular*, i.e. in a sense sufficiently random, branching processes $(\Phi_k)_{k\in\mathbb{Z}}$ by almost sure properties of their realizations without making any assumptions about stationarity or existence of moments. This enables us to classify the *clans* of (Φ_k) into the *regular part* and the *completely non-regular part*. It turns out that the completely non-regular branching processes are built up from *single-line* processes, whereas the regular ones are mixtures of left-tail trivial processes with a *Poisson family structure*.

1. Introduction. The notion of a *spatial branching process* treated here is a straightforward generalization of the classical *Galton-Watson branching model* to the situation where the branching individuals are located in a complete separable metric space A. The *offspring distribution* κ is a stochastic kernel from A to the (suitably metrisized) set of all configurations of individuals on A, describing the random daughter population of an individual with given position. We

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consider two-sided infinite Markov sequences of branching populations (Φ_k) with offspring distribution κ, so called κ-processes (see [2] for details). This means we assume the branching population to have an *infinite history* (entrance law), which is much weaker than assuming equilibrium, but still leads to important structural consequences. These are closely connected with the notion of *regularity* of a κ-process. This notion is introduced in full generality in Section 3: A κ-process is called regular, if for each bounded Borel set B of the phase space A the number of individuals in the long-ago population Φ_{-n} with a given positive *a priori* chance to have offspring in B (at some given finite time) tends to zero stochastically as we go to minus infinity.

If (Φ_k) is of first order (has finite expected number of individuals in bounded sets), then there is a first order reformulation of the notion of regularity in terms of intensity measures (Theorem 3.7) which coincides with the definition in [2].

The notion of regularity has two 'historical' sources. Both of them concern the special case of a branching dynamics κ without any proper branching, i.e. where κ is a substochastic shift $\sigma(K)$ according to a substochastic kernel K in the phase space (see Section 2).

At the one hand (see e.g. [3]), the regularity of $(\Phi_k) \sim \mathbf{H}$ ensures the applicability of the Poisson approximation yielding that $\mathbf{H}$ is *Cox*ian, i.e. is a mixture of distributions of Poissonian $\sigma(K)$-processes. At the other hand, Shiga and Takahashi proved in [10], that a Poissonian $\sigma(K)$-process (Φ_k) is left-tail trivial (extremal in the set of all distributions of $\sigma(K)$-processes) iff it is regular.

Based on [2] and [8] we prove as a final result combining both lines dashed off above, that a $\sigma(K)$-process (Φ_k) is regular iff almost all of its left-tail trivial components are Poissonian (Theorem 4.3).

This result is of a greater significance than it might appear at the first sight: Based on ideas of J. Kerstan, it was pointed out in [2] that an arbitrary branching dynamics κ becomes a substochastic shift $\boldsymbol{\kappa}$ by lifting it to the "family level" (with the disadvantage that $\boldsymbol{\kappa}$ acts on a significantly more complicated phase space), cf. Section 2. With this procedure, the κ-process $(\Phi_k) \sim \mathbf{H}$ is transformed into the corresponding "family process" $(\boldsymbol{\Phi}_k) \sim \mathbf{G}_{\mathbf{H}}$, and these family processes are a special class of $\boldsymbol{\kappa}$-processes. It turns out (cf. Proposition 3.2 and 2.3), that (Φ_k) is regular resp. left-tail trivial iff the family process has the corresponding property.

The definition of regularity cited above is elementary and makes it easy to refer to the given literature, but it does not reveal too much of the very nature of this randomness property. So it is a starting point for deeper investigations.

Just as in [6] and [8], we have to make use of genealogical relations between "individuals" appearing in a κ-process (Φ_k), not only for heuristics but also as main ingredients in definitions and proofs. We refer in Section 2 to the so-called refined branching dynamics κ^o, acting in an extended phase space, where each individual is not only characterized by its original position but by two additional "marks" from $[0,1]$, representing its own and its mother's name. The names of the individuals are chosen at random from $[0,1]$, independently of each other. This construction makes it possible to read off all genealogical relations almost surely. There is a unique correspondence between κ-processes $(\Phi_k) \sim \mathbf{H}$ and their refinements $(\Phi_k^o) \sim \mathbf{H}^o$ which preserves the properties of regularity and left-tail triviality (cf. Remark 1 and 2.1). Moreover, the introduction of (Φ_k^o) opens a simplified approach to the family process, if compared with the way outlined in [2].

In (Φ_k^o) almost surely any position of the refined phase space can be occupied at most once. So almost surely any individual at any time m is uniquely characterized by its refined position a_m^o. Then there exists a.s. the ancestral line $(a_m^o, a_{m-1}^o, a_{m-2}^o, \ldots)$ of that individual. Projection to the original phase space defines the ancestral line $(a_m, a_{m-1}, a_{m-2}, \ldots)$ of the corresponding individual in (Φ_k).

A main result of this paper (Corollary 6.4) states that a κ-process (Φ_k) is regular iff for each bounded Borel subset B of the phase space and a.s. for all ancestral lines $(a_m, a_{m-1}, a_{m-2}, \ldots)$ of an individual in Φ_m we have the convergence relation

$$\kappa^{[n]}_{(a_{m-n})}\,(\chi(B) > 0) \underset{n\to\infty}{\longrightarrow} 0.$$

Here $\kappa^{[n]}_{(a)}$ denotes the probability distribution of the nth daughter generation χ of an individual at position a. More general, we prove the assertion (Proposition 6.2 and Theorem 6.3) that for an arbitrary κ-process $(\Phi_k) \sim \mathbf{H}$ we have almost surely weak convergence of $\kappa^{[n]}_{(a_{m-n})}$ along any ancestral line, and the weak limits can be identified using the left-tail trivial components of $\mathbf{H}$.

If a bounded Borel subset B of the phase space, a time m and a constant $c > 0$ are given, we construct a "sieve" for the individuals appearing in a κ-process (Φ_k) : An individual at a_k from Φ_k is supposed to pass the sieve iff its ancestral line (a_{k-n}) fulfils the relation

$$\overline{\lim_{n\to\infty}}\, \kappa^{[m-k+n]}_{(a_{k-n})}\,(\chi(B) > 0) < c.$$

Obviously, *all* related individuals from (Φ_k) pass the sieve if only one of them passes it (for related individuals the ancestral lines coalesce), i.e. whole "clans" in (Φ_k) pass or do not pass the sieve. The latter form a sub-process $(\Phi_{k,B,m,c})_{k\in\mathbb{Z}}$ of (Φ_k). According to 5.5 such a sub-process comprises almost surely only finitely many clans. It turns out that the regular part $(\Phi_{k,\mathrm{reg}})$ of (Φ_k) just consists of those clans passing *each* sieve of the given structure, whereas the *completely non-regular part* $(\Phi_k - \Phi_{k,\mathrm{reg}})$ can a.s. be built up from a countable collection of *κ-processes with a.s. finitely many clans.* So in a certain sense the κ-processes with a.s. finitely many clans are complementary to the regular ones. By Theorem 8.1 we give an "elementary" characterization of κ-processes with a.s. finitely many clans.

The structural results presented in Section 7 give a more or less satisfactory picture of the nature of κ-processes: Each κ-process can be represented as mixture of extremal processes, which are *independent superpositions* of their corresponding regular and completely non-regular parts. The regular parts are extremal κ-processes *of Poisson type* (i.e. the family process is Poissonian), whereas the completely non-regular parts are *independent superpositions* of extremal *single-line* κ-processes (i.e. representing a single clan migrating randomly through the family phase space according to the substochastic shift $\boldsymbol{\kappa\!\!\!/}$).

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2. Basic notions. We shall use basic notions and notations as introduced in [2], [6] and only some of them we recall here briefly.

Let (A, ρ_A) be a complete separable metric space, $\mathcal{A}$ the corresponding Borel σ-field and $\mathcal{B}$ the subsystem of bounded Borel sets. (A, ρ_A) is the phase space where the branching populations are located. A population is given by a counting measure Φ on $\mathcal{A}$, i.e. a measure fulfilling $\Phi(B) \in \{0, 1, 2, \dots\}$ for $B \in \mathcal{B}$. The number $\Phi(B)$ counts those individuals of the population Φ which are located in B. We denote the empty population by o. The set of all counting measures on $\mathcal{A}$ will be denoted by M, and we write $\mathcal{M}$ for the σ-field on M generated by the mappings $\Phi \in \mathsf{M} \longmapsto \Phi(B)$, $B \in \mathcal{B}$. A random population is described by a probability distribution P on $\mathcal{M}$.

A branching dynamics on A is defined by a clustering field κ on A, i.e. a stochastic kernel assigning a distribution $\kappa_{(a)}$ on $\mathcal{M}$ to each $a \in A$. This is the distribution of the random daughter population of an individual located at a. So, if $\Phi = \sum_{i\in I} \delta_{a_i}$ is a population in M, the distribution of the random daughter population of Φ is defined as convolution $\kappa_{(\Phi)} := \mathop{\ast}_{i\in I} \kappa_{(a_i)}$, i.e. as the distribution of the superposition of independent daughter populations Ψ_i of the individuals belonging to Φ. Observe that $\kappa_{(\Phi)}$ is well-defined only in the case that for each $B \in \mathcal{B}$ we have almost surely

$$\sum_{i \in I} \Psi_i(B) < +\infty,$$

which means that the daughter population of Φ belongs to M a.s. We denote the set of all $\Phi \in \mathsf{M}$ fulfilling this condition by ${}_{\kappa}\mathsf{M}$. We have ${}_{\kappa}\mathsf{M} \in \mathcal{M}$.

Let P be a probability distribution on $\mathcal{M}$ fulfilling $P\left({}_{\kappa}\mathsf{M}\right) = 1$. We define the *clustered distribution* P_κ by

$$P_\kappa := \int P\left(d\Phi\right)\ \kappa_{(\Phi)},$$

being the distribution of the daughter population of a random population distributed according to P.

If κ is a clustering field, we define the clustering powers $\kappa^{[n]}$, $n \geq 0$: For $a \in A$ let by induction

$$\kappa_{(a)}^{[0]} := \delta_{\delta_a} \quad \text{and} \quad \kappa_{(a)}^{[n+1]} := \left(\kappa_{(a)}^{[n]}\right)_\kappa,$$

where for each n we have to suppose that $\kappa_{(a)}^{[n]}\left({}_{\kappa}\mathsf{M}\right) = 1$.

If K is a substochastic kernel on A, the following set-up defines a clustering field:

$$\sigma(K)_{(a)} := (1 - K(a, A))\delta_o + \int K(a, da')\ \delta_{\delta_{a'}}.$$

The clustering field σ represents a random shift on A with possible extinction. An individual at position a will have exactly one descendant in B with probability $K\left(a, B\right)$, whereas with probability $1 - K\left(a, A\right)$ it will have no descendant at all.

This paper deals with so-called κ-processes.

A *κ-process* is a two-sided infinite Markov sequence $(\Phi_k)_{k \in \mathbb{Z}}$ (where $\mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}$) with states in M. It is given by a probability distribution $\mathbf{H}$ on the σ-field $\mathcal{M}^{\otimes \mathbb{Z}}$ which fulfils $\mathbf{H}\left(\left({}_{\kappa}\mathsf{M}\right)^{\mathbb{Z}}\right) = 1$ and the transition probabilities of which satisfy the branching law

$$\mathbf{H}\left(\Phi_{k+1} \in (\cdot) \,|\Phi_k\right) = \kappa_{(\Phi_k)}$$

for almost all Φ_k with respect to the distribution $\mathbf{H}_k := \mathbf{H}\left(\Phi_k \in (\cdot)\right), k \in \mathbb{Z}$. Vice versa, given such a sequence of marginals fulfilling

$$\mathbf{H}_k\left({}_{\kappa}\mathsf{M}\right) = 1 \ \text{ and } \ \left(\mathbf{H}_k\right)_\kappa = \mathbf{H}_{k+1}, \qquad \text{for } \ k \in \mathbb{Z},$$

then there is a unique κ-process $\mathbf{H}$ admitting those marginals.

Let, for any $n \in \mathbb{Z}$, $\mathcal{F}_n$ denote the sub-σ-field of $\mathcal{M}^{\otimes\mathbb{Z}}$ generated by the projection $(\Phi_k)_{k\in\mathbb{Z}} \longmapsto (\Phi_k)_{k\leq n}$. We write $\mathcal{F}_{-\infty}$ for the left tail field:

$$\mathcal{F}_{-\infty} := \bigcap_{n\in\mathbb{Z}} \mathcal{F}_n.$$

A κ-process $(\Phi_k)_{k\in\mathbb{Z}} \sim \mathbf{H}$ is said to be *left-tail trivial* if $\mathbf{H}|_{\mathcal{F}_{-\infty}}$ takes values in $\{0,1\}$ only.

Investigating κ-processes we consider family relations between individuals in their realizations. Just as in [6], to do this in a rigorous manner, we introduce the notion of the *refined process* $\mathbf{H}^o$ connected with the κ-process $\mathbf{H}$ with phase space (A, ρ_A). This is again a branching process on another (refined) phase space (A^o, ρ_{A^o}) with a new branching dynamics κ^o which are chosen to make it possible to read off all family relations from the realizations of the κ^o-process a.s., and in such a way that the former κ-process can be obtained by a simple projection. We choose $A^o := A \times [0,1] \times [0,1]$ and

$$\rho_{A^o}((a,x,y),(a',x',y')) := \rho_A(a,a') + \mid x - x' \mid + \mid y - y' \mid .$$

Let M^o be the set of all counting measures on (A^o, ρ_{A^o}). Consider an individual in the original process located at $a \in A$. In the refined process it will be located at position (a,x,y), where the two marks $x, y \in [0,1]$ are the name of its mother x and its own name y. The individual will transmit its own name y to all individuals of its daughter generation. So its daughter population Ψ^o (distributed according to $\kappa^o_{((a,x,y))}$) can be written as $\Psi^o = \sum\limits_{i\in I} \delta_{(a_i,y,z_i)}$ where $\Psi = \sum\limits_{i\in I} \delta_{a_i}$ is distributed according to $\kappa_{(a)}$, and the z_i, $i \in I$, are i.i.d. equidistributed on $[0,1]$ (given I). It is not hard to understand that this refinement of the original process indeed reveals all the family relations for almost every realization $(\Phi^o_k)_{k\in\mathbb{Z}}$. To any individual at position a_k in Φ^o_k we almost surely can trace back -with a little help of marks- the *ancestral line* $(a_{k-n})_{n\geq 0}$, i.e. the locations of its ancestors.

Let us denote any object relating to the refined process by a superscript ^{o}. We may consider $\mathcal{M}^{\otimes\mathbb{Z}}$ as a sub-σ-field of $(\mathcal{M}^o)^{\otimes\mathbb{Z}}$ (with $\mathbf{H}$ being the restriction of $\mathbf{H}^o$ to this sub-field) and $\mathcal{F}_{-\infty}$ as a sub-σ-field of $\mathcal{F}^o_{-\infty}$

As is well known (cf. e.g. [2], 4.1.1.) a κ-process is *extremal* (in the sense that it cannot be represented as a mixture of different κ-processes) iff it is left-tail trivial, and to each branching dynamics κ there exists a stochastic kernel $\underline{K}$ from $(\mathsf{M}^{\mathbb{Z}}, \mathcal{F}_{-\infty})$ to$(\mathsf{M}^{\mathbb{Z}}, \mathcal{M}^{\otimes\mathbb{Z}})$ with

(1) *For all* $(\Phi_k) \in \mathsf{M}^{\mathbb{Z}}$, $\underline{K}((\Phi_k), (\cdot))$ *is the distribution of a left-tail trivial* κ*-process.*

(2) *For each* κ*-process* $(\Phi_k) \sim \mathbf{H}$ *we have* $\mathbf{H}$*-almost surely*

$$\mathbf{H}((\cdot)|\mathcal{F}_{-\infty})((\Phi_k)) = \underline{K}((\Phi_k), (\cdot)).$$

Given such a stochastic kernel for κ, it is easy to find a version of the kernel $\underline{K}^o$, which is even $\mathcal{F}_{-\infty}$-measurable (and not only $\mathcal{F}^o_{-\infty}$-measurable). In fact, by construction it is obvious that the refinement operation on κ-processes is one-to-one, surjective and interchanges with the formation of mixtures. So we immediately get from the identity of left-tail triviality and extremality

2.1 *A* κ*-process* $(\Phi_k)_{k\in\mathbb{Z}} \sim \mathbf{H}$ *is left-tail trivial iff its refinement* $(\Phi^o_k)_{k\in\mathbb{Z}} \sim \mathbf{H}^o$ *has this property.*

as well as

2.2 *If* κ *is a clustering field on* A *and* $\underline{K}$ *is a stochastic kernel from* $(\mathsf{M}^{\mathbb{Z}}, \mathcal{F}_{-\infty})$ *to* $(\mathsf{M}^{\mathbb{Z}}, \mathcal{M}^{\otimes\mathbb{Z}})$ *fulfilling* (1) *and* (2)*, then by the set-up* $\underline{K^o}((\Phi^o_k), (\cdot)) := (\underline{K}((\Phi_k), (\cdot)))^o$ *we get a corresponding stochastic kernel for the refined clustering field* κ^o*.*

The refined process can be used to construct for the κ-process $(\Phi_k)_{k\in\mathbb{Z}} \sim \mathbf{H}$ the so-called *family process* $(\boldsymbol{\Phi}_k)_{k\in\mathbb{Z}} \sim \mathbf{G_H}$. We call two individuals from Φ_k (more precisely: from Φ^o_k) related if their ancestral lines coalesce, i.e. if they have a common ancestor. This means that Φ_k is composed of (non-empty) *families* $\chi_{k,i} \in \mathsf{M}\backslash\{o\}$, with $\Phi_k = \sum_{i\in I_k} \chi_{k,i}$. Making use of the fact that $\mathbb{A} := \mathsf{M}\backslash\{o\}$, equipped with a suitable metric $\rho_{\mathbb{A}}$ (generating the vague topology, see [5], Section 3.3), is in turn a complete separable metric space, we define

$$\boldsymbol{\Phi}_k := \sum_{i\in I_k} \delta_{\chi_{k,i}}$$

giving a sequence of random counting measures on $\mathbb{A}$. For each $m \in \mathbb{Z}$, the counting measure $\boldsymbol{\Phi}_m$ can be represented as

$$\boldsymbol{\Phi}_m = \boldsymbol{\Phi}_m\left((\Phi^o_k)_{k\in\mathbb{Z}}\right),$$

where $\boldsymbol{\Phi}_m(\cdot)$ is an $\mathcal{F}^o_m$-measurable function defined on $(\mathsf{M}^o)^{\mathbb{Z}}$ with values in the set of counting measures on $\mathbb{A}$ (with the convention that we assign the zero measure on $\mathbb{A}$ to any sequence $(\Phi^o_k)_{k\in\mathbb{Z}}$ in $(\mathsf{M}^o)^{\mathbb{Z}}$ which has the property that, for some $m' \le m$ there exists an individual in $\Phi^o_{m'}$ which has not exactly one ancestor in $\Phi^o_{m'-1}$.)

We recall that a subset of $\mathbb{A}$ is $\rho_{\mathbb{A}}$-bounded iff it is contained in some set $\{\Phi \in \mathbb{A} : \Phi(B) > 0\}$ with $B \in \mathcal{B}$. Hence, if we define $\operatorname{sp}\boldsymbol{\Phi} := \sum_{i\in I} \chi_i$ for a counting measure $\boldsymbol{\Phi} = \sum_{i\in I} \delta_{\chi_i}$ on $\mathbb{A}$, we get a counting measure on A.

Now the dynamics generated by κ on the family level, i.e. on the phase space $\mathbb{A}$, is defined by the following substochastic kernel on $\mathbb{A}$

$$\mathbb{K}(\chi,(\cdot)) = \begin{cases} \kappa_{(\chi)}\left((\cdot)\setminus\{o\}\right), & \text{for } \chi \in {}_{\kappa}\mathsf{M}\setminus o \\ \delta_o, & \text{for } \chi \in \mathbb{A}\setminus {}_{\kappa}\mathsf{M} \end{cases}.$$

So the family dynamics is a *substochastic shift* in $\mathbb{A}$ which is described by the new clustering field $\boldsymbol{\kappa} = \sigma(\mathbb{K})$. The family process $(\boldsymbol{\Phi}_k)_{k\in\mathbb{Z}} \sim \mathbf{G_H}$ is a $\boldsymbol{\kappa}$-process on $\mathbb{A}$ and we have the projection property $\mathbf{H} = \mathbf{G_H}\left((\operatorname{sp}\boldsymbol{\Phi}_k)_{k\in\mathbb{Z}} \in (\cdot)\right)$.

The mapping

$$(\Phi_k^o)_{k\in\mathbb{Z}} \longmapsto \left(\boldsymbol{\Phi}_m\left((\Phi_k^o)_{k\in\mathbb{Z}}\right)\right)_{m\in\mathbb{Z}},$$

which transforms $\mathbf{H}^o$ into $\mathbf{G_H}$, is measurable with respect to both left-tail fields. This combined with 2.1 leads to (cf. [2], Proposition 4.1.15)

2.3 *A κ-process $(\Phi_k) \sim \mathbf{H}$ is left-tail trivial iff its family process $(\boldsymbol{\Phi}_k) \sim \mathbf{G_H}$ has this property.*

For a κ-process $(\Phi_k)_{k\in\mathbb{Z}} \sim \mathbf{H}$, the stochastic kernel $\underline{K}$ yields the (essentially unique) representation of $\mathbf{H}$ as a mixture of left-tail trivial components

$$\mathbf{H} = \int \mathbf{H}(d\,(\Phi_k))\underline{K}((\Phi_k),(\cdot)).$$

This representation can be lifted to the family level.

2.4 *The family process $(\boldsymbol{\Phi}_k)_{k\in\mathbb{Z}} \sim \mathbf{G_H}$ has the representation*

$$\mathbf{G_H} = \int \mathbf{H}\,(d\,(\Phi_k))\,\mathbf{G}_{\underline{K}((\Phi_k),(\cdot))}$$

as mixture of left-tail trivial components.

P r o o f. By 2.3 the probability laws $\mathbf{G}_{\underline{K}((\Phi_k),(\cdot))}$ are left-tail trivial. We have by 2.2

$$\begin{aligned} \mathbf{H}^o &= \int \mathbf{H}^o\,(d\,(\Phi_k^o))\,(\underline{K}((\Phi_k),(\cdot)))^o \\ &= \int \mathbf{H}\,(d\,(\Phi_k))\,(\underline{K}((\Phi_k),(\cdot)))^o \end{aligned}$$

Applying the map $(\Phi_k^o)_{k\in\mathbb{Z}} \longmapsto \left(\boldsymbol{\Phi}_m\left((\Phi_k^o)_{k\in\mathbb{Z}}\right)\right)_{m\in\mathbb{Z}}$ yields the desired result. □

Observe that $\mathbf{G}_{\underline{K}((\operatorname{sp}\boldsymbol{\Phi}_k),(\cdot))}$ is *not* a kernel playing the same universal role for the clustering field $\boldsymbol{\kappa}$ as $\underline{K}$ does for κ. The point is, not every $\boldsymbol{\kappa}$-process is the family process of some κ-process (e.g., for any κ-process $(\Phi_k)_{k\in\mathbb{Z}} \sim \mathbf{H}$ fulfilling $\Phi_m(A) > 0$ a.s. for each $m \in \mathbb{Z}$, the trivial lifting $(\boldsymbol{\delta}_{\Phi_k})_{k\in\mathbb{Z}}$ to the phase space $\mathbb{A}$ is a $\boldsymbol{\kappa}$-process without paying any attention to the family structure of (Φ_k).)

Given a family χ_m from a population $\boldsymbol{\Phi}$ of the family process $(\boldsymbol{\Phi}_k)$, there is almost surely an ancestral line $(\chi_{m-n})_{n\geq 0}$, with $\chi_{m-n-1} \in \operatorname{supp} \boldsymbol{\Phi}_{m-n-1}$ for $n \geq 0$ comprising all mother individuals of χ_{m-n}. (Just as in the case of the original κ-process $(\Phi_k) \sim \mathbf{H}$, this has a rigorous meaning if we consider $(\boldsymbol{\Phi}_k)$ as a measurable function of (Φ_k^o).) Since the branching dynamics $\boldsymbol{\kappa}$ of the family process is a substochastic shift, χ_m may have exactly one or no descendant χ_{m+1} in $\boldsymbol{\Phi}_{m+1}$. The counting measure χ_{m+1} comprises all positions of daughters of individuals from χ_m. So we get step by step further descendants χ_{m+n} of χ_m belonging to $\boldsymbol{\Phi}_{m+n}$, respectively, where this sequence may break off after a finite number of steps. The constructed sequence (χ_k) of families with $\chi_k \in \operatorname{supp} \boldsymbol{\Phi}_k$, which is almost surely infinite to the left but possibly terminates to the right, will be called a *clan.* For any individual of any family in the clan, the family χ_k comprises all individuals of Φ_k being related to the given individual.

3. Regular κ-processes.

Definition 3.1. *A κ-process $(\Phi_k) \sim \mathbf{H}$ is called regular, if for each $m \in \mathbb{Z}$, $B \in \mathcal{B}$ and $\varepsilon > 0$ we have*

$$\lim_{n\to\infty} \mathbf{H}\left(\Phi_{m-n}(\{a \in A : \kappa_{(a)}^{[n]}(\Psi(B) > 0) > \varepsilon\}) > 0\right) = 0.$$

Regularity means that for each $m \in \mathbb{Z}$ and $B \in \mathcal{B}$ the random array with columns

$$(P_{n,i})_{i\in I_{m-n}} := \left(\kappa_{(a_{m-n,i})}^{[n]}(\Psi(B) \in (\cdot))\right)_{i\in I_{m-n}},$$

is 'infinitesimal in probability'. Here we used the representation $\Phi_k = \sum_{i\in I_k} \delta_{a_{k,i}}$. According to Theorem 3.4 it will be infinitesimal even almost surely in that case.

If a branching dynamics κ satisfies the global deconcentration property

$$\lim_{n\to\infty} \sup_{a\in A} \kappa_{(a)}^{[n]}(\Psi(B) > 0) = 0 \qquad \text{for each} \quad B \in \mathcal{B}$$

(cf. [3] and [2], section 4.4), then all κ-processes are regular.

Remark 1. Obviously, the refined κ^o-process $(\Phi_k^o)_{k\in\mathbb{Z}} \sim \mathbf{H}^o$ is regular iff the original κ-process $(\Phi_k)_{k\in\mathbb{Z}} \sim \mathbf{H}$ has this property.

However, it is not completely obvious that regularity is a property shared with the family process:

Proposition 3.2. *A κ-process $(\Phi_k)_{k\in\mathbb{Z}} \sim \mathbf{H}$ is regular iff the corresponding family process $(\boldsymbol{\Phi}_k)_{k\in\mathbb{Z}} \sim \mathbf{G}_{\mathbf{H}}$ is regular.*

Before we prove this, we give some motivation. In fact, at first sight it seems that regularity of the family process is a stronger property. Taking into account the concept of boundedness in the phase space $\mathbb{A}$ it means that, for any bounded subset B of A, it becomes more and more unlikely (going backward in time) to find a family at time $m-n$ which has a good *a priori* chance to have any descendants in B at time m. Due to independence of branching, the chance for the family is something like the sum of the chances for the individuals (belonging to that family) to put a descendant to B. So why shouldn't a large or infinite family have good chances despite of only tiny chances for the individuals? The explanation lies in the very nature of families: All individuals of the same clan present at time m in B have a common ancestor, going backward in time far enough, or they wouldn't be related. So for large n it should be very unlikely to find among the finitely many individuals in B at time m any two which are related without having a common ancestor at time $m-n$. At the other hand, assume we have a family at $m-n$ with good chances to put a descendant to B at m. If now each single individual's chances were tiny (and independent) to be a B-*survivor*, the Poisson law would lead to the conclusion that the probability for at least *two* B-survivors should be of the same magnitude as the total B-surviving chance which was assumed to be pretty good for the family. But two B-survivors would mean *no common ancestor* at $m-n$ for the corresponding clan's B-individuals (at time m). We give now a rigorous version of these heuristics.

Proof. For $B \in \mathcal{B}$ let $B^{\square} := \{\Phi \in \mathbb{A} : \Phi(B) > 0\}$. As mentioned above, any $\rho_{\mathbb{A}}$-bounded set is contained in a set of this type. Let $n > 0$ and $\boldsymbol{\Phi}$ be a counting measure on $\mathbb{A}$ with $\mathrm{sp}\boldsymbol{\Phi} \in {}_{\kappa}\mathsf{M}\setminus\{o\}$. We have

$$\boldsymbol{\kappa}^{[n]}_{\boldsymbol{\Phi}}(\boldsymbol{\Psi}(B^{\square}) > 0) = \kappa^{[n]}_{(\mathrm{sp}\,\boldsymbol{\Phi})}(\Psi(B) > 0).$$

So the family process is regular iff, for each $m \in \mathbb{Z}$, $B \in \mathcal{B}$ and $\varepsilon > 0$ the relation

$$\lim_{n\to\infty} \mathbf{G}_{\mathbf{H}}\left(\boldsymbol{\Phi}_{m-n}(\{\Phi \in \mathbb{A} : \kappa^{[n]}_{(\Phi)}(\Psi(B) > 0) > \varepsilon\}) > 0\right) = 0$$

is valid. In this case we surely have

$$\lim_{n\to\infty} \mathbf{G}_{\mathbf{H}}\left(\boldsymbol{\Phi}_{m-n}(\{\Phi \in \mathbb{A} : \Phi(\{a \in A : \kappa^{[n]}_{(a)}(\Psi(B) > 0) > \varepsilon\}) > 0\}) > 0\right) = 0$$

which (by construction of $\mathbf{G_H}$) is the same as

$$\lim_{n\to\infty} \mathbf{H}\left(\Phi_{m-n}(\{a \in A : \kappa_{(a)}^{[n]}(\Psi(B) > 0) > \varepsilon\}) > 0\right) = 0$$

i.e. the regularity of $\mathbf{H}$. So the regularity of $\mathbf{G_H}$ implies that of $\mathbf{H}$.

To prove the other implication we make use of a technical

Lemma 3.3. *Let I be an at most countable set and $(\xi_i)_{i\in I}$ an independent family of random elements of $\{0,1\}$. Assume that, for some $\varepsilon > 0$, $0 < \eta < 1 - e^{-\varepsilon} - \varepsilon e^{-\varepsilon}$, we have*

$$\text{Prob}\left(\sum_{i\in I} \xi_i \geq 1\right) > \varepsilon \quad \textit{and} \quad \text{Prob}\left(\sum_{i\in I} \xi_i > 1\right) < \eta.$$

Then there is some $i_0 \in I$ with

$$\text{Prob}(\xi_{i_0} = 1) > 1 - e^{-\varepsilon} - \varepsilon e^{-\varepsilon} - \eta$$

and (consequently)

$$\text{Prob}\left(\sum_{i\neq i_0} \xi_i > 0\right) < \eta(1 - e^{-\varepsilon} - \varepsilon e^{-\varepsilon} - \eta)^{-1}.$$

P r o o f o f t h e l e m m a. From

$$\text{Prob}(\xi_i = 1 \ \ \textit{infinitely often}) \leq \text{Prob}\left(\sum_{i\in I} \xi_i > 1\right) < \eta < 1$$

we obtain with the Borel-Cantelli lemma

$$\lambda := \sum_{i\in I} \text{Prob}(\xi_i = 1) < +\infty,$$

and we have

$$\varepsilon < \text{Prob}\left(\sum_{i\in I} \xi_i \geq 1\right) = \text{Prob}\left(\bigcup_{i\in I}\{\xi_i = 1\}\right) \leq \sum_{i\in I} \text{Prob}\,(\xi_i = 1) = \lambda.$$

In view of $\lambda < +\infty$ we find some $i_0 \in I$ with

$$\text{Prob}(\xi_{i_0} = 1) = \max_{i\in I} \text{Prob}(\xi_i = 1),$$

and for this i_0 the estimate given in the lemma is valid. In fact, for any finite subset I' of I with $i_0 \in I'$ we have, putting

$$\lambda' := \sum_{i \in I'} \operatorname{Prob}(\xi_i = 1)$$

the estimate (cf. [1], Theorem 2.M)

$$\left| \operatorname{Prob}\left(\sum_{i \in I'} \xi_i > 1 \right) - (1 - e^{-\lambda'} - \lambda' e^{-\lambda'}) \right| \leq \operatorname{Prob}(\xi_{i_0} = 1)$$

and hence we derive, in view of

$$\operatorname{Prob}\left(\sum_{i \in I'} \xi_i > 1 \right) \leq \operatorname{Prob}\left(\sum_{i \in I} \xi_i > 1 \right) < \eta$$

the relation

$$1 - e^{-\lambda'} - \lambda' e^{-\lambda'} < \operatorname{Prob}(\xi_{i_0} = 1) + \eta.$$

The set I' was an arbitrary finite subset of I containing i_0, so we have

$$1 - e^{-\lambda} - \lambda e^{-\lambda} \leq \operatorname{Prob}(\xi_{i_0} = 1) + \eta.$$

The function $1 - e^{-x} - x e^{-x}$ is strictly increasing for $x \geq 0$. This gives the desired result in view of $\varepsilon < \lambda$. □

We go on with the proof of Proposition 3.2.

Let $(\Phi_k) \sim \mathbf{H}$ be regular, let $m \in \mathbb{Z}$, $B \in \mathcal{B}$ and $\varepsilon > 0$. Choose some number $\eta \in (0, 1 - e^{-\varepsilon} - \varepsilon e^{-\varepsilon})$ and put $c_{\varepsilon,\eta} = 1 - e^{-\varepsilon} - \varepsilon e^{-\varepsilon} - \eta$. We get by the preceding lemma

$$\mathbf{G_H}\left(\boldsymbol{\Phi}_{m-n}(\{\Phi \in \mathbb{A} : \kappa^{[n]}_{(\Phi)}(\Psi(B) > 0) > \varepsilon\}) > 0 \right)$$

$$\leq \mathbf{G_H}\left(\boldsymbol{\Phi}_{m-n}\left(\left\{ \Phi \in \mathbb{A} : \Phi(\{a \in A : \kappa^{[n]}_{(a)}(\Psi(B) > 0) > c_{\varepsilon,\eta}\}) > 0 \right\} \right) > 0 \right) +$$

$$+ \mathbf{G_H}\left(\boldsymbol{\Phi}_{m-n}\left(\left\{ \sum_{i \in I} \delta_{a_i} \in \mathbb{A} : \bigotimes_{i \in I} \kappa^{[n]}_{(a_i)}\left(\sum_{i \in I} (\Psi_i(B) \wedge 1) > 1 \right) \geq \eta \right\} \right) > 0 \right).$$

The first term on the right-hand side equals

$$\mathbf{H}\left(\Phi_{m-n}(\{a \in A : \kappa^{[n]}_{(a)}(\Psi(B) > 0) > c_{\varepsilon,\eta}\}) > 0 \right)$$

which tends to zero as $n \to \infty$ by the regularity of $\mathbf{H}$. As for the second term, observe that the event E considered there means that there is some family in $\mathbf{\Phi}_{m-n}$ with a probability at least η that two individuals of that family have offspring in B at time m. Then we have by the Markov property of the branching evolution

$$\mathbf{G_H}\left(\left\{\begin{array}{c}\textit{there is some family in } \mathbf{\Phi}_{m-n} \textit{ with two individuals}\\ \textit{of that family having offspring in } B \textit{ at time } m\end{array}\right\}\right) \geq \eta \cdot \mathbf{G_H}(E)$$

from which we conclude that the second term $\mathbf{G_H}(E)$ can be estimated from above by

$$(*) \qquad \eta^{-1}\mathbf{G_H}\left(\left\{\begin{array}{c}\textit{there is some family in } \mathbf{\Phi}_{m-n} \textit{ with two individuals}\\ \textit{of that family having offspring in } B \textit{ at time } m\end{array}\right\}\right).$$

Now by the definition of the family process, for any m and any bounded B there is a finite random time $\nu(m, B)$ such that any pair of related individuals in Φ_m with location in B has a common ancestor at time $m - \nu(m, B)$. So the expression in $(*)$ tends to zero as $n \to \infty$ showing that $\mathbf{G_H}$ is regular. □

The next theorem gives a characterization of regularity in terms of almost sure properties of realizations.

Theorem 3.4. *For any κ-process the following statements are equivalent:*

(1) *(Φ_k) is regular.*

(2) *For all $m \in \mathbb{Z}$ and $B \in \mathcal{B}$ we have*

$$\sup_{a \in \operatorname{supp} \Phi_{m-n}} \kappa^{[n]}_{(a)}(\Psi(B) > 0) \underset{n\to\infty}{\longrightarrow} 0$$

in probability.

(3) *For all $m \in \mathbb{Z}$ and $B \in \mathcal{B}$ we have almost surely*

$$\lim_{n\to\infty} \sup_{a \in \operatorname{supp} \Phi_{m-n}} \kappa^{[n]}_{(a)}(\Psi(B) > 0) = 0.$$

(4) *For all $m \in \mathbb{Z}$ and $B \in \mathcal{B}$ we have almost surely for each clan (χ_k) in $(\mathbf{\Phi}_k)$*

$$\underline{\lim}_{n\to\infty} \kappa^{[n]}_{(\chi_{m-n})}(\Psi(B) > 0) = 0.$$

(5) *For all $m \in \mathbb{Z}$ and $B \in \mathcal{B}$ we have almost surely for each clan (χ_k) in $(\mathbf{\Phi}_k)$*

$$\lim_{n\to\infty} \kappa^{[n]}_{(\chi_{m-n})}(\Psi(B) > 0) = 0.$$

We are going to show in Section 6 that in (5) clans may be replaced by ancestral lines of individuals.

Observe that in (4) and (5) the expression under the limit operator makes sense for n being large enough.

The proof of Theorem 3.4 is based on the following

Lemma 3.5. *Let $(\Phi_k) \sim \mathbf{H}$ be a $\sigma(K)$-process given by a substochastic kernel K. Then for all $m \in \mathbb{Z}$ and $B \in \mathcal{B}$ the sequence*

$$\sup_{a \in \operatorname{supp} \Phi_{m-n}} \sigma(K)^{[n]}_{(a)}(\Psi(B) > 0), \quad n = 1, 2, \ldots$$

converges $\mathbf{H}$-almost surely.

Proof. For abbreviation we put

$$s_n(\Phi, B) := \sup_{a \in \operatorname{supp} \Phi} \sigma(K)^{[n]}_{(a)}(\Psi(B) > 0), \quad n = 0, 1, 2, \ldots\ .$$

Obviously the quantities $s_n(\Phi_{m-n}, B)$, $n \geq 1$, are integrable with respect to $\mathbf{H}$. We have, putting $\Phi_{m-n-1} = \sum\limits_{i \in I} \boldsymbol{\delta}_{a_i}$,

$$\begin{aligned}
& \mathbf{E}\left[s_n(\Phi_{m-n}, B) \,|\, \mathcal{F}_{m-n-1} \right] \\
= \; & \mathbf{E}\left[s_n(\Phi_{m-n}, B) \,|\, \Phi_{m-n-1} \right] \\
= \; & \int \sigma(K)_{(\Phi_{m-n-1})}(d\chi) \sup_{a \in \operatorname{supp} \chi} \sigma(K)^{[n]}_{(a)}(\Psi(B) > 0) \\
\geq \; & \sup_{j \in I} \int K(a_j, da) K^{[n]}(a, B) \\
= \; & \sup_{j \in I} K^{[n+1]}(a_j, B) \\
= \; & \sup_{a \in \operatorname{supp} \Phi_{m-n-1}} \sigma(K)^{[n+1]}_{(a)}(\Psi(B) > 0) \\
= \; & s_{n+1}(\Phi_{m-n-1}, B).
\end{aligned}$$

So the sequence $s_n(\Phi_{m-n}, B)$, $n = 1, 2, \ldots$, is a reverse sub-martingale with respect to $(\mathcal{F}_{m-n})_{n \geq 1}$. Hence it converges almost surely (cf. for instance [9]). □

Proof of Theorem 3.4.

1. Obviously (4) is a consequence of (5), (2) is a consequence of (3), and (2) is equivalent to (1).

2. We show that (3) and (5) are consequences of (1). If (1) is fulfilled, then by Proposition 3.2 the corresponding family process $(\boldsymbol{\Phi}_k)_{k\in\mathbb{Z}} \sim \mathbf{G_H}$ is regular, too. Hence the sequence

$$\sup_{\chi\in\operatorname{supp}\boldsymbol{\Phi}_{m-n}} \kappa^{[n]}_{(\chi)}(\Psi(B)>0), \quad n=1,2,\ldots \tag{3.1}$$

tends to zero in probability. On the family level the branching dynamics is a substochastic shift. Hence Lemma 3.5 yields the almost sure convergence of this sequence to zero. This implies (5) and (3).

3. Finally, we have to show that (4) implies (1). In fact, assume that (Φ_k) is not regular. Consequently, there is a number $m\in\mathbb{Z}$, a set $B\in\mathcal{B}$ and some $\varepsilon>0$ such that with a positive probability δ there exists a random sequence $0<\nu_1<\nu_2<\ldots$ and a sequence of clans $(\chi^1_k),(\chi^2_k),\ldots$ with

$$\kappa^{[\nu_i]}_{(\chi^i_{m-\nu_i})}(\Psi(B)>0)>\varepsilon, \qquad i=1,2,\ldots, \tag{3.2}$$

since otherwise the sequence (3.1) would tend to zero almost surely implying regularity of both the family and the basic process.

We will prove that, for an arbitrary κ-process, a sequence of clans fulfilling (3.2) consists almost surely of only finitely many *different* clans.

In fact, assume the opposite. Then without any loss of the generality we may even assume that this sequence does not contain any clan more than once. Therefore, for an arbitrary integer $M>0$, we find a natural number N with the following property: With probability at least $\delta/2$ we have

$$\boldsymbol{\Phi}_{m-N}(\{\chi_{m-N}\in\mathbb{A}: \textit{there is some } 0<n<N \textit{ with } \kappa^{[n]}_{(\chi_{m-n})}(\Psi(B)>0)>\varepsilon\})>M.$$

Now, given $\boldsymbol{\Phi}_{m-N}=\sum_{i\in I_{m-N}}\delta_{\chi^i_{m-N}}$, in view of the definition of the family process to each $i\in I_{m-N}$ we may assign measurably a triple $[\varsigma^i,n^i,w^i]$ where we put

$$[\varsigma^i,n^i]$$

$$=\begin{cases} [\chi^i_{m-n^i},n^i], n^i \textit{ being the largest integer } \le N \textit{ with } \kappa^{[n^i]}_{(\chi^i_{m-n^i})}(\Psi(B)>0)>\varepsilon \\ [o,0], \qquad\qquad \textit{if there is no such integer}\end{cases}$$

and $w^i = 1$ if $\chi^i_m(B) > 0$ and $w^i = 0$ else. Since the evolution of families, given $\mathbf{\Phi}_{m-N}$, is independent and Markov, we find that the collection of triples $\{[\varsigma^i, n^i, w^i]\}_{i\in I_{m-N}}$ is independent and we have for each l in $\{1, 2, \ldots, N-1\}$

$$\mathrm{Prob}(w^i = 1|\varsigma_i \neq o, n^i = l) > \varepsilon.$$

(Observe that $m-n^i$ is a Markov time for the Markov sequence $(\chi^i_{m-N+k})_{0<k<N}$).

Hence we have

$$\mathbf{H}(\Phi_m(B) > M\varepsilon) \geq \mathrm{Prob}(\sum_{i\in I_{m-N}} w^i > M\varepsilon) \geq$$
$$\mathrm{Prob}(\textit{there are at least } M \textit{ of the } \varsigma^i \neq o)\times$$
$$\times\mathrm{Prob}(\sum_{i\in I_{m-N}} w^i > M\varepsilon \mid \textit{there are at least } M \textit{ of the } \varsigma^i \neq o).$$

By assumption, the first factor is at least $\delta/2$, whereas for the second factor we get

$$\mathrm{Prob}\left(\sum_{i\in I_{m-N}} w^i > M\varepsilon \mid \textit{there are at least } M \textit{ of the } \varsigma^i \neq o\right) \geq \sum_{k>M\varepsilon} B(k, M, \varepsilon)$$

where $B(k, M, \varepsilon)$ is the probability of k successes in a Binomial distribution with M trials and success probability ε. This tends to $1/2$ for M tending to infinity by the Moivre-Laplace theorem in contradiction to the fact that $\Phi_m(B)$ is a.s. finite.

Assume now (4) to be fulfilled. By the preceding consideration we may conclude that there exists with a positive probability δ an a.s. finite collection of clans $(\chi^1_k), (\chi^2_k), \ldots, (\chi^L_k)$ such that for each of them we have

$$\lim_{n\to\infty} \kappa^{[n]}_{(\chi^j_{m-n})}(\Psi(B) > 0) = 0 \tag{3.3}$$

as well as

$$\overline{\lim_{n\to\infty}}\, \kappa^{[n]}_{(\chi^j_{m-n})}(\Psi(B) > 0) > \varepsilon. \tag{3.4}$$

This is easily seen to be impossible. In fact, condition on this event (having a probability of at least δ) and pick up one of these clans by chance. This gives a family process which comprises almost surely one clan (χ_k), and this clan fulfils (3.3) and (3.4). On the one hand, from (3.4) it is obvious that the event $\chi_m(B) > 0$ has a probability of at least ε: Simply go back in time far enough to some time $m-N$, trace the evolution and wait until, for some $\nu < N$, $\kappa^{[n]}_{(\chi_{m-\nu})}(\Psi(B) > 0) > \varepsilon$. If N is large enough, this will happen with a probability arbitrarily close to one. This ν being again a Markov time, you find the total chance for the clan to have a descendant in B at time m to be greater than ε. But on the other hand with that very argument you conclude from (3.3), that for the picked up clan the chances are zero to visit B at m: For an arbitrarily small $\varepsilon' > 0$ you find some N such that, tracing the evolution since $m-N$ with a probability arbitrarily close to one you find some (Markov) time $\nu < N$, fulfilling $\kappa^{[n]}_{(\chi_{m-\nu})}(\Psi(B) > 0) < \varepsilon'$, leading to the conclusion that the total probability for the clan to have individuals in B at time m is arbitrarily small. □

As an immediate consequence of the preceding theorem we get

3.6 *A κ-process $(\Phi_k) \sim \mathbf{H}$ is regular iff for $\mathbf{H}$-almost all (Φ_k) the left-tail trivial component $\underline{K}((\Phi_k), (\cdot))$ has this property.*

A κ-process $(\Phi_k) \sim \mathbf{H}$ is said to be *of first order*, if all intensity measures $\Lambda_{\mathbf{H}_m}$ of $\mathbf{H}_m$, $m \in \mathbb{Z}$,

$$\Lambda_{\mathbf{H}_m}(C) := \int \mathbf{H}_m(d\Phi)\Phi(C), \qquad C \in \mathcal{A},$$

are measures which are finite on bounded Borel sets, i.e. $\Lambda_{\mathbf{H}_m}(C) < +\infty$ for $C \in \mathcal{B}$.

Obviously, $(\Phi_k) \sim \mathbf{H}$ is of first order iff the refined process $(\Phi^o_k) \sim \mathbf{H}^o$ has this property. In this case, the family process $(\boldsymbol{\Phi}_k)_{k\in\mathbb{Z}} \sim \mathbf{G}_{\mathbf{H}}$ is obviously of first order, too. (The converse is not generally true, in the case of the family process being of first order the original process may not be of first order.)

The *intensity kernel* J of the clustering field κ is defined as the family of intensity measures determined by κ,

$$J(a, C) := \int \kappa_{(a)}(d\Psi)\Psi(C), \quad a \in A,\ C \in \mathcal{A},$$

and we have for the n-th convolution power of J

$$J^{[n]}(a, C) = \int \kappa^{[n]}_{(a)}(d\Psi)\Psi(C)$$

as well as $\Lambda_{\mathbf{H}_m} * J = \Lambda_{\mathbf{H}_{m+1}}$, $m \in \mathbb{Z}$.

In [2] the notion of regularity was defined for κ-processes of first order by means of intensity measures and the intensity kernel. It turns out that this concept, being based on first order characteristics only, is compatible with the notion of regularity introduced here for arbitrary κ-processes.

Theorem 3.7. *For a κ-process of first order the following statements are equivalent:*
(1) (Φ_k) *is regular.*
(2) *For any $m \in \mathbb{Z}$, $B \in \mathcal{B}$ and $\varepsilon > 0$ we have*

$$\lim_{n\to\infty} \Lambda_{\mathbf{H}_{m-n}}(\{a \in A : J^{[n]}(a,B) > \varepsilon\}) = 0.$$

(3) *For any $m \in \mathbb{Z}$ and $B \in \mathcal{B}$ we have*

$$\lim_{n\to\infty} \sup_{a\in \operatorname{supp}\Phi_{m-n}} J^{[n]}(a,B) = 0$$

$\mathbf{H}$-*almost surely.*
(4) *For any $m \in \mathbb{Z}$ and $B \in \mathcal{B}$ we have*

$$\lim_{n\to\infty} \sup_{\chi\in \operatorname{supp}\boldsymbol{\Phi}_{m-n}} \int \chi(da) J^{[n]}(a,B) = 0$$

$\mathbf{G}_{\mathbf{H}}$-*almost surely.*

The proof shows that the theorem remains valid if we replace the almost sure convergences in (3) and (4) by convergences in probability.

Proof of Theorem 3.7.

1. Clearly (3) is a consequence of (4).

2. We are going to show that (3) implies (2). Let $m \in \mathbb{Z}$ and $B \in \mathcal{B}$. Obviously from (3) we conclude

$$\lim_{n\to\infty} \mathbf{H}(\Phi_{m-n}(\{a \in A : J^{[n]}(a,B) > \varepsilon\}) > 0) = 0$$

for each $\varepsilon > 0$. Let $x > 0$. We get for any $\varepsilon > 0$ and $n \geq 1$

$$\begin{aligned}\Lambda_{\mathbf{H}_{m-n}}(\{a &\in A : J^{[n]}(a,B) > \varepsilon\}) \\ &= \int \mathbf{H}(d(\Phi_k))\Phi_{m-n}(\{a \in A : J^{[n]}(a,B) > \varepsilon\}) \\ &= \int \mathbf{H}(d(\Phi_k))\Phi_{m-n}(\ldots)\mathbf{1}_{(0,x)}(\Phi_{m-n}(\ldots)) \\ &\quad + \int \mathbf{H}(d(\Phi_k))\Phi_{m-n}(\ldots)\mathbf{1}_{[x,\infty)}(\Phi_{m-n}(\ldots)).\end{aligned}$$

The first term in this sum can be estimated from above by

$$x \cdot \mathbf{H}(\Phi_{m-n}(\{a \in A : J^{[n]}(a, B) > \varepsilon\}) > 0)$$

which tends to zero as $n \to \infty$ for each x. The second term is not greater than

$$\int \mathbf{H}(d(\Phi_k))\frac{1}{\varepsilon}\int \Phi_{m-n}(da)J^{[n]}(a, B)\mathbf{1}_{[x,\infty)}\left(\frac{1}{\varepsilon}\int \Phi_{m-n}(da)J^{[n]}(a, B)\right)$$

$$= \frac{1}{\varepsilon}\int \mathbf{H}(d(\Phi_k))\left(\int \Phi_{m-n}(da)\int \kappa_{(a)}^{[n]}(d\Psi)\Psi(B)\right)\cdot$$

$$\cdot\mathbf{1}_{[\varepsilon x,\infty)}\left(\int \Phi_{m-n}(da)\int \kappa_{(a)}^{[n]}(d\Psi)\Psi(B)\right)$$

$$= \frac{1}{\varepsilon}\int \mathbf{H}(d(\Phi_k))\mathbf{E}(\Phi_m(B)|\mathcal{F}_{m-n})\mathbf{1}_{[\varepsilon x,\infty)}(\mathbf{E}(\Phi_m(B)|\mathcal{F}_{m-n})).$$

This tends to

$$\frac{1}{\varepsilon}\int \mathbf{H}(d(\Phi_k))\mathbf{E}(\Phi_m(B)|\mathcal{F}_{-\infty})\mathbf{1}_{[\varepsilon x,\infty)}(\mathbf{E}(\Phi_m(B)|\mathcal{F}_{-\infty}))$$

as $n \to \infty$. Hence the last integral becomes arbitrarily small for large x. This proves

$$\lim_{n\to\infty} \Lambda_{\mathbf{H}_{m-n}}(\{a \in A : J^{[n]}(a, B) > \varepsilon\}) = 0.$$

i.e. (2).

3. We have

$$\mathbf{H}\left(\Phi_{m-n}(\{a \in A : \kappa_{(a)}^{[n]}(\Psi(B) > 0) > \varepsilon\}) > 0\right)$$

$$\leq \mathbf{H}\left(\Phi_{m-n}(\{a \in A : \int \kappa_{(a)}^{[n]}(d\Psi)\Psi(B) > \varepsilon\}) > 0\right)$$

$$\leq \int \mathbf{H}(d(\Phi_k))\Phi_{m-n}(\{a \in A : \int \kappa_{(a)}^{[n]}(d\Psi)\Psi(B) > \varepsilon\})$$

$$= \Lambda_{\mathbf{H}_{m-n}}(\{a \in A : J^{[n]}(a, B) > \varepsilon\}),$$

hence (1) is a consequence of (2).

4. Finally, we prove that (4) is a consequence of (1).

In view of

$$\int \chi(da)J^{[n]}(a,B) = \int \chi(da) \int \kappa^{[n]}_{(a)}(d\Psi)\Psi(B) = \int \kappa^{[n]}_{(\chi)}(d\Psi)\Psi(B)$$

we get for $y > 0$

$$\begin{aligned}
& \mathbf{G_H}\left(\boldsymbol{\Phi}_{m-n}(\{\chi \in \mathbb{A} : \int \chi(da)J^{[n]}(a,B) > \varepsilon\}) > 0\right) \\
\leq\ & \mathbf{G_H}\left(\boldsymbol{\Phi}_{m-n}(\{\chi \in \mathbb{A} : \int \kappa^{[n]}_{(\chi)}(d\Psi)\Psi(B)\mathbf{1}_{(0,y)}(\Psi(B)) > \frac{\varepsilon}{2}\}) > 0\right) \\
& +\mathbf{G_H}\left(\boldsymbol{\Phi}_{m-n}(\{\chi \in \mathbb{A} : \int \kappa^{[n]}_{(\chi)}(d\Psi)\Psi(B)\mathbf{1}_{[y,\infty)}(\Psi(B)) > \frac{\varepsilon}{2}\}) > 0\right) \\
\leq\ & \mathbf{G_H}\left(\boldsymbol{\Phi}_{m-n}(\{\chi \in \mathbb{A} : \kappa^{[n]}_{(\chi)}(\Psi(B) > 0) > \frac{\varepsilon}{2y}\}) > 0\right) \\
& + \int \mathbf{G_H}(d(\boldsymbol{\Phi}_k))\boldsymbol{\Phi}_{m-n}\left(\{\chi \in \mathbb{A} : \int \kappa^{[n]}_{(\chi)}(d\Psi)\Psi(B)\mathbf{1}_{[y,\infty)}(\Psi(B)) > \frac{\varepsilon}{2}\}\right).
\end{aligned}$$

The first one of the two terms in the last expression tends to zero as $n \to \infty$ for each $y > 0$, for Proposition 3.2 says that $(\boldsymbol{\Phi}_k) \sim \mathbf{G_H}$ is regular iff $(\Phi_k) \sim \mathbf{H}$ is so. The second term can be estimated from above by

$$\begin{aligned}
& \frac{2}{\varepsilon}\int \mathbf{G_H}(d(\boldsymbol{\Phi}_k)) \int \boldsymbol{\Phi}_{m-n}(d\chi) \int \kappa^{[n]}_{(\chi)}(d\Psi)\Psi(B)\mathbf{1}_{[y,\infty)}(\Psi(B)) \\
=\ & \frac{2}{\varepsilon}\int \mathbf{G_H}(d(\boldsymbol{\Phi}_k)) \sum_{i\in I} \int \kappa^{[n]}_{(\chi_i)}(d\Psi)\Psi(B)\mathbf{1}_{[y,\infty)}(\Psi(B))
\end{aligned}$$

(using $\boldsymbol{\Phi}_{m-n} = \sum_{i\in I} \boldsymbol{\delta}_{\chi_i}$)

$$\begin{aligned}
=\ & \frac{2}{\varepsilon}\int \mathbf{G_H}(d(\boldsymbol{\Phi}_k)) \sum_{i\in I} \int \left(\bigotimes_{j\in I} \kappa^{[n]}_{(\chi_j)}(d(\Psi_j)_{j\in I})\right) \Psi_i(B)\mathbf{1}_{[y,\infty)}(\Psi_i(B)) \\
\leq\ & \frac{2}{\varepsilon}\int \mathbf{G_H}(d(\boldsymbol{\Phi}_k)) \int \left(\bigotimes_{j\in I} \kappa^{[n]}_{(\chi_j)}(d(\Psi_j)_{j\in I})\right) \left(\sum_{i\in I} \Psi_i(B)\right) \mathbf{1}_{[y,\infty)} \left(\sum_{i\in I} \Psi_i(B)\right)
\end{aligned}$$

$$= \frac{2}{\varepsilon}\int \mathbf{G_H}(d(\boldsymbol{\Phi}_k))\int \kappa^{[n]}_{(\mathrm{sp}\,\boldsymbol{\Phi}_{m-n})}(d\Psi)\Psi(B)\mathbf{1}_{[y,\infty)}(\Psi(B))$$

$$= \frac{2}{\varepsilon}\int \mathbf{H}(d(\Phi_k))\int \kappa^{[n]}_{(\Phi_{m-n})}(d\Psi)\Psi(B)\mathbf{1}_{[y,\infty)}(\Psi(B))$$

$$= \frac{2}{\varepsilon}\int \mathbf{H}(d(\Phi_k))\Phi_m(B)\mathbf{1}_{[y,\infty)}(\Phi_m(B)),$$

and this integral tends to zero as y tends to infinity, since $\Phi_m(B)$ is integrable with respect to $\mathbf{H}$.

So from (1) we infer that for any $m \in \mathbb{Z}$, $B \in \mathcal{B}$ and $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mathbf{G_H}\left(\boldsymbol{\Phi}_{m-n}(\{\chi \in \mathbb{A} : \int \chi(da)J^{[n]}(a,B) > \varepsilon\}) > 0\right) = 0.$$

Hence

$$\sup_{\chi\in\mathrm{supp}\,\boldsymbol{\Phi}_{m-n}} \int \chi(da)J^{[n]}(a,B)$$

tends to zero in probability as $n \to \infty$.

In order to show that this convergence is even $\mathbf{G_H}$-almost sure it is enough to prove that the sequence $\sup_{\chi\in\mathrm{supp}\,\boldsymbol{\Phi}_{m-n}} \int \chi(da)J^{[n]}(a,B)$, $n = 1,2,3,\ldots$, is a reverse sub-martingale with respect to the filtration $(\mathcal{G}_n)_{n\geq 1}$ defined by $\mathcal{G}_n := \sigma(\{\boldsymbol{\Phi}_{m-k}\}_{k\geq n})$, since this would imply that the sequence converges $\mathbf{G_H}$-almost surely (cf. [9]).

In fact, we have for $n \geq 1$

$$\begin{aligned}\sup_{\chi\in\mathrm{supp}\,\boldsymbol{\Phi}_{m-n}} \int \chi(da)J^{[n]}(a,B) &\leq \int \boldsymbol{\Phi}_{m-n}(d\chi)\int \chi(da)J^{[n]}(a,B)\\ &= \int \mathrm{sp}\,\boldsymbol{\Phi}_{m-n}(da)J^{[n]}(a,B)\end{aligned}$$

and consequently

$$\int \mathbf{G_H}(d(\boldsymbol{\Phi}_k))\sup_{\chi\in\mathrm{supp}\,\boldsymbol{\Phi}_{m-n}} \int \chi(da)J^{[n]}(a,B) \leq \int \Lambda_{m-n}(da)J^{[n]}(a,B) = \Lambda_m(B)$$

meaning that the members of the sequence are integrable with respect to $\mathbf{G_H}$. Putting again $\boldsymbol{\Phi}_{m-(n+1)} = \sum_{i\in I}\boldsymbol{\delta}_{\chi_i}$, we find

$$\mathbf{E}\left[\sup_{\chi\in\mathrm{supp}\,\boldsymbol{\Phi}_{m-n}} \int \chi(da)J^{[n]}(a,B)\,\middle|\,\mathcal{G}_{n+1}\right]$$

$$
\begin{aligned}
&= \mathbf{E}\left[\sup_{\chi\in\operatorname{supp}\boldsymbol{\Phi}_{m-n}}\int\chi(da)J^{[n]}(a,B)\,\middle|\,\boldsymbol{\Phi}_{m-(n+1)}\right]\\
&= \int \boldsymbol{\kappa}_{(\boldsymbol{\Phi}_{m-(n+1)})}(d\boldsymbol{\Psi})\sup_{\chi\in\operatorname{supp}\boldsymbol{\Psi}}\int\chi(da)J^{[n]}(a,B)\\
&= \int\left(\bigotimes_{i\in I}\kappa_{(\chi_i)}(d(\Psi_i)_{i\in I})\right)\sup_{j\in I}\int\Psi_j(da)J^{[n]}(a,B)\\
&\geq \sup_{j\in I}\int\kappa_{(\chi_j)}(d\Psi_j)\int\Psi_j(da)J^{[n]}(a,B)\\
&= \sup_{j\in I}\int\chi_j(da_j)\int\kappa_{(a_i)}(d\Psi_j)\int\Psi_j(da)J^{[n]}(a,B)\\
&= \sup_{j\in I}\int\chi_j(da)J^{[n+1]}(a,B)\\
&= \sup_{\chi\in\operatorname{supp}\boldsymbol{\Phi}_{m-(n+1)}}\int\chi(da)J^{[n+1]}(a,B),
\end{aligned}
$$

which concludes the proof. □

4. The special case of substochastic shifts and the family process. If one is trying to reveal the general structure of κ-processes, it is quite useful to consider the corresponding family process, since, as it was pointed out above, on the family level the dynamics is a substochastic shift and hence particularly simple. (On the other hand, the phase space is more complicated.)

If the branching dynamics is a substochastic shift, the $\sigma(K)$-process $(\Phi_k) \sim \mathbf{H}$ is a collection of (possibly terminating) migrations in A, which are conditionally independent of each other in the future, given the population Φ_{k_0} at a starting time k_0. So, somewhat slackly spoken, knowing the remote past of the process, the further evolution is described by an at most countable collection of independent individuals migrating in A. Considering a bounded region B, each individual has some chance to visit B at a given time m. Due to independence, the actual number $\Phi_m(B)$ has good chances to be of the order of the expected number (given the remote past), which in particular implies that this expected number must be finite. If in addition we assume that the process is regular, then the Poisson law should imply that, given the remote past, $\Phi_m(B)$ has a Poisson distribution. This is the heuristic argument for the following three assertions.

Proposition 4.1. *Each left-tail trivial $\sigma(K)$-process $(\Phi_k) \sim \mathbf{H}$ is of first order.*

Proof. Let $B \in \mathcal{B}$ and $m \in \mathbb{Z}$. Then $(\Phi_{m-k} * K^{[k]})(B)$ is almost surely finite for each $k \geq 0$, since otherwise by the Borel-Cantelli lemma $\sigma(K)^{[k]}_{(\Phi_{m-k})}$ would not exist a.s. For almost all Φ_{m-k} with

$$(\Phi_{m-k} * K^{[k]})(B) - 2\sqrt{(\Phi_{m-k} * K^{[k]})(B)} \geq L > 0$$

we have now (taking into account that, given Φ_{m-k}, the quantity $\Phi_m(B)$ is a number of Bernoulli successes)

$$\begin{aligned}
&\mathbf{H}\left(\Phi_m(B) \geq L \middle| \Phi_{m-k}\right) \\
\geq\ & \mathbf{H}\left(\Phi_m(B) \geq (\Phi_{m-k} * K^{[k]})(B) - 2\sqrt{(\Phi_{m-k} * K^{[k]})(B)} \middle| \Phi_{m-k}\right) \\
\geq\ & 1 - \mathbf{H}\left(|\Phi_m(B) - (\Phi_{m-k} * K^{[k]})(B)| > 2\sqrt{(\Phi_{m-k} * K^{[k]})(B)} \middle| \Phi_{m-k}\right) \\
\geq\ & 1 - \mathbf{H}\left(|\Phi_m(B) - \mathbf{E}(\Phi_m(B)|\Phi_{m-k})| \geq 2\sqrt{\mathbf{Var}(\Phi_m(B)|\Phi_{m-k})} \middle| \Phi_{m-k}\right) \\
\geq\ & 1 - 1/4 = 3/4,
\end{aligned}$$

which implies that

$$\mathbf{H}\left(\Phi_m(B) \geq L\right) \geq 3/4 \cdot \mathbf{H}\left((\Phi_{m-k} * K^{[k]})(B) - 2\sqrt{(\Phi_{m-k} * K^{[k]})(B)} \geq L\right).$$

Now (see [8], proof of Proposition 16.1) the left-tail triviality of $\mathbf{H}$ implies almost surely for each bounded B

$$(\Phi_{m-k} * K^{[k]})(B) \underset{k\to\infty}{\longrightarrow} \mathbf{E}\Phi_m(B),$$

where the right-hand side need not necessarily be finite. If it would be infinite, however, we would get $\mathbf{H}\left(\Phi_m(B) \geq L\right) \geq 3/4$ which is a contradiction, since L was arbitrary. □

From 2.3 and the preceding result we deduce

Corollary 4.2. *The family process* $(\boldsymbol{\Phi}_k) \sim \mathbf{G_H}$ *of a left-tail trivial* κ*-process* $(\Phi_k) \sim \mathbf{H}$ *is of first order.*

Now consider an arbitrary $\sigma(K)$-process $(\Phi_k) \sim \mathbf{H}$. By 3.6 it is regular iff its left-tail trivial components (being first order processes by the preceding proposition) are almost surely regular. Consider the corresponding sequences of intensity measures $\{\Lambda_{\mathbf{H}(\Phi_m|\mathcal{F}_{-\infty})((\Phi_k))}\}_{m\in\mathbb{Z}}$. In [2] the probability distribution of this random measure-valued sequence is denoted by $\underline{\mathcal{W}}^{\mathbf{H}}$. Regularity of $\mathbf{H}$ is by Theorem 3.7 equivalent to the fact that $\underline{\mathcal{W}}^{\mathbf{H}}$-a.s. condition (2) of this theorem is fulfilled. By Theorem 4.4.5. of [2] this means that, in the terminology of section 4.5 in [2], $\underline{\mathcal{W}}^{\mathbf{H}}$ is regular. (i.e. the notions of regularity are compatible). Moreover, in [2] the notion of a κ-process of Poisson type with a given intensity sequence $(\nu_k)_{k\in\mathbb{Z}}$ was defined as weak limit, for $m \to \infty$, of κ-Markov processes starting at time m with a Poisson population with intensity measure ν_m (provided this limit process has (ν_k) as its intensity sequence, see [2], section 3.3). Now observe that in the case of a $\sigma(K)$-dynamics the Poisson character of the initial population is conserved forever; hence the process of "Poisson type" with given intensity sequence is nothing but the corresponding Poissonian $\sigma(K)$-process. Hence [2], Theorem 4.5.3 immediately yields

Theorem 4.3. *A* $\sigma(K)$*-process* $(\Phi_k) \sim \mathbf{H}$ *is regular iff almost all left-tail trivial components are Poisson processes.*

From the contemporary point of view it is reasonable to extend the definitions of *κ-processes of Poisson type* (resp. *of Cox type*, see [2], sections 3.2 and 3.3) to the case of processes not being of first order. In [11] one of us presented an example of a subcritical spatially homogeneous branching process admitting an equilibrium, which is (necessarily) not of first order, but of Poisson type in the sense of

Definition 4.4. *A* κ*-process* $(\Phi_k) \sim \mathbf{H}$ *is said to be of Poisson type, if the corresponding family process* ($\boldsymbol{\Phi} \sim \mathbf{G_H}$ *is Poissonian (i.e. all marginals are Poisson point fields).*

The preceding theorem in combination with Proposition 3.2 leads to the conclusion

Corollary 4.5. *A* κ*-process is regular iff it is a mixture of left-tail trivial processes of Poisson type.*

Let us recall the notion of a *Coxian* $\sigma(K)$*-process.*

Let N denote the set of those measures on $\mathcal{A}$ having finite values on bounded sets. The σ-field on N generated by all mappings $\nu \in \mathsf{N} \longmapsto \nu(B)$, $B \in \mathcal{B}$, will be denoted by $\mathcal{N}$.

For $\nu \in \mathsf{N}$ and a substochastic kernel K on A the convolution

$$\nu * K := \int \nu(da)K(a,(\cdot))$$

is a measure on $\mathcal{A}$ (not necessarily belonging to $\mathcal{N}$). A probability distribution $\mathbf{Q}$ on $\mathcal{N}^{\otimes\mathbb{Z}}$ represents a random sequence $(\nu_k)_{k\in\mathbb{Z}}$ of measures in N. Let, for $k \in \mathbb{Z}$, $\mathbf{Q}_k := \mathbf{Q}(\nu_k \in (\cdot))$ be the marginals of $\mathbf{Q}$. If we have, for a substochastic kernel K on A

$$\nu_k * K = \nu_{k+1}, \quad \mathbf{Q}\text{-almost surely for each } \ k \in \mathbb{Z},$$

then the sequence of probability distributions

$$\mathcal{I}(\mathbf{Q}_m) := \int \mathbf{Q}(d(\nu_k)_{k\in\mathbb{Z}})\Pi_{\nu_m} = \int \mathbf{Q}_m(d\nu)\Pi_\nu\ , \quad m \in \mathbb{Z},$$

obviously fulfils the relation

$$(\mathcal{I}(\mathbf{Q}_m))_{\sigma(K)} = \mathcal{I}(\mathbf{Q}_{m+1}),$$

so there is a unique $\sigma(K)$-process with $\mathcal{I}(\mathbf{Q}_m)$, $m \in \mathbb{Z}$, as its marginals. It is called Coxian and its probability law is denoted by $\mathcal{I}(\mathbf{Q})$.

The following assertion has been known for long under stronger assumptions (cf. e.g. Theorem 4.6 in [3]). It is now a simple consequence of Theorem 4.3.

Corollary 4.6. *Each regular $\sigma(K)$-process is Coxian.*

Definition 4.7. *A κ-process $(\Phi_k) \sim \mathbf{H}$ is said to be of Cox type, if the corresponding family process $(\mathbf{\Phi}_k) \sim \mathbf{G_H}$ is Coxian.*

This yields

Corollary 4.8. *Each regular κ-process is of Cox type.*

5. κ-processes with finitely many clans. For a κ-process $(\Phi_k) \sim \mathbf{H}$ consider the family process $(\mathbf{\Phi}_k) \sim \mathbf{G_H}$ and put

$$r((\mathbf{\Phi}_k)) := \sup_{n\in\mathbb{Z}} \mathbf{\Phi}_n(\mathbb{A}).$$

Definition 5.1. *We say a κ-process $(\Phi_k) \sim \mathbf{H}$ has finitely many clans if the condition $r((\mathbf{\Phi}_k)) < +\infty$ is fulfilled almost surely.*

Stationary processes of that kind have been investigated in [6]. In that case it is sufficient to assume that at each time there are almost surely only finitely many families, i.e.

$$\mathbf{G_H}(\mathbf{\Phi}_m(\mathbb{A}) < \infty) = 1, \quad m \in \mathbb{Z}.$$

In the general case this may be fulfilled with the κ-process not having finitely many clans in the sense defined above.

Example 5.2. Let $A = \{1\}$ and define $\kappa = \sigma(K)$ with $K(1,\{1\}) = c$ for some $c \in (0,1)$. Then $(\Pi_{\nu_n})_{n\in\mathbb{Z}}$, with $\nu_n(\{1\}) = c^n$ is an *entrance law* for κ. For the corresponding κ-process each family comprises exactly one individual. So $\mathbf{\Phi}_m(\mathbb{A})$ has a Poisson distribution with parameter c^m meaning that each $\mathbf{\Phi}_m(\mathbb{A})$ is a.s. finite for each m, but $\sup_{m\in\mathbb{Z}} \mathbf{\Phi}_m(\mathbb{A}) = +\infty$ almost surely. Moreover we have $\kappa^{[n]}_{(1)}(\Psi(\{1\}) > 0) = c^n \underset{n\to\infty}{\longrightarrow} 0$, i.e. the κ-process is regular.

Regularity and having finitely many clans are incompatible properties for κ-processes:

Proposition 5.3. *If a κ-process $(\Phi_k) \sim \mathbf{H}$ is regular and has finitely many clans then it is almost surely empty, i.e.*

$$\mathbf{H}(\Phi_m(A) = 0) = 1, \quad m \in \mathbb{Z}.$$

P r o o f. Let $m \in \mathbb{Z}$ and $B \in \mathcal{B}$. Then we have for all $n \geq 0$ and $x > 0$

$$\begin{aligned}
&\mathbf{H}\,(\Phi_m(B) > 0) \\
&= \int \mathbf{H}(d(\Phi_k))\kappa^{[n]}_{(\Phi_{m-n})}(\Psi(B) > 0) \\
&= \int \mathbf{G_H}(d(\mathbf{\Phi}_k))\kappa^{[n]}_{(\mathrm{sp}\,\mathbf{\Phi}_{m-n})}(\Psi(B) > 0) \\
&= \int \mathbf{G_H}(d(\mathbf{\Phi}_k))\kappa^{[n]}_{(\mathrm{sp}\,\mathbf{\Phi}_{m-n})}(\Psi(B) > 0)(\mathbf{1}_{[0,x)}(r((\mathbf{\Phi}_k))) + \mathbf{1}_{[x,\infty)}(r((\mathbf{\Phi}_k)))) \\
&\leq \int \mathbf{G_H}(d(\mathbf{\Phi}_k)) \int \mathbf{\Phi}_{m-n}(d\chi)\kappa^{[n]}_{(\chi)}(\Psi(B){>}0)\mathbf{1}_{[0,x)}(r((\mathbf{\Phi}_k))){+}\mathbf{G_H}(r((\mathbf{\Phi}_k)){\geq}x) \\
&\leq x \cdot \int \mathbf{G_H}(d(\mathbf{\Phi}_k)) \sup_{\chi\in\mathrm{supp}\,\mathbf{\Phi}_{m-n}} \kappa^{[n]}_{(\chi)}(\Psi(B) > 0) + \mathbf{G_H}(r((\mathbf{\Phi}_k)) \geq x).
\end{aligned}$$

If (Φ_k) is regular, then by Theorem 3.4 the integral term of the last row tends to zero as $n \to \infty$, and if (Φ_k) has finitely many clans, then the second term tends to zero as $x \to \infty$. So (Φ_k) is a.s. empty. $\square$

For the subsequent considerations we introduce the notion of a sub-process. A κ-process $(\Phi_k^{(1)})_{k\in\mathbb{Z}} \sim \mathbf{H}^{(1)}$ will be called *sub-process* of another κ-process $(\Phi_k^{(2)})_{k\in\mathbb{Z}} \sim \mathbf{H}^{(2)}$, if there exists a random coupled sequence $((\Psi_k^{(1)}, \Psi_k^{(2)}))_{k\in\mathbb{Z}}$ of elements of $\mathsf{M} \times \mathsf{M}$ with the properties

$$\mathrm{Prob}((\Psi_k^{(i)})_{k\in\mathbb{Z}} \in (\cdot)) = \mathbf{H}^{(i)}, \quad i = 1, 2$$

and

$$\mathrm{Prob}(\Psi_m^{(1)} \le \Psi_m^{(2)}) = 1, \ \ m \in \mathbb{Z}.$$

Remark 2. In the following we will consider sub-processes $(\Phi_k^{(1)})$ of a κ-process $(\Phi_k) \sim \mathbf{H}$ which are defined by selecting whole clans according to whether they fulfil a given $\mathcal{F}_{-\infty}^{o}$-property. In this case the complementary population sequence $(\Phi_k - \Phi_k^{(1)})$ forms a sub-process, too, because it is defined by selecting clans according to the complementary $\mathcal{F}_{-\infty}^{o}$-property. In the general situation it may happen that the complementary population is *not* a κ-process and consequently not a sub-process by definition. This is illustrated by the following

Example 5.4. Let $A = \{1, 2, 3\}$ and consider the $\sigma(K)$-process (Φ_k) which is uniquely defined by the stochastic kernel

$$K(i, (\cdot)) = \begin{cases} \delta_1 & \text{for } i = 1 \\ 1/2 \cdot \delta_2 + 1/2 \cdot \delta_3 & \text{for } i = 2, 3 \end{cases}$$

and by the property that $\Phi_k(\{1\} = \Phi_k(\{2, 3\}) = 1$ for all $k \in \mathbb{Z}$. It describes a population of two individuals, with the one at position 1 staying there forever and the other one migrating randomly between the locations 2 and 3. Now consider a random sub-population $(\Phi_k^{(1)})$ of (Φ_k) defined as follows: If $\Phi_0(\{3\}) = 1$ we put $\Phi_k^{(1)} = \delta_1, k \in \mathbb{Z}$, whereas for the case $\Phi_0(\{3\}) = 0$ we put $\Phi_k^{(1)} = o$, $k \in \mathbb{Z}$. It is obvious that $(\Phi_k^{(1)})$ is a $\sigma(K)$-process (not too much an interesting one, as we admit). But $(\Phi_k - \Phi_k^{(1)})$ is not a $\sigma(K)$-process, because the motion of the individual in $\{2, 3\}$ is not independent of whether there is an individual at $\{1\}$.

Let us denote by $(\Phi_{B,m,c;k})_{k\in\mathbb{Z}}$, for any $m \in \mathbb{Z}$, any $B \in \mathcal{B}$ and any $c > 0$, the sub-process of (Φ_k) consisting of all individuals of those clans which fulfil

$$\varlimsup_{n\to\infty} \kappa^{[n]}_{(\chi_{m-n})}(\Psi(B) > 0) \ge c,$$

and by $(\Phi_{k,\mathrm{reg}})$ the sub-process consisting of all individuals of those clans which fulfil

$$\lim_{n\to\infty} \kappa^{[n]}_{(\chi_{m-n})}(\Psi(B) > 0) = 0$$

for each $B \in \mathcal{B}$ and each $m \in \mathbb{Z}$. The latter will be called the *regular part of* (Φ_k), and $\Phi_{k,\mathrm{cnreg}} := (\Phi_k - \Phi_{k,\mathrm{reg}})$ the *completely non-regular part.* We denote the corresponding κ-processes by $\mathbf{H}_{\mathrm{reg}}$ and $\mathbf{H}_{\mathrm{cnreg}}$, respectively.

We will see in the next section, that (as announced in the introduction) we might have defined these sub-processes by referring to the ancestral lines of individuals instead of the whole clans.

The considerations in the third step of the proof of Theorem 3.4 show that

5.5 *Let* (Φ_k) *be an arbitrary* κ*-process. Then for any* $m \in \mathbb{Z}$*, any* $B \in \mathcal{B}$ *and any* $c > 0$*,* $(\Phi_{B,m,c;k})$ *is a sub-process with finitely many clans.*

The following theorem claims more than incompatibility, it says that non-regularity even implies the existence of a sub-process with finitely many clans. (There are trivial examples of processes having a.s. infinitely many clans, but no regular sub-process.)

Theorem 5.6. *A* κ*-process is regular iff any sub-process with finitely many clans is almost surely empty.*

P r o o f. It is obvious that a sub-process of a regular process is regular. So by the Proposition 5.3 the sub-process is a.s empty, if it has finitely many clans.

On the other hand, assume the κ-process $(\Phi_k) \sim \mathbf{H}$ to be non-regular. By assumption there exist some $m \in \mathbb{Z}$, $B \in \mathcal{B}$ and $c > 0$ such that $(\Phi_{B,m,c;k})$ is not almost surely empty, and by 5.5 this is a sub-process with finitely many clans. □

In the following section we use the notion of the *typical clan* of a κ-process with finitely many clans.

Let $(\Phi_k) \sim \mathbf{H}$ be such a process which is assumed to be not a.s. empty. So

$$\mathbf{G_H}(r((\boldsymbol{\Phi}_k)) > 0) > 0 \ \text{ and } \ \mathbf{G_H}(r((\boldsymbol{\Phi}_k)) < \infty) = 1.$$

Then a non-empty realization $(\boldsymbol{\Phi}_k)$ almost surely consists of exactly $r((\boldsymbol{\Phi}_k))$ clans $(\chi_k^1), (\chi_k^2), \ldots, (\chi_k^{r((\boldsymbol{\Phi}_k))})$. Recall that in the non-stationary situation for a process with finitely many clans, families may go extinct. So each (χ_k^r) might be defined just for k which are small enough. *Typical clan* of $(\Phi_k) \sim \mathbf{H}$ is called the

κ-process $(\widehat{\Phi}_k) \sim \widehat{\mathbf{H}}$ defined by

$$\widehat{\mathbf{H}} := \int \mathbf{G}_{\mathbf{H}}(d(\mathbf{\Phi}_k) | r((\mathbf{\Phi}_k)) > 0) \frac{1}{r((\mathbf{\Phi}_k))} \sum_{r=1}^{r((\mathbf{\Phi}_k))} \boldsymbol{\delta}_{(\widetilde{\chi}_k^r)_{k \in \mathbb{Z}}}$$

where $(\widetilde{\chi}_k^r)_{k \in \mathbb{Z}}$ denotes the two-sided infinite sequence obtained from the rth clan by continuing it to the right -in case it goes extinct- by an infinite number of empty populations o. (If the clan does not break off we leave it unchanged, of course.) So a typical clan is a clan randomly chosen from a non-empty realization.

This definition differs slightly from the one in [6], definition 7.6, by the fact that here we always have

$$\mathbf{G}_{\widehat{\mathbf{H}}}(r((\mathbf{\Phi}_k)) = 1) = 1.$$

A κ-process obeying this property will be called *single-line*.

6. Convergence results.

Lemma 6.1. *For a κ-process $(\Phi_k) \sim \mathbf{H}$, there exists to any $m \in \mathbb{Z}$ and $B \in \mathcal{B}$ and almost surely to each clan (χ_k) the limit*

$$\kappa_{B,m}((\chi_k)) := \lim_{n \to \infty} \kappa^{[n]}_{(\chi_{m-n})}(\Psi(B) > 0).$$

P r o o f. For the regular part of (Φ_k) the assertion is obvious. So it is sufficient to prove the lemma for the sub-process $(\Phi_{B,m,c;k})$ of $(\Phi_k) \sim \mathbf{H}$ where $c > 0$ is arbitrary, and we may consider the typical clan of this sub-process. Hence we may assume even that the process $(\Phi_k) \sim \mathbf{H}$ is single-line. In this case we have

$$\kappa^{[n]}_{(\Phi_{m-n})}(\Psi(B) > 0) = \mathbf{H}(\Phi_m(B) > 0 | \Phi_{m-n}) = \mathbf{H}(\Phi_m(B) > 0 | (\Phi_{m-l})_{l \geq n}).$$

Consequently, there exists a.s. the limit

$$\kappa_{B,m}((\Phi_k)) = \lim_{n \to \infty} \kappa^{[n]}_{(\Phi_{m-n})}(\Psi(B) > 0) = \mathbf{H}(\Phi_m(B) > 0 | \mathcal{F}_{-\infty}). \qquad \square$$

The assertion of Lemma 6.1 can be strengthened. Not only the probabilities $\kappa^{[n]}_{(\chi_{m-n})}(\Psi(B) > 0)$, but even the probability distributions $\kappa^{[n]}_{(\chi_{m-n})}$ do converge. The limit distributions can be specified. For this aim let us introduce some notations.

Let $B_1, B_2, \ldots$ be a sequence of elements of $\mathcal{B}$ such that to each $B \in \mathcal{B}$ there is a B_l with $B \subset B_l$. By Lemma 6.1, to a κ-process $(\Phi_k) \sim \mathbf{H}$ there exists almost surely to each clan (χ_k) the quantity

$$\kappa((\chi_k)) := \sup_{l \geq 1, m \in \mathbb{Z}} \kappa_{B_l, m}((\chi_k)).$$

Obviously its value does not depend on the special choice of the sequence (B_k).

According to Theorem 3.4, a clan (χ_k) belongs to the regular part of (Φ_k) iff $\kappa((\chi_k)) = 0$.

In Section 2 we referred to the kernel $\underline{K}$. In view of the $\mathcal{F}_{-\infty}$-measurability of this kernel the distributions $\underline{K}((\chi_k), (\cdot))$ are well-defined even for clans terminating to the right.

Proposition 6.2. *Let $(\Phi_k) \sim \mathbf{H}$ be a κ-process, $m \in \mathbb{Z}$, $s > 0$, $B_1, \ldots, B_s \in \mathcal{B}$ and $j_1, \ldots, j_s \geq 0$. Then we have almost surely for each clan (χ_k)*

$$\lim_{n\to\infty} \kappa^{[n]}_{(\chi_{m-n})} (\Psi(B_1) = j_1, \ldots, \Psi(B_s) = j_s)$$

$$= \begin{cases} \boldsymbol{\delta}_{(0,\ldots,0)}(\{(j_1, \ldots, j_s)\}) & \text{if} \quad \kappa((\chi_k)) = 0 \\ \underline{K}((\chi_k), \{\Phi_m(B_1) = j_1, \ldots, \Phi_m(B_s) = j_s\}) & \text{if} \quad \kappa((\chi_k)) > 0. \end{cases}$$

P r o o f. If (χ_k) belongs to the regular part of (Φ_k), the assertion is obvious. So we may assume that $\mathbf{H}$ has finitely many clans or is even single-line. Then the (only) clan coincides with the Markov process $(\widehat{\Phi}_k) \sim \widehat{\mathbf{H}}$ and we have

$$\begin{aligned} &\lim_{n\to\infty} \kappa^{[n]}_{(\chi_{m-n})} (\Psi(B_1) = j_1, \ldots, \Psi(B_s) = j_s) \\ = \ &\widehat{\mathbf{H}}(\widehat{\Phi}_m(B_1) = j_1, \ldots, \widehat{\Phi}_m(B_s) = j_s | \mathcal{F}_{-\infty})((\widehat{\Phi}_k)) \end{aligned}$$

which coincides with $\underline{K}((\widehat{\Phi}_k), \{\Phi_m(B_1) = j_1, \ldots, \Phi_m(B_s) = j_s\})$ by property (2) of the stochastic kernel $\underline{K}$ (cf. [2], 4.1.2 for a similar result). □

In Proposition 6.2 each clan may be replaced by the ancestral line of anyone of its individuals. This is not too much a surprise in view of the considerations used already to motivate Proposition 3.2: We saw that in the non-regular part asymptotically at most one individual at time $n \to -\infty$ of a given clan is responsible for the whole offspring in a given bounded set at time m.

Theorem 6.3. *Let $(\Phi_k) \sim \mathbf{H}$ be a κ-process. Let $m \in \mathbb{Z}$, $s > 0$, $B_1, \ldots, B_s \in \mathcal{B}$ and $j_1, \ldots, j_s \geq 0$. Then we have almost surely for each clan (χ_k)*

and each ancestral line $(a_{m'-n})_{n\geq 0}$ *of an individual of* $\chi_{m'}$*,* $m' \in \mathbb{Z}$*:*

$$\lim_{n\to\infty} \kappa^{[n]}_{(\chi_{m-n})} \left(\Psi(B_1) = j_1, \ldots, \Psi(B_s) = j_s\right)$$

$$= \lim_{n\to\infty} \kappa^{[n+m-m']}_{(a_{m'-n})} \left(\Psi(B_1) = j_1, \ldots, \Psi(B_s) = j_s\right).$$

Proof. Obviously we may again assume without any restriction of the generality that $\mathbf{H} = \widehat{\mathbf{H}}$ is single-line. We may confine ourselves to the case that at least one of the j_k's is positive. We may also assume that the limit in the first line is a.s. positive, since its positiveness is a left-tail event, and we may, if necessary, pass to the restriction of $\mathbf{H}$ to this event, which is again a κ-process. Choose some $B \supseteq \cup B_k$. Due to our assumption, we a.s. find to any sufficiently small $\eta > 0$ some n_0, such that for $n \geq n_0$ being large enough

$$0 < \widehat{\mathbf{H}}^o \left(\widehat{\Phi}_m(B_1) = j_1, \ldots, \widehat{\Phi}_m(B_s) = j_s | \mathcal{F}^o_{-\infty}\right) ((\widehat{\Phi}^o_k)) - 2\eta$$

$$\leq \widehat{\mathbf{H}}^o \left(\begin{array}{l} \widehat{\Phi}_m(B_1) = j_1, \ldots, \widehat{\Phi}_m(B_s) = j_s \textit{ and all individuals} \\ \textit{in } B \textit{ at time } m \textit{ have a common ancestor at } m - n_0 \end{array} \middle| \mathcal{F}^o_{-\infty} \right) ((\widehat{\Phi}^o_k)) - \eta$$

$$\leq \widehat{\mathbf{H}}^o \left(\begin{array}{l} \widehat{\Phi}_m(B_1) = j_1, \ldots, \widehat{\Phi}_m(B_s) = j_s \textit{ and all individuals} \\ \textit{in } B \textit{ at time } m \textit{ have a common ancestor at } m - n_0 \end{array} \middle| \widehat{\Phi}^o_{m-n} \right)$$

$$\leq \widehat{\mathbf{H}}^o \left(\begin{array}{l} \widehat{\Phi}_m(B_1) = j_1, \ldots, \widehat{\Phi}_m(B_s) = j_s \textit{ and all individuals} \\ \textit{in } B \textit{ at time } m \textit{ have a common ancestor at } m - n \end{array} \middle| \widehat{\Phi}^o_{m-n} \right)$$

$$\leq \widehat{\mathbf{H}}^o \left(\widehat{\Phi}_m(B_1) = j_1, \ldots, \widehat{\Phi}_m(B_s) = j_s | \widehat{\Phi}^o_{m-n}\right)$$

$$\leq \widehat{\mathbf{H}}^o \left(\widehat{\Phi}_m(B_1) = j_1, \ldots, \widehat{\Phi}_m(B_s) = j_s | \mathcal{F}^o_{-\infty}\right) ((\widehat{\Phi}^o_k)) + \eta.$$

So we have for large n

$$\widehat{\mathbf{H}}^o \left(\widehat{\Phi}_m(B_1) = j_1, \ldots, \widehat{\Phi}_m(B_s) = j_s | \mathcal{F}^o_{-\infty}\right) ((\widehat{\Phi}^o_k)) - 2\eta$$

$$\leq \widehat{\mathbf{H}}^o \left(\begin{array}{l} \widehat{\Phi}_m(B_1) = j_1, \ldots, \widehat{\Phi}_m(B_s) = j_s \textit{ and all individuals} \\ \textit{in } B \textit{ at time } m \textit{ have a common ancestor at } m - n \end{array} \middle| \widehat{\Phi}^o_{m-n} \right)$$

$$\leq \widehat{\mathbf{H}}^{o}\left(\widehat{\Phi}_m(B_1)=j_1,\ldots,\widehat{\Phi}_m(B_s)=j_s|\widehat{\Phi}^o_{m-n}\right)$$

$$\leq \widehat{\mathbf{H}}^{o}\left(\widehat{\Phi}_m(B_1)=j_1,\ldots,\widehat{\Phi}_m(B_s)=j_s|\mathcal{F}^o_{-\infty}\right)((\widehat{\Phi}^o_k))+\eta.$$

We recall the fact that the population at time m is the independent superposition of the populations generated by the individuals of $\widehat{\Phi}^o_{m-n}$. We can apply Lemma 3.3 which says that, from the fact that the probability of *more than one* of these individuals having offspring in B is (arbitrarily) small whereas the probability of *at least one* of the individuals having offspring in B is bounded away from zero, we may conclude that there is a specific individual responsible for the whole offspring with an overwhelming probability. So a.s. to each $\eta'>0$ for n being large enough there is an individual in $\widehat{\Phi}^o_{m-n}$ such that for its position a_{m-n} we have

$$\kappa^{[n]}_{(a_{m-n})}(\Psi(B_1)=j_1,\ldots,\Psi(B_s)=j_s)$$

$$\geq \widehat{\mathbf{H}}\left(\widehat{\Phi}_m(B_1)=j_1,\ldots,\widehat{\Phi}_m(B_s)=j_s|\widehat{\Phi}^o_{m-n}\right)-\eta'$$

as well as

$$\kappa^{[n]}_{(\widehat{\Phi}_{m-n}-\delta_{a_{m-n}})}(\Psi(B)>0)<\eta'.$$

So a.s. there is a sequence of positions (a_{m-n}) with the following properties

$$\lim_{n\to\infty}\kappa^{[n]}_{(a_{m-n})}(\Psi(B_1)=j_1,\ldots,\Psi(B_s)=j_s)$$

$$=\widehat{\mathbf{H}}\left(\widehat{\Phi}_m(B_1)=j_1,\ldots,\widehat{\Phi}_m(B_s)=j_s|\mathcal{F}_{-\infty}\right)((\widehat{\Phi}_k))$$

and

$$\lim_{n\to\infty}\kappa^{[n]}_{(\widehat{\Phi}_{m-n}-\delta_{a_{m-n}})}(\Psi(B)>0)=0.$$

Since in a single-line process any two ancestral lines eventually coalesce, we are through with the proof of the theorem if we can show that the (a_{m-n}) almost surely eventually form an ancestral line.

Assume the opposite. Then there is a positive δ with the following property: Choose an arbitrary large positive integer N. Then, tracing the evolution of our Markov single-line κ-process, with a probability of at least δ we find a time $n>N$ with $\boldsymbol{\delta}_{a_{m-n+1}}$ being among the descendants of $\widehat{\Phi}_{m-n}-\boldsymbol{\delta}_{a_{m-n}}$. Now, the offspring of $\boldsymbol{\delta}_{a_{m-n}}$ and $\widehat{\Phi}_{m-n} - \boldsymbol{\delta}_{a_{m-n}}$ being conditionally independent, we come to the conclusion that with a probability of at least

$$\delta \cdot \kappa^{[n]}_{(a_{m-n})}(\Psi(B) > 0) \cdot \kappa^{[n-1]}_{(a_{m-n+1})}(\Psi(B) > 0)$$

we have two different ancestors of the individuals living at m in B at the given time $m-n$. By assumption,

$$\widehat{\mathbf{H}}\left(\widehat{\Phi}_m(B_1) = j_1, \ldots, \widehat{\Phi}_m(B_s) = j_s | \mathcal{F}_{-\infty}\right)((\widehat{\Phi}_k))$$

$$= \lim_{n\to\infty} \kappa^{[n]}_{(a_{m-n})}(\Psi(B_1) = j_1, \ldots, \Psi(B_s) = j_s)$$

$$\leq \lim_{n\to\infty} \kappa^{[n]}_{(a_{m-n})}(\Psi(B) > 0)$$

was positive, which leads to the contradictive conclusion that in a single-line process the probability of having more than one ancestor (of the individuals in B at m) at time $m-n$ does not tend to zero. □

From the preceding theorem and Theorem 3.4 we deduce

Corollary 6.4. *A κ-process $(\Phi_k) \sim \mathbf{H}$ is regular iff almost surely for each $m \in \mathbb{Z}$, each $B \in \mathcal{B}$ and each ancestral line (a_{m-n}) of an individual of Φ_m we have the convergence*

$$\lim_{n\to\infty} \kappa^{[n]}_{(a_{m-n})}(\Psi(B) > 0) = 0.$$

(Of course, the last convergence relation yields $\lim_{n\to\infty} \kappa^{[n]}_{(\chi_{m-n})}(\Psi(B)>0) = 0$ immediately only for those clans (χ_k) which did not die out until m. So we still have to prove the validity of condition (5) of Theorem 3.4 for those, which died out until m. This is easy: Take some $m' < m$ where the clan is still alive. Then by Proposition 6.2 from $\lim_{n\to\infty} \kappa^{[n]}_{(\chi_{m'-n})}(\Psi(B) > 0) = 0$ for each bounded B we deduce $\kappa((\chi_k)) = 0$ or $\underline{K}((\chi_k), \{\Phi_{m'} = o\}) = 1$ for each bounded B. In both cases we clearly have $\lim_{n\to\infty} \kappa^{[n]}_{(\chi_{m-n})}(\Psi(B) > 0) = 0$.)

Stimulated by the paper [4] of Liggett and Port, in [7], Lemma 3.1, another property of stationary κ-processes of first order was presented, with an intensity kernel J satisfying an additional finiteness condition, which is complementary to regularity. This result is still true for non-stationary κ-processes.

Proposition 6.5. *Let $(\Phi_k) \sim \mathbf{H}$ be a κ-process of first order. If (Φ_k) is regular, then there is no sequence $(\nu_k)_{k\in\mathbb{Z}}$ of measures on $\mathcal{A}$ that would satisfy*

all of the following conditions.
(*a*) $\nu_k * J = \nu_{k+1}$ *for all* $k \in \mathbb{Z}$,
(*b*) $\nu_k \leq \Lambda_{\mathbf{H}_k}$ *for all* $k \in \mathbb{Z}$
(*c*) $0 < \sup_{k\in\mathbb{Z}} \nu_k(A) < \infty$.
If (Φ_k) *is not regular and if in addition we have* $\sup_{n\geq 1, a\in A} J^{[n]}(a, A) < \infty$, *then there exists such a sequence* $(\nu_k)_{k\in\mathbb{Z}}$ *of measures on* $\mathcal{A}$.

The finiteness condition posed on the intensity kernel J in the second statement is fulfilled for instance if $J(a, A) \leq 1$ for each $a \in A$, i.e. if each individual has an expected number of (direct) descendants not greater than one. This is fulfilled for clustering fields $\sigma(K)$ derived from a substochastic kernel K.

P r o o f o f P r o p o s i t i o n 6.5.

1. To prove the first statement, assume there is a sequence $(\nu_k)_{k\in\mathbb{Z}}$ fulfilling (a), (b) and (c). We have to show that (Φ_k) cannot be regular.

With $(\nu_k)_{k\in\mathbb{Z}}$ fulfilling (a) and (b) we get

$$(\Lambda_{\mathbf{H}_k} - \nu_k) * J = \Lambda_{\mathbf{H}_{k+1}} - \nu_{k+1}, \quad \text{for } k \in \mathbb{Z}.$$

By [2], 3.3.5, we see that there are κ-processes $(\Phi_k^{(1)}) \sim \mathbf{H}^{(1)}$ and $(\Phi_k^{(2)}) \sim \mathbf{H}^{(2)}$ with sequences of intensity measures $(\nu_k)_{k\in\mathbb{Z}}$ and $(\Lambda_{\mathbf{H}_k} - \nu_k)_{k\in\mathbb{Z}}$, respectively. The independent superposition of these two κ-processes is a κ-process $(\Phi_k^*) \sim \mathbf{H}^*$, too. The latter satisfies $\Lambda_{\mathbf{H}_k^*} = \Lambda_{\mathbf{H}_k}$ for $k \in \mathbb{Z}$, i.e. (Φ_k^*) and (Φ_k) are equivalent in the first order sense. So by Theorem 3.7 it would be enough to show that (Φ_k^*) is not regular. In fact, it cannot be regular in view of Theorem 5.6 since it has $(\Phi_k^{(1)}) \sim \mathbf{H}^{(1)}$ as its sub-process which is easily seen to be a not almost surely empty (by (c)) process with finitely many clans. Consider the respective family process $(\boldsymbol{\Phi}_k^{(1)}) \sim \mathbf{G}_{\mathbf{H}^{(1)}}$. We have (remember that the number of families is non-increasing a.s.)

$$\begin{aligned}\mathbf{E}\left[\sup_{k\in\mathbb{Z}} \boldsymbol{\Phi}_k^{(1)}(\mathbb{A})\right] &= \mathbf{E}\left[\lim_{k\to-\infty} \boldsymbol{\Phi}_k^{(1)}(\mathbb{A})\right] = \lim_{k\to-\infty} \mathbf{E}\left[\boldsymbol{\Phi}_k^{(1)}(\mathbb{A})\right] \\ &\leq \sup_{k\in\mathbb{Z}} \mathbf{E}\left[\Phi_k^{(1)}(A)\right] = \sup_{k\in\mathbb{Z}} \nu_k(A) < +\infty\end{aligned}$$

by (c), proving that $(\boldsymbol{\Phi}_k^{(1)})$ has finitely many clans.

2. Assume that (Φ_k) is not regular. Then we find a sub-process which, conditioned to non-emptyness, is single-line, with the sequence of intensity measures being majorized by $(\Lambda_{\mathbf{H}_k})$. Hence we may assume for simplicity that (Φ_k)

itself is single-line. We are going to show that in this case $(\Lambda_{\mathbf{H}_k})$ fulfils (c), provided the intensity kernel J satisfies the given finiteness condition.

Choose some $m \in \mathbb{Z}$ and some bounded set B with $\Lambda_{\mathbf{H}_m}(B) > 0$. Now we recall from the proof of Theorem 6.3 that for the realizations (Φ_k) there are a.s. two possibilities: Either we can identify a distinguished sequence of individuals' positions $(a_{m-n})_{n\geq 0}$ in the sequence of counting measures $(\Phi_{m-n})_{n\geq 0}$ such that it forms eventually an ancestral line, where we find the individual by looking for the one in Φ_{m-n} with the property that $\kappa^{[n]}_{(a_{m-n})}(\Psi(B) > 0)$ reaches its maximum. Or we have $\lim_{n\to\infty} \kappa^{[n]}_{(\Phi_{m-n})}(\Psi(B) > 0) = 0$, which implies $\Phi_m(B) = 0$ and even $\kappa^{[n]}_{(\Phi_{m-n})}(\Psi(B) > 0) = 0$ for each $n \geq 0$. Let us denote by $\Phi_{m,n}(B)$ the number of individuals in Φ_m located in B which have the property that their ancestor at time $m-n$ was located at a_{m-n},and define $p_{m,n} := J^{[n]}(a_{m-n}, B)$ or, in the second case, we put $\Phi_{m,n}(B) = 0$ and $p_{m,n} := 0$. Since by assumption all ancestral lines coalesce, and since B is bounded, we get that $\Phi_{m,n}(B) \to \Phi_m(B)$ as $n \to \infty$. Hence we get, using $\Phi_{m,n}(B) \leq \Phi_m(B)$,

$$\begin{aligned} \Lambda_{\mathbf{H}_m}(B) &= \lim_{n\to\infty} \mathbf{E}\Phi_{m,n}(B) \\ &= \lim_{n\to\infty} \mathbf{E}\left[\mathbf{E}\left[\Phi_{m,n}(B)|\Phi_{m-n}\right]\right] \\ &= \lim_{n\to\infty} \mathbf{E}p_{m,n} \\ &\leq \sup_{n\geq 1, a\in A} J^{[n]}(a, A). \end{aligned}$$

Hence $\Lambda_{\mathbf{H}_m}(B)$ is uniformly bounded which proves (c). □

In the second part of Proposition 6.5 the finiteness condition concerning J cannot be omitted. There are (even stationary) non-regular κ-processes of first order with

$$\kappa^{[n]}_{(a)}(\Psi(A) < \infty) = 1 \qquad \text{for each } a \in A$$

and

$$\sup_{n\geq 1, a\in A} J^{[n]}(a, B) < \infty \qquad \text{for each } B \in \mathcal{B},$$

for which there exists no sequence of measures $(\nu_k)_{k\in\mathbb{Z}}$ fulfilling (a), (b) and (c).

Example 6.6. Let $A := \{0, 1, 2, \ldots\}$ equipped with the metric $\rho_A(a, a')$ $:= |a - a'|$, and let

$$\kappa_a := \boldsymbol{\delta}_o \quad \text{for} \quad a \geq 1 \quad \text{and} \quad \kappa_0 := \int U(d\xi)\boldsymbol{\delta}_{\delta_0 + \ldots + \delta_\xi},$$

where U is a probability distribution on A with an infinite expectation.

We have $\kappa_a(\Psi(A) = 0) = 1$ for each $a \geq 1$ and

$$\kappa_0(\Psi(A) < \infty) = \mathrm{Prob}(\xi < \infty) = 1.$$

Moreover, we have $J^{[n]}(a, B) = 0$ for each $n \geq 1$ and $a \geq 1$, whereas

$$J^{[n]}(0, B) = \mathbf{E}\left[(\boldsymbol{\delta}_0 + \ldots + \boldsymbol{\delta}_\xi)(B)\right] \leq \#B \quad \text{for each} \quad n \geq 1 \quad \text{and} \quad B \in \mathcal{B},$$

and consequently

$$\sup_n \geq 1, a \in A J^{[n]}(a, B) \leq \#B < \infty \quad \text{for each} \quad B \in \mathcal{B}.$$

Obviously the probability distribution $P := \kappa_0$ is an equilibrium for κ. It defines a stationary single-line (and hence completely non-regular) κ-process of first order. Assuming the existence of a sequence of measures $(\nu_k)_{k\in\mathbb{Z}}$ fulfilling (a), (b) and (c), we get

$$\nu_{k+1}(A) = \sum_{a\in A} \nu_k(\{a\}) J(a, A) = \nu_k(\{0\}) J(0, A) = \nu_k(\{0\}) \cdot \infty,$$

which contradicts (c), unless $\nu_k(\{0\}) = 0$ for each k. But this also contradicts (c), because then we would have $\nu_k(A) = 0$ for each k.

7. A structural result for general κ-processes. Based on the results of the preceding sections we give a general structural result for κ-processes stating that any such process is a mixture of extremal processes being convolutions (i.e. independent superpositions) of a regular Poisson type component with an at most countable convolution of extremal single-line processes.

Theorem 7.1. *For any κ-process $(\Phi_k) \sim \mathbf{H}$ we have the representation*

$$\mathbf{H} = \int \mathbf{H}^o(d(\Phi_k^o)) \underline{K}((\Phi_{k,\mathrm{reg}}), (\cdot)) * \mathop{\ast}_{\text{clans } (\chi_k) \text{ in } (\Phi_{k,\mathrm{cnreg}})} \underline{K}((\chi_k), (\cdot))$$

of $\mathbf{H}$ as mixture of convolutions of left-tail trivial regular κ-processes $\underline{K}((\Phi_{k,\mathrm{reg}}), (\cdot))$ of Poisson type with left-tail trivial single-line κ-processes $\underline{K}((\chi_k), (\cdot))$.

The proof of this theorem is based on some preparing results.

Let, for a $\mathcal{F}^o_{-\infty}$-measurable subset C of $(\mathsf{M}^o)^{\mathbb{Z}}$ and for a κ-process $\mathbf{H}$, $\mathbf{H}_{\mathsf{C}}$ be the distribution of the sub-process consisting of all clans belonging to C (observe that this makes sense also for clans terminating to the right).

Assume that a countable partition $\{\mathsf{C}_n\}_{n=0}^{\infty}$, of $(\mathsf{M}^o)^{\mathbb{Z}}$ into $\mathcal{F}^o_{-\infty}$-measurable subsets is given. Then any κ-process has a representation as mixture of convolutions of left-tail trivial processes with clans in C_n, $n = 0, 1, 2 \ldots$.

Lemma 7.2. *Let* $(\Phi_k) \sim \mathbf{H}$ *be a* κ*-process. The we have*

$$\mathbf{H} = \int \mathbf{H}^o(d(\Phi^o_k)) \mathop{\ast}_{n\geq 0} \underline{K}((\Phi_{\mathsf{C}_n;k}), (\cdot)),$$

where

$$(\Phi_{\mathsf{C}_n;k}) := \sum_{\text{clans } (\chi^o_k) \text{ in } \mathsf{C}_n} (\chi_k) \sim \mathbf{H}_{\mathsf{C}_n}$$

is the sub-process consisting of all clans belonging to C_n *and* $\underline{K}((\Phi_{\mathsf{C}_n;k}), (\cdot))$ *the corresponding left-tail trivial component.*

Proof. We have to prove that $\mathbf{H}^o$-a.s.

$$\underline{K}((\Phi_k), (\cdot)) = \mathop{\ast}_{n\geq 0} \underline{K}((\Phi_{\mathsf{C}_n;k}), (\cdot))$$

is valid. Let two integers $n_1 < n_2$ and a collection $\{\mathsf{D}_n\}_{n=0}^{\infty}$ of sets in $\mathcal{M}^{\otimes\{n_1,n_1+1,\ldots,n_2\}}$ be given. By 2.2 and the definition of the kernel $\underline{K}$ we have $\mathbf{H}^o$-a.s for each $j \geq 0$

$$\begin{aligned}
&\underline{K^o}\left((\Phi^o_k), (\Psi_{\mathsf{C}_n;k}) \in \mathsf{D}_n, n \geq 0\right) \\
=\ &\mathbf{H}^o((\Psi_{\mathsf{C}_n;k}) \in \mathsf{D}_n, n \geq 0 | \mathcal{F}^o_{-\infty})((\Phi^o_k)) \\
=\ &\lim_{m\to\infty} \mathbf{H}^o((\Psi_{\mathsf{C}_n;k}) \in \mathsf{D}_n, n \geq 0 | \mathcal{F}^o_{-m})((\Phi^o_k)) \\
=\ &\lim_{m\to\infty} \mathbf{H}^o\left((\Psi_{\mathsf{C}_n;k}) \in \mathsf{D}_n, n \geq 0, n \neq j | \mathcal{F}^o_{-m}\right)((\Phi^o_k)) \\
&\cdot \mathbf{H}^o\Big((\Psi_{\mathsf{C}_j;k}) \in \mathsf{D}_j | \mathcal{F}^o_{-m})((\Phi^o_k)\Big) \\
=\ &\lim_{m\to\infty} \mathbf{H}^o\left((\Psi_{\mathsf{C}_n;k}) \in \mathsf{D}_n, n \geq 0, n \neq j | \mathcal{F}^o_{-m}\right)((\Phi^o_k)) \\
&\cdot \lim_{m\to\infty} \mathbf{H}^o\Big((\Psi_{\mathsf{C}_j;k}) \in \mathsf{D}_j | \mathcal{F}^o_{-m}\Big)((\Phi^o_k)) \\
=\ &\lim_{m\to\infty} \mathbf{H}^o_{(\mathsf{M}^o)^{\mathbb{Z}}\setminus\mathsf{C}_j}\left((\Psi_{\mathsf{C}_n;k}) \in \mathsf{D}_n, n \geq 0, n \neq j | \mathcal{F}^o_{-m}\right)((\Phi^o_{(\mathsf{M}^o)^{\mathbb{Z}}\setminus\mathsf{C}_j;k}))
\end{aligned}$$

$$\cdot \lim_{m\to\infty} \mathbf{H}^o_{\mathsf{C}_j} \left((\Psi_{\mathsf{C}_j;k}) \in \mathsf{D}_j | \mathcal{F}^o_{-m} \right) ((\Phi^o_{\mathsf{C}_j;k}))$$

$$= \mathbf{H}^o_{(\mathsf{M}^o)^{\mathbb{Z}}\setminus \mathsf{C}_j} ((\Psi_{\mathsf{C}_n;k}) \in \mathsf{D}_n, n \geq 0, n \neq j | \mathcal{F}^o_{-\infty}) ((\Phi^o_{(\mathsf{M}^o)^{\mathbb{Z}}\setminus \mathsf{C}_j;k}))$$

$$\cdot \mathbf{H}^o_{\mathsf{C}_j} \left((\Psi_{\mathsf{C}_j;k}) \in \mathsf{D}_j | \mathcal{F}^o_{-\infty} \right) ((\Phi^o_{\mathsf{C}_j;k}))$$

$$= \underline{K^o} \left((\Phi^o_{(\mathsf{M}^o)^{\mathbb{Z}}\setminus \mathsf{C}_j;k}), (\Psi_{\mathsf{C}_n;k}) \in \mathsf{D}_n, n \geq 0, n \neq j \right)$$

$$\cdot \underline{K^o} \left((\Phi^o_{\mathsf{C}_j;k}), (\Psi_{\mathsf{C}_j;k}) \in \mathsf{D}_j \right).$$

In view of the fact that $\mathcal{M}^{\otimes \mathbb{Z}}$ is countably generated by sets in $\bigcup_{n_1<n_2} \mathcal{M}^{\otimes\{n_1,n_1+1,\ldots,n_2\}}$, this proves that a.s. with respect to $\underline{K^o}((\Phi^o_k),(\cdot))$ the sequence $\{(\Psi_{\mathsf{C}_n;k})\}_{n\geq 0}$ is completely independent and the distribution of $(\Psi^o_{\mathsf{C}_j;k})$ is $\underline{K^o}((\Phi^o_{\mathsf{C}_j;k}),(\cdot))$. From the relation $\sum_{n\geq 0} (\Psi_{\mathsf{C}_n;k}) = (\Psi_k)$ we finally deduce the desired result. □

As a first application of the preceding lemma we deduce from Corollary 4.5

Corollary 7.3. *For any κ-process $(\Phi_k) \sim \mathbf{H}$ we have the representation*

$$\mathbf{H} = \int \mathbf{H}^o(d(\Phi^o_k)) \underline{K}((\Phi_{k,\mathrm{reg}}),(\cdot)) * \underline{K}((\Phi_{k,\mathrm{cnreg}}),(\cdot))$$

of $\mathbf{H}$ as mixture of convolutions of left-tail trivial regular κ-processes $\underline{K}((\Phi_{k,\mathrm{reg}}),(\cdot))$ which are of Poisson type with left-tail trivial completely non-regular κ-processes $\underline{K}((\Phi_{k,\mathrm{cnreg}}),(\cdot))$.

Corollary 7.4. *For any left-tail trivial κ-process $(\Phi_k) \sim \mathbf{H}$ the distribution $\mathbf{H}$ is the convolution $\mathbf{H}_{\mathrm{reg}} * \mathbf{H}_{\mathrm{cnreg}}$ of the distributions of its regular (and hence Poisson type) and completely non-regular parts.*

Proof of Theorem 7.1.

We have to analyze the structure of the completely non-regular components $\underline{K}((\Phi_{k,\mathrm{cnreg}}),(\cdot))$. Let $\{B_n\}_{n\geq 0}$ be an ascending sequence of sets in $\mathcal{B}$ such that each bounded B is covered by some B_n. We define

$$\mathsf{C}_0 \ : \ = \{(\Phi^o_k) \in (\mathsf{M}^o)^{\mathbb{Z}} : \lim_{m'\to-\infty} \kappa^{[m']}_{(\Phi^o_{m-m'})}(\Psi(B_l)>0) = 0 \ \text{ for all } l \geq 0, m \in \mathbb{Z}\},$$

$$\widetilde{\mathsf{C}}_0 \ : \ = \emptyset$$

$$\widetilde{\mathsf{C}}_n \quad : \quad = \{(\Phi_k^o) \in (\mathsf{M}^o)^{\mathbb{Z}} : \lim_{m' \to -\infty} \sup_{-n \le m \le n} \kappa^{[m']}_{(\Phi^o_{m-m'})}(\Psi(B_n) > 0) \ge 1/n\},\ n \ge 0,$$

$$\mathsf{C}_n \quad : \quad = \widetilde{\mathsf{C}}_n \backslash \widetilde{\mathsf{C}}_{n-1},\ n \ge 1.$$

By Lemma 7.2 we have the representation

$$\text{(o)} \qquad \mathbf{H} = \int \mathbf{H}^o(d(\Phi_k^o)) \mathop{\ast}_{n \ge 0} \underline{K}((\Phi_{\mathsf{C}_n;k}), (\cdot)),$$

where the κ-process $\underline{K}((\Phi_{\mathsf{C}_0;k}), (\cdot))$ is a.s. regular and left-tail trivial, whereas for each $n \ge 1$, the κ-process $\underline{K}((\Phi_{\mathsf{C}_n;k}), (\cdot))$ is a.s. the distribution of a left-tail trivial completely non-regular κ-process with all of its clans belonging to C_n a.s. Just as in the third step of the proof of Theorem 3.4 we see that a.s. $\underline{K}((\Phi_{\mathsf{C}_n;k}), (\cdot))$ is the distribution of a κ-process with finitely many clans, where the number of clans is given by the left-tail measurable quantity $r = r((\Phi^o_{\mathsf{C}_n;k}))$. We have a.s. for each $n \ge 1$, arbitrary integers $n_1 < n_2$ and an arbitrary D in $(\mathcal{M}^o)^{\otimes\{n_1, n_1+1, \ldots, n_2\}}$

$$\begin{aligned}
&\underline{K^o}\left((\Phi^o_{\mathsf{C}_n;k}), (\Psi_k^o) \in \mathsf{D}\right) \\
= \ & \mathbf{H}^o_{\mathsf{C}_n}\left((\Psi_k^o) \in \mathsf{D} | \mathcal{F}^o_{-\infty}\right)\left((\Phi^o_{\mathsf{C}_n;k})\right) \\
= \ & \lim_{m \to -\infty} \mathbf{H}^o_{\mathsf{C}_n}\left((\Psi_k^o) \in \mathsf{D} | \mathcal{F}^o_{m}\right)\left((\Phi^o_{\mathsf{C}_n;k})\right) \\
= \ & \lim_{m \to -\infty} \left(\mathop{\ast}_{\text{clans } (\chi_k^o) \text{ in } \mathsf{C}_n} \mathbf{U}^o_{(\chi_m^o)} \right) \left((\Psi^o_{n_1}, \ldots, \Psi^o_{n_2}) \in \mathsf{D}\right)
\end{aligned}$$

where in the last row the convolution extends over all clans in (Φ_k^o) belonging to C_n and $\mathbf{U}^o_{(\chi_m^o)}$ denotes the Markov chain starting at time m a κ^o-branching evolution, the starting configuration being given by χ_m^o. This is an a.s. finite convolution, and by the method used in the proof of Proposition 6.2 we find that the individual convolution factor tends weakly towards $\underline{K^o}((\chi_k^o), (\cdot))$. Hence we may continue the chain of equations by

$$\begin{aligned}
&\underline{K^o}\left((\Phi^o_{\mathsf{C}_n;k}), (\Psi_k^o) \in \mathsf{D}\right) \\
= \ & \left(\mathop{\ast}_{\text{clans } (\chi_k^o) \text{ in } \mathsf{C}_n} \underline{K^o}((\chi_k^o), (\cdot)) \right) \left((\Psi^o_{n_1}, \ldots, \Psi^o_{n_2}) \in \mathsf{D}\right).
\end{aligned}$$

So we have almost surely for each n

$$\underline{K^o}\Big((\Phi^o_{\mathsf{C}_n;k}),(\cdot)\Big)$$

$$= \mathop{\ast}_{\text{clans }(\chi^o_k)\text{ in }\mathsf{C}_n} \underline{K^o}\left((\chi^o_k),(\cdot)\right).$$

Observe that the totality of all clans of (Φ_k) belonging to all C_n forms the completely non-regular part $(\Phi_{k,\mathrm{cnreg}})$. So the last equation in connection with (o) and Corollary 7.3 leads us to the desired structural assertion. □

We conclude with a counterpart to Corollary 6.4, which revealed the nature of left-tail trivial regular processes as Poisson type processes.

Proposition 7.5. *A left-tail trivial κ-process $(\Phi_k) \sim \mathbf{H}$ is completely non-regular iff $\mathbf{H}$ is the convolution $\mathop{\ast}\limits_{i\in I} \mathbf{H}_i$ of an at most countable collection of left-tail trivial single-line κ-processes.*

P r o o f. 1. Assume that $\mathbf{H}$ is completely non-regular and left-tail trivial. By Theorem we may represent $\mathbf{H}$ as a mixture of κ-processes and since $\mathbf{H}$ is assumed to be extremal, this mixture is trivial in the sense that almost all components in this representation coincide with $\mathbf{H}$. So we find some sequence $(\Phi^o_k)_{k\in\mathbb{Z}}$ with (observe that the regular part of $(\Phi^o_k)_{k\in\mathbb{Z}}$ is $\mathbf{H}^o$-almost surely empty)

$$\mathbf{H} = \mathop{\ast}_{\text{clans }(\chi_k)\text{ in }(\Phi_{k,\mathrm{cnreg}})} \underline{K}\left((\chi_k),(\cdot)\right),$$

representing $\mathbf{H}$ as an at most countable convolution of left-tail trivial single-line κ-processes.

2. Assume there is a representation $\mathbf{H} = \mathop{\ast}\limits_{i\in I} \mathbf{H}_i$ as convolution of an at most countable collection of left-tail trivial single-line κ-processes. Then $\mathbf{H}$ is completely non-regular since a.s. each clan (as realization of some of the single-line processes $\mathbf{H}_i$) belongs to the completely non-regular part. $\mathbf{H}$ is left-tail-trivial as an at most countable convolution of left-tail trivial processes (see [2], Proposition 4.1.9.) □

8. Feeding sequences of regions. We defined in Section 5 the notion of a κ-process with finitely many clans by referring to the corresponding family process. In [6], Theorem 6.1, for the case of stationary κ-processes this notion could be characterized by properties of the original κ-process. This characterization remains valid in the non-stationary case as well. The following theorem even gives a characterization by almost sure properties of the realizations. It turns out, that there exists a sequence of regions in the phase space eventually comprising exactly as many individuals as there are clans and which asymptotically feed the whole population:

Theorem 8.1. *For any κ-process with finitely many clans, there is a sequence $S_{-1}, S_{-2}, \ldots \in \mathcal{A}$, such that we have almost surely*

$$\Phi_{-n}(S_{-n}) \underset{n\to\infty}{\longrightarrow} r((\mathbf{\Phi}_k))$$

and

$$\kappa^{[l+n]}_{(\Phi_{-n}((\cdot)\backslash S_{-n}))}(\Psi(C) > 0) \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{for each } l \in \mathbb{Z} \quad \text{and } C \in \mathcal{B}.$$

Vice versa, if there is, for a κ-process $(\Phi_k) \sim \mathbf{H}$, a sequence $S_{-1}, S_{-2}, \ldots \in \mathcal{A}$, with the sequence

$$\mathbf{H}(\Phi_{-n}(S_{-n}) \in (\cdot)), \quad n = 1, 2, \ldots$$

being weakly compact, and the following convergence relation being valid in probability

$$\kappa^{[l+n]}_{(\Phi_{-n}((\cdot)\backslash S_{-n}))}(\Psi(C) > 0) \underset{n\to\infty}{\longrightarrow} \qquad \text{for each } l \in \mathbb{Z} \quad \text{and } C \in \mathcal{B},$$

then $(\Phi_k) \sim \mathbf{H}$ has finitely many clans.

P r o o f. 1. Let us assume that $(\Phi_k) \sim \mathbf{H}$ is a single-line κ-process. Choose an increasing sequence $(B_k)_{k=1,2,\ldots}$ of sets in $\mathcal{B}$ such that for each bounded B there is some $B_k \supseteq B$. Fix some number k and consider all individuals of all populations Φ_l, $-k \le l \le k$, which are located within B_k. These are finitely many individuals, so there is a random time N_k with all these individuals having a common ancestor in Φ_{-N_k}.

So for each k we see as in the proof of Theorem 6.3 that there is a.s. eventually at most one individual (say at position a) in Φ_{-n} with

$$\sup_{-k\le l\le k} \kappa^{[l+n]}_{(a)}(\Psi(B_k) > 0) \ge 2^{-k},$$

and if we choose any ancestral line (a_{-n}), then we even eventually have

$$\sup_{-k\le l\le k} \kappa^{[l+n]}_{(\Phi_{-n}-\delta_{a_{-n}})}(\Psi(B_k) > 0) < 2^{-k}.$$

Let us denote by $S_{k,n}$ the set $\{a \in A : \sup_{-k\le l\le k} \kappa^{[l+n]}_{(a)}(\Psi(B_k) > 0) \ge 2^{-k}\}$. (Observe that, for fixed n, the sequence of sets $S_{k,n}$ is non-decreasing.) Hence there exists a number $N(k)$ such that

$$\mathbf{H}\left(\Phi_{-n}(S_{k,n}) > 1 \text{ for at least one } n \ge N(k)\right) < 2^{-k}$$

as well as

$$\mathbf{H}\left(\begin{array}{c}\sup_{-k\le l\le k}\kappa^{[l+n]}_{(\Phi_{-n}((\cdot)\setminus S_{k,n}))}(\Psi(B_k)>0)\ge 2^{-k+1}\\ \textit{for at least one } n\ge N(k)\end{array}\right)<2^{-k}.$$

(Here in the last line we had to take into account the possibility that the common ancestor at time $-n$ belongs to the complement of $S_{k,n}$.)

Of course we may choose the function $N(k)$ to increase in k. In this case for sufficiently large n there is a unique $k(n)$ with $N(k(n))\le n<N(k(n)+1)$. Put $S_{-n}:=S_{k(n),n}$. Almost surely, the ancestral line (a_{-n}) of any individual in $(\Phi_k)_{k\in\mathbb{Z}}$ has the property $a_{-n}\in S_{-n}$ for n being large enough. In fact, due to our assumption (a single-line process is a.s. non-empty) there is one k and one l, $-k\le l\le k$ with $\Phi_l(B_k)>0$. Hence we have, conditioned on this event almost surely, for these k,l, that $s:=\lim\limits_{n\to\infty}\kappa^{[l+n]}_{(a_{-n})}(\Psi(B_k)>0)>0$. So for $k'\ge k$ being chosen in such a way that $2^{-k'}<\frac{s}{2}$ and sufficiently large $n\ge N(k')$ we have $k(n)\ge k'$ and consequently

$$\kappa^{[l+n]}_{(a_{-n})}\left(\Psi(B_{k(n)})>0\right)\ge\kappa^{[l+n]}_{(a_{-n})}(\Psi(B_{k'})>0)\ge\frac{s}{2}>2^{-k'}\ge 2^{-k(n)}$$

proving that $a_{-n}\in S_{k(n),n}=S_{-n}$.

On the other hand we have

$$\begin{aligned}&\mathbf{H}\left(\Phi_{-n}(S_{-n})>1\ \textit{infinitely often}\right)\\ \le\ &\mathbf{H}\left(\begin{array}{c}\textit{for infinitely many } k\ \textit{ we have } \Phi_{-n}(S_{k,n})>1\\ \textit{for at least one } n\ge N(k)\end{array}\right)=0\end{aligned}$$

in view of the Borel-Cantelli lemma. This proves that $\Phi_{-n}(S_{-n})=1$ almost surely for sufficiently large n.

Now choose some $l\in\mathbb{Z}$ and $C\in\mathcal{B}$. We have

$$\begin{aligned}&\mathbf{H}\left(\kappa^{[l+n]}_{(\Phi_{-n}((\cdot)\setminus S_{-n}))}(\Psi(C)>0)\ \textit{does not tend to zero as } n\to\infty\right)\\ \le\ &\mathbf{H}\left(\kappa^{[l+n]}_{(\Phi_{-n}((\cdot)\setminus S_{-n}))}(\Psi(C)>0)\ge 2^{-k(n)+1}\ \textit{infinitely often}\right)\\ \le\ &\mathbf{H}\left(\begin{array}{c}\textit{for infinitely many } k \textit{ we have}\\ \kappa^{[l+n]}_{(\Phi_{-n}((\cdot)\setminus S_{-n}))}(\Psi(C)>0)\ge\ 2^{-k+1}\\ \textit{for at least one } n\ge N(k)\end{array}\right)=0\end{aligned}$$

again in view of the Borel-Cantelli lemma.

This proves the first part of the theorem, since for a process with finitely many clans the typical clan has eventually exactly one individual in S_{-n} so that $\Phi_{-n}(S_{-n}) \underset{n\to\infty}{\longrightarrow} r((\mathbf{\Phi}_k))$ a.s.

2. To prove the second part of the theorem, observe that the existence of a sequence $S_{-1}, S_{-2}, \ldots \in \mathcal{A}$ with the given properties for the process (Φ_k) implies that the sequence $S^{\square}_{-1}, S^{\square}_{-2}, \ldots$ (where $S^{\square}_{-k} := \{\Phi \in \mathbb{A} : \Phi(S_{-k}) > 0\}, k = 1, 2, \ldots)$ obeys the same properties if the family process $(\mathbf{\Phi}_k) \sim \mathbf{G_H}$ is considered. So we may assume without any loss of the generality that κ describes a substochastic shift in A. We have to prove that the number of individuals $\Phi_k(A)$ is a.s. uniformly bounded. In fact, assume the opposite. Then there is some $\delta > 0$ such that to each $M > 0$ there exists a bounded set C and some $l \in \mathbb{Z}$ with $\mathbf{H}(\Phi_l(C) > M) > \delta$. It is easy to see that the second property of the sequence $S_{-1}, S_{-2}, \ldots$ implies that, for sufficiently large n, $\mathbf{H}(\Phi_{-n}(S_{-n}) > M) > \delta/2$. In fact, this property says that for large n the individuals of Φ_l lying in C are with a probability arbitrarily close to one descendants of those individuals of Φ_{-n} located in S_{-n}. Since κ describes a substochastic shift, the number of descendants is not greater than $\Phi_{-n}(S_{-n})$.

With M being arbitrary, the relation $\mathbf{H}(\Phi_{-n}(S_{-n}) > M) > \delta/2$ can be fulfilled for all sufficiently large n only at the cost of a contradiction to the tightness of $\{\mathbf{H}(\Phi_{-n}(S_{-n}) \in (\cdot))\}_{n=1,2,\ldots}$. $\square$

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