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ISOMORPHISM PROBLEMS FOR THE BAIRE FUNCTION SPACES OF TOPOLOGICAL SPACES

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Dedicated to the memory of Professor D. Doitchinov

Abstract. Let a compact Hausdorff space X contain a non-empty perfect subset. If $\alpha < \beta$ and β is a countable ordinal, then the Banach space $B_\alpha(X)$ of all bounded real-valued functions of Baire class α on X is a proper subspace of the Banach space $B_\beta(X)$. In this paper it is shown that:

1. $B_\alpha(X)$ has a representation as $C(b_\alpha X)$, where $b_\alpha X$ is a compactification of the space PX – the underlying set of X in the Baire topology generated by the G_δ-sets in X.

2. If $1 \le \alpha < \beta \le \Omega$, where Ω is the first uncountable ordinal number, then $B_\alpha(X)$ is uncomplemented as a closed subspace of $B_\beta(X)$.

These assertions for $X = [0,1]$ were proved by W. G. Bade [4] and in the case when X contains an uncountable compact metrizable space – by F.K.Dashiell [9]. Our argumentation is one non-metrizable modification of both Bade's and Dashiell's methods.

1. Preliminary results and definitions. We consider only completely regular spaces. We shall use the notation and terminology from [11, 4, 17, 21].

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In particular, βX is the Stone-Čech compactification of the space X, νX is the Hewitt realcompactification of the space X, $w(X)$ is the weight of the space X, the cardinality of a set Y is denoted by $|Y|$, $\mathrm{cl}_X H$ or $\mathrm{cl}H$ denotes the closure of a set H in X, the symbol R will denote the topological field of real numbers, $N = \{1, 2, \ldots\}$ is a discrete subspace of the positive integers of R, $C(X)$ is the space of all continuous bounded functions on the space X.

A space is realcompact if it is homeomorphic to a closed subspace of a product of real lines R.

Let S be a set and $B(S)$ be the space of all real-valued bounded functions on S. The space $B(S)$ is a Banach space with the supremum norm

$$\|f\| = \sup\{|f(x)| : x \in S\}.$$

If $c \in R$, then $c_S \in B(S)$ with $c_S(x) = c$ for every $x \in S$.

Let $E \subseteq B(S)$. Then T_E is the topology on S generated by E and it has the subbase consisting of all sets of the form $f^{-1}(U)$, where $f \in E$ and U is an open subset of R. The space E separates the set S if for every pair of distinct points $x, y \in S$ there is $f \in E$ such that $f(x) \neq f(y)$. The space (S, T_E) is completely regular if and only if E separates the set S.

Let a subspace E of $B(S)$ separates the set S. Then the mapping $v_E : S \to R^E$, where $v_E(x) = (f(x) : f \in E)$, is an embedding of (S, T_E) in R^E. The closure $\beta_E S$ of the subspace $S = v_E(S)$ in R^E is a compactification of the space (S, T_E).

Let X be a dense subset of the spaces Y and Z. The symbol $Y\,{}_X{>}\,Z$ means that there exists a continuous mapping $g : Y \to Z$ such that $g(x) = x$ for all $x \in X$.

Property 1.1. *Let $F \subseteq E \subseteq B(S)$ and F separate the set S. Then $\beta_E S\,{}_S{>}\,\beta_F S$.*

Property 1.2. *Let $E \subseteq B(S)$ separate the set S. Then $\beta_E S$ is the smallest compactification of the space (S, T_E) such that there exists an extender $e_E : E \to C(\beta_E S)$ such that $e_E(f)|S = f$ for every $f \in E$.*

Let $\{f, f_n : n \in N\} \subseteq B(S)$. We have $u - \lim f_n = f$ if $\lim \|f - f_n\| = 0$ and $p - \lim f_n = f$ if $\lim f_n(x) = f(x)$ for each $x \in S$. If $A \subseteq B(S)$, then $[A]_u = \{f \in B(S) : f = u - \lim f_n$ for some sequence $\{f_n \in A : n \in N\}\}$ is the u-closure of A and $[A]_p = \{f : f = t{-}\lim f_n$ for some sequence $\{f_n \in A : n \in N\}\}$ is the p-closure of A in $B(S)$.

Let $E \subseteq B(S)$. Denote $p_0 E = E$ and $p_\alpha E = [\cup\{p_\beta E : \beta < \alpha\}]_p$ for all $\alpha \leq \Omega$. By construction $p_\Omega E = [p_\Omega E]_p = \cup\{p_\alpha E : \alpha < \Omega\}$. The set E is closed in $B(S)$ if and only if $E = [E]_u$.

Property 1.3. *The space E separates the set S if and only if $p_\Omega E$ separates the set S.*

For every $f \in B(S)$ we denote $Z(f) = f^{-1}(0)$ and $CZ(f) = S \setminus Z(f)$. If $E \subseteq B(S)$, then $Z(E) = \{Z(f) : f \in E\}$ and $CZ(E) = \{CZ(f) : f \in E\}$.

Fix a space X. Let $B_\alpha(X) = p_\alpha C(X)$ for all $\alpha \leq \Omega$. The functions in $B_\alpha(X)$ are called the Baire functions of class α on the space X. We put

$$\begin{aligned} Z_\alpha(X) &= Z(B_\alpha(X)), \\ CZ_\alpha(X) &= CZ(B_\alpha(X), \\ \text{and } A_\alpha(X) &= Z_\alpha(X) \cap CZ_\alpha(X). \end{aligned}$$

The class $Z_\alpha(X)$ (class $CZ_\alpha(X)$) is a multiplicative (additive) class α of the Baire sets of the space X. The sets in $A_\alpha(X)$ are called the sets of ambiguous or two-sided Baire sets of class α.

Fix a space X. Let PX be the set X with the topology generated by the G_δ-sets in X. The topology of the space PX is called the Baire topology of the space X. If $B_1(X) \subseteq E \subseteq B_\Omega(X)$, then $PX = (X, T_E)$. If $\alpha \leq \Omega$, then $Z_\alpha(X), CZ_{1+\alpha}(X), A_{1+\alpha}(X)$ are open bases of the space PX. Denote $b_\alpha X = \beta_{B_\alpha(X)} X$ for every $\alpha \leq \Omega$. The compact space $b_\alpha X$ is called the maximal ideal space of the α-th Baire class $B_\alpha(X)$.

Property 1.4. *For every $\alpha \leq \Omega$ there exists a unique isomorphism $e_\alpha : B_\alpha(X) \to C(b_\alpha X)$ such that $e_\alpha(f)|X = f$ for each $f \in B_\alpha(X)$.*

Property 1.5. *Let $0 \leq \alpha \leq \beta \leq \Omega$. Then there exists a unique continuous mapping $\pi^\beta_\alpha : b_\beta X \to b_\alpha X$ such that $\pi^\beta_\alpha(x) = x$ for every $x \in X$ and a unique canonical linear isometric embedding $e^\beta_\alpha : C(b_\alpha(X)) \to C(b_\beta(X))$ indused by the mapping π^β_α, i.e. $f = e^\beta_\alpha(e_\alpha(f)|X$ for all $f \in B_\alpha(X)$.*

Property 1.6. *If $\alpha > 0$, then:*

1. *$H \to \mathrm{cl}_{b_\alpha X} H$ defines a Boolean isomorphism of the field $A_\alpha(X)$ onto the field of clopen (closed and open) sets in $b_\alpha X$.*

2. *$\dim b_\alpha X = 0$, i.e. the compact $b_\alpha X$ is totally disconnected.*

2. Baire complemented Banach spaces. Let E be a Banach space. The space E is canonical embedded in the second dual E^{**} of E.

For every set $H \subseteq E^{**}$ denote by $w^*_1(H)$ the set of all limits in E^{**} of w^*-convergent sequences in H.

Denote $w^*_0 E = E \subseteq E^{**}$ and $w^*_\alpha E = w^*_1(\cup\{w^*_\beta E : \beta < \alpha\})$ for every $\alpha \leq \Omega$. By construction, $w^*_\Omega E = \cup\{w^*_\alpha E : \alpha < \Omega\}$. The space $w^*_\alpha E$ is called the α-Baire space for E (see [9, 15]).

The Banach space E is called α-Baire complemented if there exists a continuous linear projection from $w^*_\alpha E$ onto $E = w^*_0 E$. The space E is Baire complemented if E is a complemented subspace of the space $w^*_1 E$.

The following Properties were proved in [9, 15, 24].

Property 2.1. *$w^*_\alpha E$ is a closed subspace of the space E^{**} for every $\alpha \geq 0$.*

Property 2.2. *If E, F are isomorphic Banach spaces, $\alpha \geq 1$ and E is α-Baire complemented, then F is an α-Baire complemented space, too.*

Property 2.3. *If $\alpha \geq 1$, E is an α-Baire complemented Banach space and F is a complemented Banach subspace of E, then F is α-Baire complemented.*

Property 2.4. *If X is a compact space, then $B_\alpha(X) = w^*_\alpha C(X)$ for every $\alpha \geq 0$.*

Corollary 2.5. *$w^*_\alpha C(X) = B_\alpha(\beta X)$ for every space X and $\alpha \geq 0$.*

Corollary 2.6. *Let X be a space, $0 \leq \alpha \leq \Omega$ and $D_{\alpha+1}(X) = w^*_1 B_\alpha(X)$ be the first Baire space of the Banach space $B_\alpha(X)$. Then $D_{\alpha+1}(X) = B_1(b_\alpha X)$.*

Proposition 2.7. *Let X be a pseudocompact space. Then $w^*_\alpha C(X) = B_\alpha X$ for every $\alpha \leq \Omega$.*

P r o o f. In virtue of P. R. Meyer's theorem [15, Theorem 7], every $f \in B_\alpha(X)$ has a unique extension to an $m(f) \in B_\alpha(\nu X)$. There exists a unique one-to-one isometric linear mapping $m : B_\Omega(X) \to B_\Omega(\nu X)$ with:

1. $m(f)|X = f$ for every $f \in B_\Omega(X)$.
2. $m(f \cdot g) = m(f) \cdot m(g)$.
3. $\|m(f)\| \leq \|f\|$, m is a homeomorphism in the topologies of u-convergence and p-convergence. The space X is pseudocompact if and only if $\nu X = \beta X$. The Corollary 2.5 completes the proof. □

Example 2.8. Let X be an infinite discrete space. Then $B_\alpha(X) = C(X)$ for all $\alpha \leq \Omega$ and $D_1(X) \neq C(X)$. From Corollary 2.5 the spaces $D_1(X)$ and $B_1(\beta X)$ are isometrically isomorphic.

Example 2.9. Let X be an infinite scattered compact space [8, 22]. Recall that a space is scattered if its every non-empty subspace contains at least one isolated point. In this case $B_\alpha(X) = B_1(X) = D_1(X)$ for every $\alpha \geq 1$ and $D_2(X) \neq D_1(X)$(see [5, 15]).

Proposition 2.10. *Let X be a space and $\alpha \geq 1$. Then $B_\alpha(X) \subseteq D_{\alpha+1}(X)$ and there exists a linear continuous mapping $p : D_{\alpha+1}(X) \to B_{\alpha+1}(X)$ such that $p(f) = f$ for every $f \in B_\alpha(X)$ and $\|p(g)\| \leq \|g\|$ for every $g \in D_{\alpha+1}(X)$.*

P r o o f. In virtue of Corollary 2.6 we consider that $D_{\alpha+1}(X) = B_1(b_\alpha(X)$. The mapping p, defined by letting $p(f) = f|X$ for every $f \in B_1(b_\alpha X)$, has the required properties. □

Remark 2.11. For every limit ordinal α we put $D_\alpha(X) = B_\alpha(X)$.

Corollary 2.12. *Let X be a space and $0 \leq \alpha \leq \beta \leq \Omega + 1$. If $B_\alpha(X)$ is complemented in $B_\beta(X)$, then $B_\alpha(X)$ is complemented in $D_\beta(X)$, too.*

Corollary 2.13. *The space $B_\Omega(X)$ is complemented in $D_{\Omega+1}(X)$.*

Remark 2.14. For every α there exists a canonical embedding of the Banach space $D_\alpha(X)$ in $D_{\alpha+1}(X)$.

Question 2.15. *Let $D_1(X) = B_1(X)$. Is it true that X is a pseudocompact space?*

Question 2.16. *Let $0 \leq \alpha < \beta \leq \Omega$, β be not a limit ordinal and $D_\alpha(X)$ be complemented in $D_\beta(X)$. Is it true that $B_\alpha(X)$ is complemented in $B_\beta(X)$?*

3. The convergent sequences of the maximal ideal spaces. The following theorem answers a question of F. K. Dashiell [9].

Theorem 3.1. *Let $\alpha \geq 1$ and X be an infinite space. Then for every infinite closed subspace Y of $b_\alpha X$ the set $Y \setminus \nu X$ contains a copy of βN.*

P r o o f. In virtue of P. R. Meyer's theorem (see the Proof of Proposition 2.7.), it is sufficient to prove the theorem for a realcompact space $X = \nu X$. Then PX is a realcompact space, too. In the compactification $b_\alpha X$ of the space PX we have points of two types.

Type 1. $x \in X$.

In this case for every sequence $\{U_n : n \in N\}$ of neighbourhoods of the point x in $b_\alpha X$ there exists an open set U in $b_\alpha X$ such that $x \in U \subseteq \cap\{U_n : n \in N\}$, i.e. x is a P-point of the space $b_\alpha X$.

Type 2. $x \in b_\alpha X \setminus X$.

In this case there exists a sequence $\{W_n(x) : n \in N\}$ of clopen subsets of $b_\alpha X$ such that $x \in W(x) = \cap\{W_n(x) : n \in N\} \subseteq b_\alpha X \setminus X$ and $b_\alpha X = W_1(x)$, i.e. x is not a P-point of the space $b_\alpha X$.

Let Y be an infinite closed subspace of $b_\alpha X$ and $\alpha \geq 1$. In the P-space every compact subset is finite. Therefore there exists an accumulation point $y_0 \in Y \setminus X$ of Y. Fix a sequence $\{H_n : n \in N\}$ of clopen subsets of $b_\alpha X$ with:

1. $y_0 \in H_{n+1} \subset H_n \subset W_n(y_0)$ for every $n \in N$.
2. $Y_n = Y \cap (H_n \setminus H_{n+1} \neq \emptyset$ for every $n \in N$.
3. $X_n = X \cap (H_n \setminus H_{n+1} \neq \emptyset$ for all $n \in N$.
4. $H_1 = W_1(y_0) = b_\alpha X$.

Fix $z_n \in Y_n$ and $x_n \in X_n$. Denote $L = \{z_n : n \in N\}$. Then $Z = CL_Y L$ is a compactification of the discrete space L and $Z \subseteq Y \setminus X$.

Consider the continuous function $h : b_\alpha X \to R$, where $h^{-1}(0) = \cap\{H_n : n \in\} = H$ and $h^{-1}(n^{-1}) = H_n \setminus H_{n+1}$ for each $n \in N$. By construction, $X = \cup\{V_n : n \in N\}$. In virtue of Property 1.6, we have $g = h|X \in B_\alpha(X)$. Let M be a subset of L. We put $N(M) = \{n \in N : y_n \in M\}$. Then $V(M) = \cup\{X_n : n \in N(M)\} = g^{-1}(\{n^{-1} : n \in N(M)\}) \in A_\alpha(X)$ and $W(M) = X \setminus V(M) = g^{-1}\{n^{-1} : n \in N \setminus N(M)\} \in A_\alpha(X)$. From Property 1.6, $\mathrm{cl}V(M)$ and $\mathrm{cl}W(M)$ are clopen subsets of $b_\alpha X$, $\mathrm{cl}V(M) \cap \mathrm{cl}V(M) = \emptyset$ and $\mathrm{cl}_Z M = Z \cap \mathrm{cl}V(M)$. Hence $\mathrm{cl}_Z M \cap \mathrm{cl}_Z(L \setminus M) = \emptyset$ and the spaces Z and $\beta L = \beta N$ are homeomorphic. □

Corollary 3.2. *Let $\alpha \geq 1$ and X be an infinite space. Then $|Y| \geq 2^c$ for every infinite closed subspace Y of $b_\alpha X$, where c is the cardinal number assigned to the set of all real numbers.*

Corollary 3.3. *Let $\alpha \geq 1$ and X be an infinite space. Then the maximal ideal space $b_\alpha X$ does not contain non-trivial convergent sequences.*

4. On Baire separated sets. A subset A of a space X is called a D-set if there exist a separable metric space Y and a continuous mapping $f : X \to Y$ such that $A = f^{-1}(f(A))$. Every Baire set is a D-set.

Lemma 4.1. *Let $\{H_n : n \in N\}$ be a sequence of D-sets of a space X. Then there exist a separable metric space Y and a continuous mapping $f : X \to Y$ such that $H_n = f^{-1}(f(H_n))$ for every $n \in N$. Moreover, if $H_n \in Z_\alpha(X)$ or $H_n \in CZ_\alpha(X)$, then $f(H_n) \in Z_\alpha(Y)$ or $f(H_n) \in CZ_\alpha(Y)$ respectively.*

P r o o f. For every $n \in N$ fix a separable space Y_n and a continuous mapping $f_n : X \to Y_n$ such that $H_n = f^{-1}(f(H_n))$. Let $f : X \to Y = f(X) \subseteq \prod\{Y_n : n \in N\}$ be the diagonal product of mappings $\{f_n : n \in N\}$, where $f(x) = (f_n(x) : n \in N)$ for all $x \in X$. The mapping f has the required properties. □

Definition 4.2. *Two subsets A and B of a space X are called α-Baire separated if there exists a set $L \in A_\alpha(X)$ such that $A \subseteq L \subseteq X \setminus B$.*

Theorem 4.3. *Let $\alpha \geq 1$, $f : X \to Y$ be a continuous mapping of a pseudocompact space X onto a space Y and A, B be disjoint D-sets of Y. The sets A and B are α-Baire separated in Y if and only if the sets $A_1 = f^{-1}(A)$ and $B_1 = f^{-1}(B)$ are α-Baire separated in X.*

P r o o f. It is obvious that A_1 and B_1 are D-sets in X. If $L \in A_\alpha(Y)$, $L_1 = f^{-1}(L)$ and $A \subseteq L \subseteq Y \setminus B$, then $L_1 \in A_\alpha(X)$ and $A_1 \subseteq L_1 \subseteq X \setminus B_1$.

Now assume that $H \in A_\alpha(X)$ and $A_1 \subseteq H \subseteq X \setminus B_1$.

Case 1. X is a compact metric space and $\alpha \geq \omega$.

In this case there exists a mapping $g : Y \to X$ such that $g(y) \in f^{-1}(y)$ for every $y \in Y$ and $g^{-1}(U)$ is a F_σ-set of Y for every open subset U of X. In this case $L = g^{-1}(H) \in A_\alpha(Y)$ and $A \subseteq L \subseteq Y \setminus B$.

Case 2. X is a compact metric space.

We consider the function $h : X \to [0, 1]$ for which $H = f^{-1}(0)$ and $X \setminus H = f^{-1}(1)$. By J. Saint Raimond's Lemma [20, Lemma 3] there exists a mapping $g : Y \to X$ such that $g(y) \in f^{-1}(y)$ for every $y \in Y$ and $\varphi = h \cdot g \in B_\alpha(Y)$. Then $\varphi^{-1}(0) \in A_\alpha(Y)$ and $A \subseteq \varpi - 1(0) \subseteq Y \setminus B$.

Case 3. X is a pseudocpompact space.

There are the separable metric spaces Z, S_1 and continuous mappings $g : Y \to Z, h_1 : X \to S_1$ such that $Z = g(Y)$, $A = g^{-1}(g(A))$, $B = g^{-1}(g(B))$, $H = h_1^{-1}(h_1(H))$ and $h_1(H) \in A_\alpha(S_1)$. Consider the mapping $h : X \to S = h(X) \subseteq Z \times S_1$, where $h(x) = (g(x), h_1(x))$ for every $x \in X$, and the continuous mappings $\varpi : S \to Z$ and $\psi : S \to S_1$, where $\varpi(z, s) = z$ and $\psi(z, s) = s$ for all $(z.s) \in S$. By construction, S and Z are compact metric spaces, $H_1 = h(H) = \psi^{-1}(h_1(H)) \in A_\alpha(S)$, $A_2 = g(A)$ and $B_2 = g(B)$ are disjoint subsets of the space Z and $\varphi^{-1}(A_2) \subseteq H_1 \subseteq S \setminus \varphi^{-1}(B_2)$. In virtue of cases 1 and 2 there exists a set $L_1 \in A_\alpha(Z)$ such that $A_2 \subseteq L_1 \subseteq Z \setminus B_2$. Then $L = g^{-1}(L_1) \in A_\alpha(Y)$ and $A \subseteq L \subseteq Y \setminus B$. □

Now we shall develope one non-metrizable modification of Bade's method from [4].

A subset L of a space X is called F_σ-scattered if L is a union of a countable family of compact scattered subsets.

A continuous image of an F_σ-scattered space is F_σ-scattered.

From R. Telgarski's theorem [8, 22] an F_σ-scattered subset of a first countable space is countable and metrizable.

If L is an F_σ-scattered D-set in X, then $L \in CZ_1(X)$.

Theorem 4.4. *Let X be a non-scattered compact space, H be a Baire non-F_σ-scatterd subset of X and $1 \leq \alpha < \Omega$. Then there exist a compact set $H_0 \in Z_0(X)$ and disjoint sets $A, B \in CZ_\alpha(X)$ such that:*

1. *$A \cup B \subseteq H_0 \subseteq H$.*
2. *A and B are not α-Baire separated.*
3. *If $A' \subseteq A$ and $B' \subseteq B$ are any Baire subsets with $A \setminus A'$ and $B \setminus B'$ F_σ-scattered, then A' and B' are not α-Baire separated.*

P r o o f. By Lemma 4.1 there exist a metrizable compact space Y and a continuous mapping $f : X \to Y$ such that $H = f^{-1}(f(H))$ and $f(H)$ is a Borel subset of Y.

If $f(H)$ is an uncountable Borel set in Y, then $f(H)$ contains the Cantor set C (see [17, p. 446]). In this case we put $H_0 = f^{-1}(C)$.

If the set $f(H)$ is countable, then $H_0 = f^{-1}(y)$ is a non-scattered compact subset of X for some $y \in f(H)$.

There exists a continuous mapping g of the compact H_0 onto the closed interval $[0, 1]$.

Case 1. $\alpha \geq 2$.

In virtue of N. N. Luzin's Lemma (see [18, p. 204] or [14, p. 274]) there exist two disjoint sets $A_1, B_1 \in CZ_\alpha([0, 1])$ which are not α-Baire separated in $[0, 1]$. We put $A = g^{-1}(A_1)$ and $B = g^{-1}(B_1)$. Then $A, B \in CZ_\alpha(H_0) \subseteq CZ_\alpha(X)$. By Theorem 4.3, the sets A, B are not α-Baire separated in X. Let $C \subset A$, $D \subset B$, $L_1 \in A_\alpha(X)$, $C \subseteq L_1 \subseteq X \setminus D$ and $C_1 = A \setminus C$, $D_1 = B \setminus D$ are F_σ-scattered. Then $L = (L_1 \setminus D_1) \cup C_1 \in A_\alpha(X)$ and $A \subseteq L \subseteq X \setminus B$.

Case 2. $\alpha = 1$.

Let $\{V_1, V_2, \ldots\}$ be a base of open sets for $[0, 1]$. Choose perfect nowhere dense closed subsets $\{A_n, B_n : n \in N\}$ of $[0, 1]$ such that:

1. $A_n \cap B_n = \emptyset$ for every $n \in N$.
2. $A_1 \cup B_1 \subseteq V_1$.
3. $A_n \cup B_n \subseteq V_n \setminus (\{A_i \cup B_i : i < n\}$, for every $n \geq 2$,

We put $A = \cup\{g^{-1}(A_n) : n \in N\}$ and $B = \cup\{g^{-1}(B_n) : n \in N\}$. Then $A \cap B = \emptyset$ and A, $B \in CZ_1(X)$. Suppose that there are Baire sets C, D and L of X such that $L \in A_1(X)$, $C \subseteq A$, $D \subseteq B$, $C \subseteq L \subseteq X \setminus D$ and $A \setminus C$, $B \setminus D$ are F_σ-scattered. Every set $H \in A_1(X)$ is a G_δ-subset and a Čech complete space.

There exists a closed subspace Z of X_0 such that $g(Z) = [0, 1]$ and $h = g|Z : Z \to [0, 1]$ is irreducible, i.e. $h(F) \neq [0, 1]$ for every proper closed subset F of Z. Then $U = L \cap Z$ and $V = Z \setminus L$ are dense G_δ-subsets of Z. By the Baire category theorem two dence G_δ-sets in compact space must intersect. □

5. *F*-spaces and the maximal ideal spaces. A space is extremally disconnected if the closure of every its open subset is open. A space X is an F'-space if the closure of every functionally open set $H \in CZ_0(X)$ is open. A space X is an F-space if every two disjoint functionally open sets are functionally separated. Every extremally disconnected space is an F'-space ind every F'-space is an F-space (see [12]).

Theorem 5.1 (see [16, 6, 7]). *$b_\Omega X$ is an F'-space for every space X.*

Proof. By construction, $H \in CZ_0(b_\Omega(X))$ if and only if $H \cap X \in B_\Omega(X) = A_\Omega(X)$. Therefore, from Property 1.6, cl$H$ is open in $b_\Omega X$ for every $H \in CZ_0(b_\Omega X)$. □

A space X is called strongly non-F if there exists a non-empty subset L of X such that for each point $x \in L$ there exist two disjoint open sets $U, V \in CZ_0(X)$ with $x \in \mathrm{cl}_X(U \cap L) \cap \mathrm{cl}_X(V \cap L)$ (see [9]).

Let $\phi : X \to Y$ be a continuous mapping of X onto Y. Define $\phi^o : C(Y) \to C(X)$ by the formula $\phi^o(f) = f \cdot \phi$. The projection constant $p(\phi)$ is the infimum of $\|u\|$ of all linear projection $u : C(X) \to \phi^o(C(Y))$. We have $p(\phi) = \infty$ if and only if $\phi^o(C(Y))$ is uncomplemented in $C(X)$ (see [4, 10, 21]).

Theorem 5.2. *Let $\phi : X \to Y$ be a continuous mapping onto a strongly non-F-space Y, X_1 be a dense subspace of X and $\mathrm{cl}_X(X_1 \cap \phi^{-1}(U))$ be open in X for every $U \in CZ_0(Y)$. Then $p(\phi) = \infty$ and $\phi^o(C(Y))$ is uncomplemented in $C(X)$.*

Proof. We assume that $X = \beta X$. Then X and Y are compact spaces.

There exists a non-empty subset L of Y such that for every point $y \in L$ there are two disjoint sets $V_y, W_y \in CZ_0(Y)$ with $y \in \mathrm{cl}_Y(L \cap V_y) \cap \mathrm{cl}_Y(L \cap Wy)$.

Define $M_1(\phi) = Y$ and inductively define $M_{n+1}(\phi) = \{y \in Y :$ there exist nets $B = \{b_\mu \in M_n(\phi) : \mu \in M\}, C = \{c_\eta \in M_n(\phi) : \eta \in H\}$ such that $y = \lim b_\mu = \lim c_\eta$ and $\phi^{-1}(y) \cap \mathrm{cl}_X(\phi^{-1}(B))$, $\phi^{-1}(y) \cap \mathrm{cl}_X(\phi^{-1}(C))$ are non-empty disjoint sets$\}$. By construction, $M_{n+1}(\phi) \subseteq M_n(\phi)$ for every $n \in N$ (sse [4, 9]). Let $y \in L \cap \mathrm{cl}_Y(M_n(\phi))$. Then there exist nets $B \subseteq V_y \cap M_n(\phi)$ and $C \subseteq W_y \cap M_n(\phi)$ such that $y = \lim B = \lim C$. Since $\mathrm{cl}_X(X_1 \cap \phi^{-1}(V_y))$ and $\mathrm{cl}_X(X_1 \cap \phi^{-1}(W_y))$ are disjoint open sets and ϕ is a closed mapping, we have $L \subseteq M(\phi) = \cap\{M_n(\phi) : n \in N\}$. From S. Z. Ditor's Theorem [10, 4, 9], if $M(\phi) \neq \emptyset$, then $p(\phi) = \infty$. □

Theorem 5.3 (see [9] for $\alpha = 0$)**.** *Let X be a compact space, $\alpha < \beta \leq \Omega$ and $b_\alpha X$ be a strongly non-F-space. Then $p(\pi_\alpha^\beta) = \infty$ and $B_\alpha(X)$ is uncomplemented in $B_\beta(X)$.*

Proof. If $U \in CZ_0(b_\alpha X)$, then $U \cap X \in CZ_\alpha(X) \subseteq A_\beta(X)$ and $\mathrm{cl}_{b_\beta X}(U \cap X)$ is open in $b_\beta X$. Theorem 5.2 completes the proof. □

Theorem 5.4. *Let X be a pseudocompact space and βX be a non-scattered space. Then for every countable ordinal $\alpha > 0$ the maximal ideal space $b_\alpha X$ is strongly non-F.*

Proof. In virtue of Proposition 2.6, we have $B_\eta(X) = B_\eta(\beta X)$ for every $\eta \leq \Omega$. Therefore, it is sufficient to prove the theorem for compact spaces $(X = \beta X)$.

Assume that $0 < \alpha \leq \Omega$. Define $L = \{x \in b_\alpha X :$ there exist two disjoint open F_σ-sets U, V in $b_\alpha X$ such that for every clopen neighbourhood W of x in $b_\alpha X$ the sets $W \cap U \cap X$ and $W \cap V \cap X$ are not F_σ-scattered in $X\}$.

Fix a non-F_σ-scattered Baire set H of X. By Theorem 4.4 there exist two disjoint sets $H_1, H_2 \in CZ_\alpha(X)$ such that $H_1 \cup H_2 \subseteq H$ and if $C_1 \subseteq H_1$ and $C_2 \subseteq H_2$ are any Baire sets with $H_1 \setminus C_1$ and $H_2 \setminus C_2$ F_σ-scattered, then C_1 and C_2 are not α-Baire separated in X. There exist two disjoint open F_σ-sets U, V in $b_\alpha X$ such that $U \cap X = H_1$ and $V \cap X = H_2$. We put $F = \mathrm{cl}H_1 \cap \mathrm{cl}H_2 = \mathrm{cl}U \cap \mathrm{cl}V$. The set F is closed and non-empty. We claim that $F \cap L \neq \emptyset$.

Case 1. $\alpha \geq 2$.

In this case we prove that $F \subseteq L$. Let $x \in F$ and W be a clopen neighbourhood of x in $b_\alpha X$. Suppose that $H_3 = W \cap H_1$ is F_σ-scattered. Then $H_3 \in A_2(X) \subseteq A_\alpha X$ and $\mathrm{cl}H_3 \cap \mathrm{cl}H_2 = \emptyset$. By construction, $x \in \mathrm{cl}H_1 \cap \mathrm{cl}H_2 \cap \mathrm{cl}H_3$. Hence $x \in L$.

Case 2. $\alpha = 1$.

Suppose that $F \cap L = \emptyset$. Then for every $x \in F$ there exists a clopen neighbourhood Ux of x in $b_1 X$ such that $H_1 x = Ux \cap H_1$ or $H_2 x = Ux \cap H_2$ is F_σ-scattered. The set F is compact, so there exists a finite cover $\{Ux_1, \ldots, Ux_n\}$ of F. Then $C_1 = H_1 \setminus \cup\{Ux_i : H_1 \cap Ux_i$ is F_σ-scattered$\}$ and $C_2 = H_2 \setminus \cup\{Ux_j : H_2 \cap Ux_j$ is F_σ-scattered$\}$ are Baire sets, $H_1 \setminus C_1$ and $H_2 \setminus C_2$ are F_σ-scattered and $\mathrm{cl}C_1 \cap \mathrm{cl}C_2 = \emptyset$. Therefore C_1 and C_2 are 1-Baire separated in X. Hence $F \cap L \neq \emptyset$.

Consequently $L \neq \emptyset$, L is dense in itself and L satisfies the conditions of the definition of the strongly non-F-space. □

Corollary 5.5 ([4] for $X = [0,1]$, [9] if X contains an uncountable compact metrizable space)**.** *Let $0 < \alpha < \eta \leq \Omega$, X be a pseudocompact space and βX be non-scattered. Then $p(\pi^\eta_\alpha) = \infty$ and $B_\alpha(X)$ is uncomplemented in $B_\eta(X)$.*

Corollary 5.6. *Let $0 < \alpha < \eta \leq \Omega$, X be a pseudocompact space and βX is non-scattered. Then:*

1. *$B_\alpha(X)$ is uncomplemented in $D_{\alpha+1}(X)$, i.e. the Banach space $B_\alpha(X)$ is not Baire complemented.*
2. *$B_\alpha(X)$ is uncomplemented in $D_\eta(X)$.*

6. Extentions of Baire functions.

Lemma 6.1. *For every $\alpha \leq \Omega$ and $f \in B_\alpha(X)$ there exists a countable subset $E(f) \subseteq C(X)$ such that $f \in p_\alpha E(f)$.*

P r o o f. If f is continuous, then we put $E(f) = \{f\}$. Suppose that $\alpha \geq 1$ and for $f \in \cup\{B_\eta(X) : \eta < \alpha\} = B^-_\alpha(X)$ the set $E(f)$ is constructed. For $f \in B_\alpha(X)$ fix a sequence $\{f_n \in B^-_\alpha(X) : n \in N\}$ such that $f = p - \lim f_n$. In this case we put $E(f) = \cup\{E(f_n) : n \in N\}$. □

Theorem 6.2. *Let Y be a compact subspace of a space X. Then for every $\alpha \le \Omega$ and every function $f \in B_\alpha(Y)$ there exists a function $e(f) \in B_\alpha(X)$ such that $f = e(f)|Y$.*

Proof. For $\alpha = 0$ the existence of $e(f)$ follows by the P. S. Urysohn's Lemma [11, p. 63]. Let $\alpha \ge 1$ and $f \in B_\alpha(Y)$. There exists a countable family $E(f) = \{f_n \in C(Y) : n \in N\}$ such that $f \in p_\alpha E(f)$: Consider the continuous mapping $g : X \to Z = g(X) \subseteq \prod\{R_n = R : n \in N\}$, where $g(x) = (e(f_n)(x) : n \in N)$ for all $x \in X$. By construction, the set $g(Y)$ is compact and Z is metrizable. Since $g(x) = g(y)$ provided $x, y \in Y$ and $f(x) = f(y)$, there exists a function $h \in B_\alpha(g(Y))$ for which $f(x) = h(g(x))$ for every $x \in Y$. Now we put $e(f)(x) = h(g(x))$ if $x \in g^{-1}(g(Y))$ and $e(f)(x) = 0$ if $x \notin g^{-1}(g(Y))$. $\square$

Corollary 6.3. *Let Y be a compact subspace of a space X and $\alpha \le \Omega$. Then $b_\alpha Y = \mathrm{cl}_{b_\alpha X} Y$.*

Corollary 6.4. *Let Y be a non-scattered compact subspace of a space X and $o < \alpha < \beta \le \Omega$. Then:*

1. *$p(\phi) = \infty$, where $\phi = \pi^\beta_\alpha : b_\beta X \to B_\alpha X$.*
2. *$b_\alpha X$ is a strongly non-F-space.*
3. *$B_\alpha X$ is uncomplemented in $B_\beta X$.*

7. On Theorem of the B. B. Wells.

Theorem 7.1 (B. B. Wells [23])**.** *If a space X contains an infinite compact metrizable space, then for every $\beta \ge 1$ the space $C(X)$ is not complemented in $B_\beta(X)$ and $P(\pi^\beta_0) = \infty$. In particular, the space $C(X)$ is not Baire complemented.*

Proof. There exists a subspace Y of X homeomorphic to a convergent sequence $\{0, 1, \ldots, n^{-1}, \ldots\}$ and a linear operator $u : B_1(Y) \to B_1(X)$ such that:

1. $u(C(Y)) \subseteq C(X)$.
2. $\|u(f)\| = \|f\|$ for every $f \in B_1(Y)$.
3. $f = u(f)|Y$ for all $f \in B_1(Y)$.

Then $b_1 Y = \mathrm{cl}_{b_1 X}$ and the operator $v : C(b_1 Y) \to C(b_1 X)$, where $v(f) = e_1(u(f|Y))$ for every $f \in C(b_1 Y)$, satisfies the following properties:

4. v is linear and $\|v\| = 1$.
5. $v(f)|b_1 Y = f$.
6. $b_1 Y$ is the Stone-Čech compactification of the discrete countable space PY.

The space $C(Y)$ is not complemented in $C(b_1 Y) = B_1(Y)$ (see [1, 2, 19, 21]). Since $C(Y)$ is complemented in $C(X)$ and $B_1(Y)$ is complemented in $B_1(X)$, the space $C(X)$ is not complemented in $B_1(X)$. $\square$

The spaces X, Y are called u-equivalent – notation $X \sim Y$, if the Banach spaces $C(X)$ and $C(Y)$ are linearly homeomorphic. The symbol $X + Y$ denotes the discrete sum of the spaces X and Y. We have $C(X + Y) = C(X) \times C(Y)$.

From Propositions 2.3, 2.4 and Theorem 7.1 it follows.

Corollary 7.2. *Let X, Y be spaces, Z be an infinite metrizable compact space and $X \sim Y + Z$. Then:*

1. $p(\pi_0^\beta) = \infty$ *for every* $\beta > 0$.
2. $C(X)$ *is not complemented in* $B_\beta(X)$ *for all* $\beta > 0$.
3. $C(X)$ *is not Baire complemented.*

8. On a scattered spaces.

Theorem 8.1. *For an infinite compact space X the following assertions are equivalent:*

1. X *is scattered.*
2. b_1X *is an* F*-space.*
3. *For some* $\alpha < \Omega$ *the space* $b_\alpha X$ *is an* F*-space.*
4. b_1X *is an* F'*-space.*
5. $B_1(X) = B_\alpha(X)$ *for some* $\alpha \geq 2$.

P r o o f. Implication $1 \to 5 \to 1$ are proved in [5, 15]. Implications $4 \to 2 \to 3$ are obvious. Implications $5 \to 4$ and $3 \to 1$ follows from Theorems 5.1 and 5.5 respectively. □

Remark 8.2. If X is an infinite pseudocompact scattered space, then:

1. X contains an infinite compact metrizable space.
2. X is not an F-space.
3. $C(X)$ is not complemented in $B_\alpha(X)$ for each $\alpha \geq 1$.
4. $C(X)$ is not Baire complemented.

9. On the F. K. Dashiell's theorem.

Theorem 9.1 [9, Theorem 2.11]**.** *For a compact space X the following assertions are equivalent:*

1. X *is an* F*-space.*
2. $C(X)$ *is Baire complemented by a projection of norm* 1.
3. *There exists a linear multiplicative norm* 1 *projection* $u : B_\Omega(X) \to C(X)$.

Corollary 9.2. *For a compact space X the following are equivalent:*

1. X *is an* F*-space.*
2. *There exists a closed subspace* X_1 *of* b_1X *such that* $\pi_0^1(X_1) = X$ *and* $\pi_0^1|X_1 \to X$ *is homeomorphism.*

3. *There exists a closed subspace X_Ω of $b_\Omega X$ such that $\pi_0^\Omega(X_\Omega) = X$ and $\pi_0^\Omega|X_\Omega$ is homeomorphism.*

4. *There exists a sequence of compact subspaces $\{X_\alpha \subseteq b_\alpha X : \alpha \leq \Omega\}$ such that:*

4.1. *$\pi_0^\alpha(X_\alpha) = X$ and $\pi_0^\alpha|X_\alpha$ is a homeomorphism.*

4.2. *$\pi_\alpha^\beta(X_\beta) = X_\alpha$ and $\pi_\alpha^\beta|X_\beta$ is a homeomorphism.*

Theorem 9.3. *Let $\psi : X \to Y$ be a continuous mapping of a space X onto a dense subspace of a space Y and $p(\psi) < \infty$. Then:*

1. *If $C(X)$ is Baire complemented, then $C(Y)$ is Baire complemented, too.*
2. *If $\alpha \leq \Omega$ and $C(X)$ is complemented in $B_\alpha(X)$, then $C(Y)$ is complemented in $B_\alpha(Y)$.*

P r o o f. There is a continuous operator $u : C(X) \to C(Y)$ such that $u(f \cdot \psi) = f$ for every $f \in C(Y)$. In particular, $C(Y)$ is linearly homeomorphic with the complemented subspace $\psi^o(C(Y))$ of the space $C(X)$. If $v : B_\alpha(X) \to C(X)$ is a linear projection, then $w : B_\alpha(Y) \to C(Y)$, where $w(f) = u(v(f \cdot \psi))$, is a linear projection, too. □

Corollary 9.4. *For a compact space Y the following are equivalent:*

1. *$C(Y)$ is complemented in $B_\Omega(Y)$.*
2. *There exist an F'-space X and a continuous mapping $\psi : X \to Y$ such that $\psi(X) = Y$ and $p(\psi) < \infty$.*
3. *There exist an F'-space X and a complemented subspace E of $C(X)$ linearly homeomorphic to $C(Y)$.*

Question 9.5. *Let Y be an infinite compact space and $C(Y)$ be Baire complemented. Is it true that $C(Y)$ is complemented in $B_2(Y)$ or in $B_\Omega(Y)$?*

10. On Baire saturated spaces. A space X is called a Baire saturated space with a Baire nucleus Y if Y is a dense subspace of X and $\{f|Y : f \in C(X)\} = \{f|Y : f \in B_1(X)\}$.

Example 10.1 Let Y be an infinite P-space,i.e. $PX = X$. Then Y is a Baire nucleus of the Baire saturated space $X = \beta Y$.

Example 10.2. For every infinite space X the spaces PX and $P\nu X$ are Baire nucleus of compact space $b_\Omega X$.

Lemma 10.3. *If Y is a Baire nucleus of the space X, then Y is a P-space.*

P r o o f. Suppose now that Y is a Baire nucleus of X, $\{U_n : n \in N\}$ be a sequence of open subsets on X and $y \in \cap\{Y \cap U_n : n \in N\} = U$. There exists a sequence of continuous functions $\{f_n : X \to [0,1] : n \in N\}$ foe which:

1. $f_n(y) = 0$ and $f_{n+1}(x) \geq f_n(x)$ for all $x \in X$ and $n \in N$.
2. $X \setminus U_n \subseteq f_n^{-1}(1)$ for all $n \in N$.

Then we have $f = p - \lim f_n$ for some $f \in B_1(X)$. By construction, the function $g = f|Y$ is continuous, $V = g^{-1}(-1, 1)$ is open in Y and $y \in V \subseteq U$. □

Lemma 10.4. *Let X be a Baire saturated space with a Baire nucleus Y. Then there exists a unique linear multiplicative norm 1 projection $u : B_\Omega(X) \to C(Y)$ such that $u(f)|Y = f|Y$ for every $f \in B_\Omega(X)$.*

Proof. From the definition, $E = \{f|Y : f \in C(X)\} = \{f|Y : f \in B_1(X)\}$. By a simple induction and Lemma 10.3, we obtain that $E = \{f|Y : f \in B_\Omega(X)$. For every $g \in E$ there is a unique function $v(g) \in C(X)$ such that $v(g)|Y = g$. Now we put $u(f) = v(f|Y)$ for every $f \in B_\Omega(X)$. □

Remark 10.5. Every separable dense in itself space is not Baire saturated.

11. The embeddings of spaces $B_\alpha(X)$.

Theorem 11.1. *Suppose that X is a space with one of the following properties:*

1. *X contains a non-scattered compact subspace.*
2. *X is normal and contains a closed pseudocompact subspace Y for which βY is not scattered.*
3. *X contains a subspace Y for which $Z = \mathrm{cl}_{\beta X} Y$ is not scattered and every continuous function $F \in C(X)$ is bounded on Y.*

Then for every countable ordinal number $\alpha \geq 1$ we have:

a. $b_\alpha X$ is a strongly non-F-space.

b. $B_\alpha(X)$ is not Baire complemented.

c. If $B_{\alpha+1}(X)$ is a subspace of a linear topological space E, then $B_\alpha(X)$ is not complemented in E.

d. $B_\alpha(X)$ is not ilinear homeomorphic to any complemented subspace of $B_\Omega(X')$ for some compact space X'.

e. $B_\alpha(X)$ is not linear homeomorphic to any complemented subspace $C(X'$ for any F'-space X'.

f. $B_\alpha(X)$ is not linear homeomorphic to any complemented subspace of same Baire complemented Banach space E.

Proof. Let every function $F \in C(X)$ is bounded on Y and $Z = \mathrm{cl}_{\beta X} Y$ is not scattered, where Y is a subspace of X. In this case $Z = \mathrm{cl}_{\nu X} Y$ and Z is non-scattered compact subspace of a space νX. From Corollary 6.4, $b_\alpha \nu X$ is a strongly non-F-space. From P. R. Meyer's Theorem (see the proof of Proposition 2.7), $b_\alpha X = b_\alpha \nu X$. Therefore $B_\alpha(X) = B_\alpha(\nu X)$ is not a complemented subspace of spaces $D_{\alpha+1}(X)$ and $B_{\alpha+1}(X)$. The assertions a, b, c are proved. From Property 2.3 and Theorems 5.1 and 9.1, $B_\alpha(X)$ is not linear homeomorphic to any complemented subspace of a Baire complemented Banach space E. This proves the assertions d, e, f. $\square$

In [9, Corollary 3.7] the assertions d, e of Theorem 11.1 are formuled for a compact space X which contains an uncountable metrizable compact space.

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