ON A CLASS OF D–CAUCHY FILTERS

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ABSTRACT. In a quasi-uniform space, firmly D-Cauchy filters are introduced and they role in constructing firm extensions is investigated.

Let $\mathcal{U}$ be a quasi-uniformity on a set $X$. A pair $(t, s)$ of filters in $X$ is said to be Cauchy [2, 1.1] iff, for any given entourage $U \in \mathcal{U}$, there are $T \in t$ and $S \in s$ such that $T \times S \subset U$. According to [4], a filter $s$ in $X$ is said to be $D$-Cauchy (Cauchy in [4]) iff it admits a cofilter, i.e. a (proper) filter $t$ such that $(t, s)$ is a Cauchy filter pair. Our purpose is to investigate a class of D-Cauchy filters and, in particular, its role in extension problems.

1. Preliminaries. For quasi-uniform spaces in general, see the monograph [5]. We denote by $U^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ the conjugate of $\mathcal{U}$ [5, p. 1] and by $\mathcal{U}^{tp}$ the topology induced by $\mathcal{U}$ [5, p. 3]; we write $U^{-1}^{tp}$ for $(U^{-1})^{tp}$. If $X \subset Y$, the quasi-uniformity $\mathcal{V}$ on $Y$ is said to be an extension of $\mathcal{U}$ iff the restriction

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\( \mathcal{V}|X = \{V \cap (X \times X) : V \in \mathcal{V}\} \) of \( \mathcal{V} \) to \( X \) equals \( \mathcal{U} \) and \( X \) is \( \mathcal{V}^{1p} \)-dense. The extension \( \mathcal{V} \) of \( \mathcal{U} \) is said to be firm \([2, \text{p. 50}]\) iff \( X \) is dense for \( \sup\{\mathcal{V}^{-1p}, \mathcal{V}^{1p}\} \).

For \( p \in Y - X \) let \( s^+(p) \) be a (proper) filter in \( X \). We say that the quasi-uniformity \( \mathcal{V} \) on \( Y \) is compatible with \((\mathcal{U}, s^+)\) iff \( \mathcal{V} \) is an extension of \( \mathcal{U} \) and \( s^+(p) \) is, for each \( p \in Y - X \), the trace on \( X \) of the \( \mathcal{V}^{1p} \)-neighbourhood filter of \( p \). If \( s^-(p) \) is, again for \( p \in Y - X \), a (proper) filter in \( X \), \( \mathcal{V} \) is said to be compatible with \((\mathcal{U}, s^-, s^+)\) iff it is compatible with \((\mathcal{U}, s^+)\) and \( \mathcal{V}^{-1} \) is compatible with \((\mathcal{U}^{-1}, s^-)\). It is well-known that, for the existence of \( \mathcal{V} \) compatible with \((\mathcal{U}, s^+)\), it is necessary that every filter \( s^+(p) \) should be \((\mathcal{U}^{-}) \) round \([1, 1]\) \([1, 1.1]\); a \( \mathcal{V} \) compatible with \((\mathcal{U}, s^-, s^+)\) exists iff the filter pairs \((s^-(p), s^+(p))\) are round and Cauchy, i.e. each \( s^+(p) \) is \( \mathcal{U} \)-round, each \( s^-(p) \) is \( \mathcal{U}^{-1} \)-round, and each pair \((s^-(p), s^+(p))\) is \( \mathcal{U} \)-Cauchy \([3, 6.1]\). Therefore we can easily deduce the following theorem, implicitly contained in \([3, 11.2]\):

**Theorem 1.1 (J. Deák).** A firm extension compatible with \((\mathcal{U}, s^-, s^+)\) exists iff the filter pairs \((s^-(p), s^+(p))\) are linked \([3, 7.1]\), round and Cauchy. If so then there is exactly one firm extension compatible with \((\mathcal{U}, s^-, s^+)\).

### 2. Firmly D-Cauchy filters.

In Theorem 1.1, we have to do with linked Cauchy filter pairs. Therefore it is natural to say that a \( D \)-Cauchy filter \( s \) is **firmly \( D \)-Cauchy** (in a quasi-uniform space \((X, \mathcal{U})\)) iff it admits a cofilter \( t \) such that \((t, s)\) is linked (i.e. \( T \cap S \neq \emptyset \) whenever \( T \in t, S \in s \)).

In order to characterize firmly \( D \)-Cauchy filters, let us recall that, if \( s \) is any filter in a quasi-uniform space \((X, \mathcal{U})\), there is a finest \( \mathcal{U} \)-round filter \( \mathcal{U}(s) \) coarser than \( s \), the \( \mathcal{U} \)-envelope of \( s \), composed of all sets \( U(S) \) where \( S \in s, U \in \mathcal{U} \) \([1, 4.6]\). In accordance with \([1]\) (but differently from \([4]\) or \([5]\)), let us say that the filter \( s \) is **Cauchy** in \((X, \mathcal{U})\) iff \((s, s)\) is a Cauchy filter pair, i.e. iff, for any \( U \in \mathcal{U} \), there is \( S \in s \) such that \( S \times S \subset U \).

**Lemma 2.1.** In a quasi-uniform space \((X, \mathcal{U})\) a filter \( s \) is firmly \( D \)-Cauchy iff there is a proper Cauchy filter \( \mathcal{V} \) such that \( \mathcal{U}(\mathcal{V}) \subset s \subset \mathcal{V} \).

**Proof.** If \( \mathcal{V} \) is a proper Cauchy filter then \((\mathcal{V}, \mathcal{U}(\mathcal{V}))\) is a Cauchy filter pair, hence clearly \((\mathcal{V}, \mathcal{U}(\mathcal{V}))\) and \((\mathcal{V}, s)\) are Cauchy provided \( \mathcal{U}(\mathcal{V}) \subset s \). Further, \( s \subset \mathcal{V} \) implies that \((\mathcal{V}, s)\) is linked.

Suppose on the other hand that \((t, s)\) is linked and Cauchy. Then \( \mathcal{V} = t(\cap)s = \{T \cap S : T \in t, S \in s\} \) is a proper Cauchy filter. For any \( U \in \mathcal{U} \) and
$R \in \mathfrak{r}$, let $T \in \mathfrak{t}$ and $S \in \mathfrak{s}$ satisfy $T \times S \subset U$ and choose $x \in T \cap S \cap R$. Now $S \subset U(x) \subset U(R)$ so that $U(R) \in \mathfrak{s}$, $U(\mathfrak{r}) \subset \mathfrak{s}$, and obviously $\mathfrak{s} \subset \mathfrak{r}$. □

**Corollary 2.2.** In a quasi-uniform space $(X, \mathcal{U})$, a filter $\mathfrak{s}$ is round and firmly $D$–Cauchy iff $\mathfrak{s} = \mathcal{U}(\mathfrak{r})$ for some proper Cauchy filter $\mathfrak{r}$.

**Proof.** Choose $\mathfrak{r}$ according to Lemma 2.1. By the roundness of $\mathfrak{s}$ we have $\mathfrak{s} \subset \mathcal{U}(\mathfrak{r})$. □

**Corollary 2.3.** A round filter pair $(\mathfrak{t}, \mathfrak{s})$ is linked and Cauchy iff there is a proper Cauchy filter $\mathfrak{r}$ such that $\mathfrak{t} = \mathcal{U}^{-1}(\mathfrak{r})$, $\mathfrak{s} = \mathcal{U}(\mathfrak{r})$.

**Proof.** If $(\mathfrak{t}, \mathfrak{s})$ is round, linked and Cauchy then $\mathfrak{r} = \mathfrak{t}(\cap)\mathfrak{s}$ is a proper Cauchy filter such that $\mathcal{U}(\mathfrak{r}) \subset \mathfrak{s} \subset \mathfrak{r}$. By Corollary 2.2, $\mathfrak{s} = \mathcal{U}(\mathfrak{r})$. Similarly $\mathfrak{t} = \mathcal{U}^{-1}(\mathfrak{r})$.

Conversely, for a proper Cauchy filter $\mathfrak{r}$, the filter pair $(\mathcal{U}^{-1}(\mathfrak{r}), \mathcal{U}(\mathfrak{r}))$ is clearly linked and Cauchy. □

We obtain easily some extension theoretical applications.

**Theorem 2.4.** Let $(X, \mathcal{U})$ be a quasi-uniform space, $X \subset Y$ and $\mathfrak{s}^{+}(p)$ be a filter in $X$ for $p \in Y - X$. A firm extension compatible with $(\mathcal{U}, \mathfrak{s}^{+})$ exists iff each $\mathfrak{s}^{+}(p)$ is round and firmly $D$–Cauchy.

**Proof.** The necessity follows from Theorem 1.1. If each $\mathfrak{s}^{+}(p)$ is round and firmly $D$–Cauchy, then, by Corollary 2.2, $\mathfrak{s}^{+}(p) = \mathcal{U}(\mathfrak{r}(p))$ for a proper Cauchy filter $\mathfrak{r}(p)$ (depending on $p$). Define $\mathfrak{s}^{-}(p) = \mathcal{U}^{-1}(\mathfrak{r}(p))$ to obtain by Corollary 2.3 a round, linked Cauchy pair $(\mathfrak{s}^{-}(p), \mathfrak{s}^{+}(p))$. Now Theorem 1.1 furnishes the existence of the firm extension looked for. □

The filter $\mathfrak{r}$ in Corollary 2.2 is not unique for a given $\mathfrak{s}$: take a (non-$T_{0}$) topology $\mathcal{T}$ on $X$ and points $x, y \in X$, $x \neq y$ such that $x$ and $y$ have the same neighbourhood filter $\mathfrak{s}$; for any quasi-uniformity $\mathcal{U}$ inducing $\mathcal{T}$, we have $\mathcal{U}(x) = \mathcal{U}(y) = \mathfrak{s}$.

However, if $\mathfrak{s}$ is a round and firmly $D$–Cauchy filter and $\mathfrak{s} = \mathcal{U}(\mathfrak{r})$ for a Cauchy filter $\mathfrak{r}$, then $\mathcal{U}^{-1}(\mathfrak{r})$ does not depend on the choice of $\mathfrak{r}$.

**Lemma 2.5.** In a quasi-uniform space $(X, \mathcal{U})$, let $\mathfrak{r}$ and $\mathfrak{r}'$ be proper Cauchy filters such that $\mathcal{U}(\mathfrak{r}) = \mathcal{U}(\mathfrak{r}')$. Then $\mathcal{U}^{-1}(\mathfrak{r}) = \mathcal{U}^{-1}(\mathfrak{r}')$.

**Proof.** For $U \in \mathcal{U}$, $R' \in \mathfrak{r}'$, let $U_1 \in \mathcal{U}$ and $R_1 \in \mathfrak{r}$ be such that $U_1^2 \subset U$, $R_1 \times R_1 \subset U_1$. Then $U_1(R_1) \in \mathcal{U}(\mathfrak{r}) = \mathcal{U}(\mathfrak{r}') \subset \mathfrak{r}'$, so there are $x \in U_1(R_1) \cap R'$ and $y \in R_1$ such that $(y, x) \in U_1$, i.e. $(x, y) \in U_1^{-1}$, $y \in U_1^{-1}(x)$, $R_1 \subset U_1^{-1}(y) \subset$
$U^{-1}(x) \subset U^{-1}(R')$. Thus $U^{-1}(R') \in \tau$ and $U^{-1}(r') \subset \tau$; the left hand side being $U^{-1}$–round, we also have $U^{-1}(r') \subset U^{-1}(r)$. Similarly $U^{-1}(r) \subset U^{-1}(r')$. □

**Corollary 2.6.** Let $(X, \mathcal{U})$ be a quasi-uniform space, $X \subset Y$ and, for $p \in Y - X$, $s^+(p)$ be a round, firmly $D$–Cauchy filter in $X$. Then there are uniquely determined filters $s^-(p)$ such that any firm extension $\mathcal{V}$ compatible with $(\mathcal{U}, s^+)$ is compatible with $(\mathcal{U}, s^-, s^+)$.  

**Proof.** By Theorem 1.1 if $\mathcal{V}$ is compatible with $(\mathcal{U}, s^-, s^+)$, then the pairs $(s^-(p), s^+(p))$ must be linked, round and Cauchy. By Corollary 2.3, there are (proper) Cauchy filters $r_p$ such that $s^-(p) = U^{-1}(r_p)$, $s^+(p) = U(r_p)$. By Lemma 2.5, $s^-(p)$ does not depend on the choice of $r_p$. □

**Corollary 2.7.** Under the hypotheses of Corollary 2.6 there is exactly one firm extension compatible with $(\mathcal{U}, s^+)$. 

**Proof.** Corollary 2.6 and Theorem 1.1. □

**REFERENCES**


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