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CO-CONNECTED SPACES

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Dedicated to the memory of Professor D. Doitchinov

Abstract. Co-connected spaces, i.e. the spaces X for which any continuous map $X^2 \to X$ factors through a projection, are investigated. The main result: every free monoid is isomorphic to the monoid of all nonconstant continuous selfmaps of a metrizable co-connected space.

1. In [8], $\mathbb{E}$-connected spaces are introduced where $\mathbb{E}$ is a class of spaces. Let us recall that a topological space X is $\mathbb{E}$-connected if every continuous map $f : X \to E$ is constant, for all $E \in \mathbb{E}$. This nice notion not only shows analogies with properties of other structures (we do not mention the wide variety of papers where the analogous notion is used outside topology), but it also admits to simplify the formulations of some results, e.g. the famous result of [6] can be formulated such that if $\mathbb{E}$ consists of one space E, then the class of all $\mathbb{E}$-connected spaces contains a regular space with more than one point iff E is a T_1-space. And, clearly, a space

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X is connected iff it is $\mathbb{E}$-connected for a non-trivial class $\mathbb{E}$ (i.e. containing a space E with $\mathrm{card} E > 1$) of totally disconnected spaces.

The duality principle means, intuitively: turn the arrows! Hence the dual notion, $\mathbb{E}$-co-connectedness, offers the following definition: a space X is $\mathbb{E}$-co-connected if every continuous map $E \to X$, $E \in \mathbb{E}$, is constant. For some classes $\mathbb{E}$, this notion is also interesting. However, *this* dual notion of the usual connectedness, i.e. the $\mathbb{E}$-co-connectedness where $\mathbb{E}$ is a non-trivial class of totally disconnected spaces, is no more interesting: the only spaces which are $\mathbb{E}$-co-connected in this sense would be the one-point spaces and the empty space.

Duality between the class $\mathbb{C}$ of all connected spaces and the class $\mathbb{D}$ of all totally disconnected spaces is presented in [1]. Here, two operators $\mathfrak{C}$ and $\mathfrak{D}$ are introduced by

$$\mathfrak{C}(\mathbb{E}) = \{X \mid \text{ for every } E \in \mathbb{E}, \text{ every continuous } f : X \to E \text{ is constant}\},$$
$$\mathfrak{D}(\mathbb{E}) = \{Y \mid \text{ for every } E \in \mathbb{E}, \text{ every continuous } f : E \to Y \text{ is constant}\}.$$

The duality between the operators $\mathfrak{C}$ and $\mathfrak{D}$, developped in [1], is very faithful and, clearly, $\mathbb{C} = \mathfrak{C}(\mathbb{D})$ and $\mathbb{D} = \mathfrak{D}(\mathbb{C})$. The faithfulness of this duality is caused by the fact, that the arrows are turned only by our approach: both in $\mathfrak{D}(\mathbb{C}) = \mathbb{D}$ and in $\mathfrak{C}(\mathbb{D}) = \mathbb{C}$, the class of the maps is the same one, namely from the connected spaces into the totally disconnected spaces; but in $\mathfrak{D}(\mathbb{C}) = \mathbb{D}$, we approach from the side of their domains and in $\mathfrak{C}(\mathbb{D}) = \mathbb{C}$, we approach from the side of their ranges.

This formal “turning of arrows” is the real duality (as used in the category theory and its applications). However, nobody in the community of topologists calls the totally disconnected spaces by the name “co-connected spaces”. Hence this name seems still to be free. Let me offer a definition of co-connected spaces which turns some arrows, so it has some features of duality.

However, since the turning of the arrows is not only a formal one, the duality is not too faithful: the properties of the co-connected spaces are sometimes far from the “dual properties” of the connected spaces. But the co-connected spaces, obtained by this “non-formal turning of some arrows” form a class of spaces which could be of some interest and of some importance.

2. If X is a topological space and n is a natural number, $n = \{0, \dots, n-1\}$, let us denote by nX the coproduct (= the sum) of n copies of X, i.e. the underlying set of nX is just $\bigcup_{i \in n} \{i\} \times X$, every $\{i\} \times X$ is clopen (=closed-and-open) in nX and every coproduct-injection $v_i^{(n)} : X \to nX$ sending every $x \in X$

to (i, x), $i \in n$, is a homeomorphism of X onto $v_i^{(n)}(X)$. Our approach is based on the evident equivalence of the following statements:

(0) X is connected;

(1) every continuous map $f : X \to 2X$ factors through $v_0^{(2)}$ or $v_1^{(2)}$ (i.e. there exists a continuous $g : X \to X$ such that either $f = v_0^{(2)} \circ g$ or $f = v_1^{(2)} \circ g$);

(2) for every natural number $n \geq 2$ and every continuous map $f : X \to nX$ there exists continuous $g : X \to X$ such that $f = v_i^{(n)} \circ g$ for some $i \in n$;

(3) there exists a natural number $n \geq 2$ such that every continuous map $f : X \to nX$ factors through some $v_i^{(n)}$, $i \in n$.

The statements (1), (2), (3) can be dualized and we show just below that the dual statements are also equivalent. Hence it is quite natural "to dualize" also the statement (0) and to call such spaces co-connected. This is really our definition of *co-connectedness*.

3. If X is a topological space, we denote by X^n its n-th power and by $\pi_i^{(n)} : X^n \to X$ the i-th projection, $i \in n$.

We show that the following statements are equivalent:

($\tilde{1}$) every continuous map $f : X^2 \to X$ factors through $\pi_0^{(2)}$ or $\pi_1^{(2)}$ (i.e. there exists a continuous map $g : X \to X$ such that either $f = g \circ \pi_0^{(2)}$ or $f = g \circ \pi_1^{(2)}$);

($\tilde{2}$) for every natural number $n \geq 2$ and every continuous map $f : X^n \to X$ there exists continuous $g : X \to X$ such that $f = g \circ \pi_i^{(n)}$ for some $i \in n$;

($\tilde{3}$) there exists a natural number $n \geq 2$ such that every continuous map $f : X^n \to X$ factors through some $\pi_i^{(n)}$, $i \in n$.

Proof. Clearly, ($\tilde{2}$) $\Rightarrow$ ($\tilde{3}$).

($\tilde{3}$) $\Rightarrow$ ($\tilde{1}$): If ($\tilde{1}$) does not hold, i.e. there exists $f : X^2 \to X$ which does not factor through $\pi_0^{(2)}$ or $\pi_1^{(2)}$, then $f \circ \pi_{\{0,1\}}^{(n)} : X^n \to X$ does not factor through any $\pi_i^{(n)}$, $i \in n$, whenever $\pi_{\{0,1\}}^{(n)}$ denotes the map sending any $(x_0, \ldots, x_{n-1}) \in X^n$ to (x_0, x_1).

$(\tilde{1}) \Rightarrow (\tilde{2})$: We proceed by induction. Let $n \geq 2$ be a natural number and let for every space X satisfying $(\tilde{1})$, every continuous map $X^n \to X$ factors through $\pi_i^{(n)}$ for some $i \in n$. Let a continuous map

$$f : X^{n+1} \to X$$

be given. We prove that it factors through some $\pi_i^{(n+1)}$, $i \in n+1$.

a) For every $z \in X$, denote by $\varphi_z : X^n \to X^{n+1}$ the map sending every $(x_0, \ldots, x_{n-1})$ to $(x_0, \ldots, x_{n-1}, z)$. Denote by A_0 the set of all $z \in X$ for which there exists a continuous map $g_z^{(0)} : X \to X$ such that $f \circ \varphi_z = g_z^{(0)} \circ \pi_0^{(n)}$ and by A_1 the set of all $z \in X$ for which there exist $i \in \{1, \ldots, n-1\}$ and a continuous map $g_z^{(i)} : X \to X$ such that $f \circ \varphi_z = g_z^{(i)} \circ \pi_i^{(n)}$. By the induction hypothesis, $X = A_0 \cup A_1$.

b) Now, we show that for every $z \in X$,

$f \circ \varphi_z$ is a constant map iff $z \in A_0 \cap A_1$.

In fact, if $f \circ \varphi_z = g_z^{(i_0)} \circ \pi_{i_0}^{(n)}$ is constant and $i_0 \in n$, then necessarily $g_z^{(i_0)} : X \to X$ is constant, so that $f \circ \varphi_z = g_z^{(i_0)} \circ \pi_i^{(n)}$ for all $i \in n$, hence $z \in A_0 \cap A_1$. Conversely, if $z \in A_0 \cap A_1$, then there exist $i \in \{1, \ldots, n-1\}$ and continuous maps $g_z^{(0)}$, $g_z^{(i)}$ such that

$$g_z^{(0)} \circ \pi_0^{(n)} = f \circ \varphi_z = g_z^{(i)} \circ \pi_i^{(n)} .$$

Hence for every $x, y \in X$ necessarily $g_z^{(0)}(x) = g_z^{(i)}(y)$ so that $g_z^{(0)}$ is constant so that $f \circ \varphi_z$ is constant.

c) Now, we show that either $A_0 \setminus A_1 = \emptyset$ or $A_1 \setminus A_0 = \emptyset$. Let us suppose the contrary and choose $a_0 \in A_0 \setminus A_1$ and $a_1 \in A_1 \setminus A_0$. Since $a_0 \in A_0$, we get $f \circ \varphi_{a_0} = g_{a_0}^{(0)} \circ \pi_0^{(n)}$. Since $a_0 \notin A_1$, the map $g_{a_0}^{(0)} : X \to X$ must be nonconstant so that, there exists $d, d' \in X$ such that $g_{a_0}^{(0)}(d) \neq g_{a_0}^{(0)}(d')$. Hence

$$f(d, c_1, \ldots, c_{n-1}, a_0) \neq f(d', c_1, \ldots, c_{n-1}, a_0)$$

for arbitrarily chosen $(c_1, \ldots, c_{n-1}) \in X^{n-1}$. Thus, choose some $(c_1, \ldots, c_{n-1}) \in X^{n-1}$ and denote $p = f(d, c_1, \ldots, c_{n-1}, a_1)$. Since $a_1 \in A_1 \setminus A_0$, p is also equal to $f(d', c_1, \ldots, c_{n-1}, a_1)$. If p is equal to $f(d, c_1, \ldots, c_{n-1}, a_0)$, we get

$$f(d', c_1, \ldots, c_{n-1}, a_0) \neq f(d', c_1, \ldots, c_{n-1}, a_1) ;$$

if p is equal to $f(d', c_1, \ldots, c_{n-1}, a_0)$ or it is distinct from the both points $f(d, c_1, \ldots, c_{n-1}, a_0)$ and $f(d', c_1, \ldots, c_{n-1}, a_0)$, we get

$$f(d, c_1, \ldots, c_{n-1}, a_0) \neq f(d, c_1, \ldots, c_{n-1}, a_1) .$$

Let us investigate the later case (otherwise we exchange the notation of d and d'). We define a map

$$m : X^2 \to X^{n+1}$$

by

$$m(y, z) = (y, c_1, \dots, c_{n-1}, z)$$

and put $l = f \circ m : X^2 \to X$. Then l is continuous but it does not factor through $\pi_0^{(2)}$ because $l(d, a_0) \neq l(d, a_1)$ and it does not factor through $\pi_1^{(2)}$ because $l(d, a_0) \neq l(d', a_0)$.

We conclude that either $A_1 \setminus A_0 = \emptyset$, i.e. $X = A_0$, or $A_0 \setminus A_1 = \emptyset$, i.e. $X = A_1$.

d) If $X = A_0$, i.e. $f(x_0, \dots, x_{n-1}, z) = g_z^{(0)}(x_0)$, f does not depend on $x_1, \dots, x_{n-1}$. Put $h(x_0, z) = g_z^{(0)}(x_0)$. If h factors through $\pi_0^{(2)}$, then f factors through $\pi_0^{(n+1)}$; if h factors through $\pi_1^{(2)}$, then f factors through $\pi_n^{(n+1)}$. If $X = A_1$, then $f(x_0, x_1, \dots, x_{n-1}, z) = h(x_1, \dots, x_{n-1}, z)$, where $h : X^n \to X$ is continuous. By the induction hypotheses, h factors through $\pi_i^{(n)}$ for some $i \in n$. Then, clearly, f factors through $\pi_{i+1}^{(n+1)}$.

Remark. If X is a Hausdorff co-connected space, then for every *cardinal* number $\mathfrak{n} \geq 2$, every continuous $f : X^{\mathfrak{n}} \to X$ factors through a projection. In fact, let S be the subspace of $X^{\mathfrak{n}}$ consisting of all $x \in X^{\mathfrak{n}}$ which differ from a point $a = \{a_i \mid i \in \mathfrak{n}\} \in X^{\mathfrak{n}}$ in finitely many coordinates. If $f \restriction S$ is constant then f is constant hence it factors through any projection. If $f \restriction S$ is nonconstant, there exists a finite $F_0 \subseteq \mathfrak{n}$ such that

$$X^{F_0} \xrightarrow{e^{F_0}} S \xrightarrow{f \restriction S} X$$

is nonconstant, where e^{F_0} sends any $\{x_i \mid i \in F_0\}$ to $\{x_i \mid x_i \in \mathfrak{n}\}$ with $x_i = a_i$ for all $i \in \mathfrak{n} \setminus F_0$. Hence $(f \restriction S) \circ e^{F_0}$ factors through a unique $\pi_{i_0}^{(F_0)}$, $i_0 \in F_0$. Then $(f \restriction S) \circ e^F$ factors through $\pi_{i_0}^{(F)}$ for every finite F, $F_0 \subseteq F \subseteq \mathfrak{n}$, hence $f \restriction S$ factors through $\pi_{i_0}^{\mathfrak{n}}$ so that f factors through it.

4. Some statements about connected spaces can be dualized. The statement

"if X is a non-empty space and $n \geq 2$, then nX is never connected"

has its dual in the statement

"if card$X > 1$ and $n \geq 2$, then X^n is never co-connected" which is satisfied, evidently. In fact, the map $f : X^2 \times X^2 \to X^2$ given by

$$f((x_0, y_0), (x_1, y_1)) = (x_0, x_1)$$

factors neither through $\pi_0^{(2)}$ nor through $\pi_1^{(2)}$.

The assumption card$X > 1$, stronger than the requirement that X is non-empty, shows that the forming of the dual statements about connected spaces has some features of the behavior of contravariant hom-functors. This would also explain why the dual of the statement

"if X is connected, then X^n is also connected",

i.e. the statement

"if X is co-connected, then nX is also co-connected",

is not valid: contravariant hom-functors turn coproducts into products but not vice versa.

5. The main advantage of the co-connected spaces is the possibility to recover the monoid of all continuous selfmaps of the n-th power of the spaces, $n \geq 2$, from the monoid of all continuous selfmaps of the spaces themselves. In fact, let us denote by $(M, \circ)$ the monoid of all continuous selfmaps of a co-connected space X and by $(M_n, \overset{n}{\circ})$ the monoid of all continuous selfmaps of the space X^n. Then $(M_n, \overset{n}{\circ})$ is isomorphic to $(M^n \times n^n, \odot)/_\sim$ where $\odot$ is the binary operation on $M^n \times n^n$ given by

$$(m_0, m_1, \dots, m_{n-1}, \varphi) \odot (m'_0, \dots, m'_{n-1}, \varphi') = (m_0 \circ m'_{\varphi(0)}, \dots, m_{n-1} \circ m'_{\varphi(n-1)}, \varphi \circ \varphi')$$

where $\circ$ denotes also the composition of the maps, $\varphi, \varphi' : n \to n$ (i.e. $(M^n \times n^n, \odot)$ is the wreath product, see e.g. [2], of the monoid $(M, \circ)$ and the dual $(n^n, \circ)^{\mathrm{opp}}$ of $(n^n, \circ)$) and $\sim$ is the smallest equivalence on $M^n \times n^n$ (which is also the smallest congruence of $(M^n \times n^n, \odot)$) for which

$$(m_0, m_1, \dots, m_{n-1}, \varphi) \sim (m_0, m_1, \dots, m_{n-1}, \varphi')$$

whenever there exists $i \in n$ such that $\varphi(j) = \varphi'(j)$ for all $j \in n \setminus \{i\}$ and m_i is a left zero of the monoid $(M, \circ)$.

In fact, we can investigate elements of M_n as n-tuples of continuous maps $X^n \to X$ because every such n-tuple $(f_0, \dots, f_{n-1})$ determines uniquely the map $f = f_0 \dot{\times} \cdots \dot{\times} f_{n-1}$ sending any $x \in X^n$ to $(f_0(x), \dots, f_{n-1}(x))$ and, vice versa, every continuous $f : X^n \to X^n$ determines uniquely the n-tuple

$(\pi_0^{(n)} \circ f, \dots, \pi_{n-1}^{(n)} \circ f)$ and these procedures are mutually inverse. Let us define a map

$$h : (M^n \times n^n, \odot) \to (M_n, \overset{n}{\circ})$$

by the rule

$$h(m_0, \dots, m_{n-1}, \varphi) = (m_0 \circ \pi_{\varphi(0)}^{(n)}, \dots, m_{n-1} \circ \pi_{\varphi(n-1)}^{(n)}) .$$

Then, clearly, h is a homomorphism of the above monoids. Since X is co-connected, h is surjective. Hence $(M_n, \overset{n}{\circ})$ is isomorphic to $(M^n \times n^n, \odot)/_{\mathrm{Ker}\, h}$, where the kernel of h, $\mathrm{Ker}\, h$, is just the decomposition $\{h^{-1}(y) \,|\, y \in M_n\}$. But $\sim$ determines the same decomposition because $m \in M$ is a left zero of the monoid $(M, \circ)$ iff the map $m : X \to X$ is constant and this is precisely when $m \circ \pi_i^{(n)} = m \circ \pi_j^{(n)}$ for some $i, j \in n$, $i \neq j$, and this happens precisely when $m \circ \pi_i^{(n)} = m \circ \pi_j^{(n)}$ for all $i, j \in n$. Thus, $(M_n, \overset{n}{\circ})$ is really isomorphic to $(M^n \times n^n, \odot)/_{\sim}$.

6. Every co-connected space must be also connected (it is quite evident). This statement cannot be dualized, most of the connected spaces are not co-connected. In fact, co-connected spaces are quite rare. Rigid spaces (i.e. the spaces for which every continuous selfmap is either the identity or a constant) are examples of co-connected spaces: by [7], if X is a rigid space with $\mathrm{card} X > 2$ (i.e. distinct from the Sierpinski's two-point space), then every continuous map $f : X^2 \to X$ is either $\pi_0^{(2)}$ or $\pi_1^{(2)}$ or constant, hence X is co-connected. We present a theorem below which shows that there are more co-connected spaces than only the rigid ones. For rigid spaces, the nonconstant continuous selfmaps form the trivial monoid $\{1\}$. If L is a non-empty set, let us denote by L^* the free monoid over L, i.e. elements of L^* are all words $l_1 \dots l_n$ consisting of elements of L, the composition $\circ$ in L^* is just concatenation, i.e.

$$(l_1 \dots l_n) \circ (l'_1 \dots l'_m) = l_1 \dots l_n l'_1 \dots l'_m ,$$

and the unit element of L^* is just the empty word.

Theorem. *For every set L there exists a co-connected metrizable space such that its nonconstant continuous selfmaps form a monoid isomorphic to L^*. Any cardinal number $\aleph \geq 2^{\aleph_0} + \mathrm{card} L$ can be the cardinality of such space.*

The proof of this theorem is in fact an application of the construction presented in [10]. This application is outlined in 7.–10. below.

7. First, to introduce our notation, let us recall some basic notions of universal algebra (see e.g. [5]) which are used in [10]. Let (Σ, ar) be a signature of (monosorted, finitary) universal algebras, i.e. Σ is a non-empty set and $\mathrm{ar} : \Sigma \to \omega$ is its "arity function", i.e. every $\sigma \in \Sigma$ is an operation symbol of an $\mathrm{ar}\sigma$-ary operation, where ω denotes the set of all finite ordinals. Let us denote $\Sigma_n = \mathrm{ar}^{-1}(n)$, i.e. Σ_n is the set of all symbols of n-ary operations. Hence $\Sigma = \bigcup_{n=0}^{\infty} \Sigma_n$ and the sets on the right side are disjoint.

The standard construction of a free Σ-algebra $\mathbb{A} = (A, \{a_\sigma \mid \sigma \in \Sigma\})$ on a set of generators G (see e.g. [5]) describes its underlying set A as the set of all Σ-terms $\bigcup_{k=0}^{\infty} A_k$ where

$$\begin{aligned} A_0 &= G \cup \Sigma_0 && \text{(the set of Σ-terms of the depth 0)} \\ A_{k+1} &= A_k \cup \bigcup_{\sigma \in \Sigma \setminus \Sigma_0} && \{\sigma(t_0, \dots, t_{\mathrm{ar}\sigma - 1}) \mid t_i \in A_k \text{ for all } i \in \mathrm{ar}\sigma\} \\ & && \text{(the set of all Σ-terms of the depth $\le k+1$)}, \end{aligned}$$

and the operations

$$a_\sigma : A^{\mathrm{ar}\sigma} \to A\,, \quad \sigma \in \Sigma\,,$$

are defined such that $a_\sigma(*) = \sigma$ for $\sigma \in \Sigma_0$ (where A^0 is a fix one-point set $\{*\}$) and, for $\sigma \in \Sigma \setminus \Sigma_0$, a_σ is *the* map which sends the $\mathrm{ar}\sigma$-tuple $(t_0, \dots, t_{\mathrm{ar}\sigma-1})$ of Σ-terms to the Σ-term $\sigma(t_0, \dots, t_{\mathrm{ar}\sigma-1})$.

8. Let Σ_0 be infinite and let us denote by

$$\mathbb{P} = (P, \{p_\sigma \mid \sigma \in \Sigma\})$$

the initial Σ-algebra, i.e. the free Σ-algebra on the empty set of generators, i.e. $P = \bigcup_{k=0}^{\infty} P_k$ where $P_0 = \Sigma_0$ and $P_{k+1} = P_k \cup \bigcup_{\sigma \in \Sigma \setminus \Sigma_0} \{\sigma(t_0, \dots, t_{\mathrm{ar}\sigma-1}) \mid t_i \in P_k \text{ for all } i \in \mathrm{ar}\sigma\}$.

For every $n \in \omega$, we define the set $M^{(n)}$ of maps $P^n \to P$ as follows:

$$M_0^{(n)} = \{\pi_0^{(n)}, \dots, \pi_{n-1}^{(n)}\} \cup \{\mathrm{const}_\sigma^{(n)} \mid \sigma \in \Sigma_0\}$$

where $\pi_i^{(n)} : P^n \to P$ denotes the i-th projection and $\mathrm{const}_\sigma^{(n)} : P^n \to P$ denotes the constant map with the value $\sigma \in \Sigma_0 \subseteq P$;

$$M_{k+1}^{(n)} = M_k^{(n)} \cup \bigcup_{\sigma \in \Sigma \setminus \Sigma_0} \{p_\sigma \circ (f_0 \dot{\times} \cdots \dot{\times} f_{\mathrm{ar}\sigma-1}) \mid f_i \in M_k^{(n)} \text{ for all } i \in \mathrm{ar}\sigma\}$$

where $f_i : P^n \to P$ are in $M_k^{(n)}$, $f = f_0 \dot{\times} \cdots \dot{\times} f_{\mathrm{ar}\sigma-1}$ denotes the map $P^n \to P^{\mathrm{ar}\sigma}$ sending any $y \in P^n$ to $(f_0(y), \dots, f_{\mathrm{ar}\sigma-1}(y))$, $p_\sigma : P^{\mathrm{ar}\sigma} \to P$ is the σ-th operation of the initial Σ-algebra $\mathbb{P}$ and $\circ$ denotes the composition of these maps,

$$M^{(n)} = \bigcup_{k=0}^{\infty} M_k^{(n)} .$$

Let $\mathbb{A}_n$ denote the free Σ-algebra on n generators $g_0, \dots, g_{n-1}$, let A_n be its underlying set and let $\lambda : A_n \to M^{(n)}$ be the map defined inductively by

$$\lambda(g_i) = \pi_i^{(n)} , \quad \lambda(\sigma) = \mathrm{const}_\sigma^{(n)} \text{ for } \sigma \in \Sigma_0 ,$$
$$\lambda(\sigma(t_0, \dots, t_{\mathrm{ar}\sigma-1})) = p_\sigma \circ (\lambda(t_0) \dot{\times} \cdots \dot{\times} \lambda(t_{\mathrm{ar}\sigma-1})) \text{ for } \sigma \in \Sigma \setminus \Sigma_0 .$$

It is well-known and easy to see that λ is a bijection of A_n onto $M^{(n)}$.

9. In [10], for every signature (Σ, ar) with

$$\mathrm{card}\Sigma_0 \geq 2^{\aleph_0} + \mathrm{card}(\Sigma \setminus \Sigma_0) ,$$

a metric ϱ on the underlying set P of the initial Σ-algebra $\mathbb{P} = (P, \{p_\sigma \mid \sigma \in \Sigma\})$ is constructed such that the metric space $X = (P, \varrho)$ has the following property:

> for every $n \in \omega$, the set $\mathcal{C}(X^n, X)$ of all continuous maps $X^n \to X$ is precisely the set $M^{(n)}$.

We show that, for suitable choice of the signature (Σ, ar), the space $X = (P, \varrho)$ satisfies the requirements of our theorem.

10. *Now, we finish the proof of the Theorem.* Let a set L be given. Choose a signature (Σ, ar) such that

$$\Sigma_1 = L , \quad \mathrm{card}\Sigma_0 \geq 2^{\aleph_0} + \mathrm{card}L ,$$
$$\Sigma_n = \emptyset \text{ for all } n \in \omega \setminus \{0, 1\} .$$

Let $\mathbb{P} = (P, \{p_\sigma \mid \sigma \in \Sigma\})$ be the initial Σ-algebra, let ϱ be a metric on P mentioned in 9, i.e. for the metric space $X = (P, \varrho)$, the set $\mathcal{C}(X^n, X)$ is equal to $M^{(n)}$.

a) We show that the metric space X is co-connected. Thus, let $f \in \mathcal{C}(X^2, X) = M^{(2)} = \bigcup_{k=0}^{\infty} M_k^{(2)}$ be given. Let k be the smallest ordinal such that

$f \in M_k^{(2)}$. If $f \in M_0^{(2)}$, then either $f = \pi_0^{(2)}$ or $f = \pi_1^{(2)}$ or $f = \mathrm{const}_\sigma^{(2)}$ for some $\sigma \in \Sigma_0$, so f factors through $\pi_0^{(2)}$ or $\pi_1^{(2)}$ in any case. We proceed by induction. Let us suppose that $f \in M_{k+1}^{(2)}$, i.e. $f = p_\sigma \circ (f_0 \dot{\times} \cdots \dot{\times} f_{\mathrm{ar}\sigma-1})$ for some $\sigma \in \Sigma \setminus \Sigma_0$ and $f_i \in M_k^{(2)}$, $i \in \mathrm{ar}\sigma$. Since $\Sigma_2 = \Sigma_3 = \ldots = \emptyset$, necessarily $\sigma \in \Sigma_1$; hence $f = p_\sigma \circ f_0$ and f_0 factors through $\pi_0^{(2)}$ or $\pi_1^{(2)}$, by the induction hypothesis. Thus, f also factors through $\pi_0^{(2)}$ or $\pi_1^{(2)}$.

b) We show that the nonconstant continuous selfmaps of X form a monoid isomorphic to L^*. We show that every nonconstant continuous selfmap $f : X \to X$ is of the form $f = p_{\sigma_k} \circ \ldots \circ p_{\sigma_1}$, $\sigma_1, \ldots, \sigma_k \in L$, or $f = \mathbf{1}$, where $\mathbf{1} : X \to X$ is the identity map. We have $f \in \mathcal{C}(X, X) = M^{(1)} = \bigcup_{k=0}^{\infty} M_k^{(1)}$. If $f \in M_0^{(1)}$, then $f \in \{\pi_0^{(1)}\} \cup \{\mathrm{const}_\sigma^{(1)} \mid \sigma \in \Sigma_0\}$; since f is nonconstant, necessarily $f = \pi_0^{(1)} = \mathbf{1}$; then we proceed by induction in k: if $f \in M_{k+1}^{(1)}$, then necessarily $f = p_\sigma \circ f_0$ for a nonconstant $f_0 \in M_k^{(1)}$; by the induction hypothesis, $f_0 = p_{\sigma_{k-1}} \circ \ldots \circ p_{\sigma_1}$ or $f_0 = \mathbf{1}$, hence f is equal to $p_\sigma \circ p_{\sigma_{k-1}} \circ \ldots \circ p_{\sigma_1}$ or to p_σ. If $\lambda : \mathbb{A}_1 \to M^{(1)}$ is as in 8, λ^{-1} is a bijection of the set of all nonconstant elements of $M^{(1)}$ onto $(\Sigma_1)^* = L^*$ so that λ restricted to all the Σ-terms of the form $(\sigma_k(\sigma_{k-1} \ldots (\sigma_1(g_0) \ldots)$ gives an isomorphism of the monoid L^* onto the monoid of all nonconstant continuous selfmaps of X.

11. Should compact Hausdorff co-connected spaces be called co-continua? Let us present some facts about them.

Proposition. *Let X be a metrizable realcompact co-connected space. Then its β-compactification βX is also co-connected.*

P r o o f. Let a metrizable realcompact co-connected space X be given.

a) Let $f : \beta X \times \beta X \to \beta X$ be a continuous map. We prove that either f is constant or it maps $X \times X$ into X. Thus, let us suppose that there exists a point $(x_0, x_1) \in X \times X$ such that $z = f(x_0, x_1)$ is in $\beta X \setminus X$. We prove that f is constant. Let $\{x^{(n)}\}$ be a sequence of elements of $X \times X$ which converges to $x = (x_0, x_1)$. Then $\{f(x^{(n)})\}$ converges to $z \in \beta X \setminus X$. Since X is realcompact, each non-discrete closed set in $\beta X \setminus X$ has the cardinality at least $2^{\aleph_0}$ (see e.g. [4], Th. 9.11). Hence there exists n_0 such that $f(x^{(n)}) = z$ for all $n \geq n_0$, so that $f^{-1}(z)$ contains an open neighbourhood of x in $X \times X$. Thus, $X \times X \cap f^{-1}(z)$ must be open in $X \times X$. However, it must be also closed in $X \times X$. Since co-connected space is always connected, $X \times X$ is connected, hence $f \restriction X \times X$ is constant, hence f is constant.

b) If $f : \beta X \times \beta X \to \beta X$ si constant, then it factors through the both projections. If it maps $X \times X$ into X, then $f \restriction X \times X$ factors through a projection $\pi_i^{(2)}$, $i \in 2$, i.e. $f \restriction X \times X = g \circ \pi_i^{(2)}$ for some continuous $g : X \to X$. Then f factors through the i-th projection of $\beta X \times \beta X$, evidently. $\square$

Remark. Let L be a non-empty set of a nonmeasurable cardinality. Then there exists a compact Hausdorff co-connected space X such that its non-constant continuous selfmaps form a monoid isomorphic to the free monoid L^*.

In fact, by the Theorem, there exists a metrizable co-connected space X with $\mathrm{card} X = 2^{\aleph_0} + \mathrm{card} L$ such that all its nonconstant continuous selfmaps form a monoid isomorphic to L^*. Since $\mathrm{card} X$ is nonmeasurable, it is realcompact (see e.g. [4], 15.24), hence βX is also co-connected, by the above Proposition. And every continuous selfmap $\beta X \to \beta X$ is either constant or sends X into X (the reasonning is just the same as in the proof of the above Proposition).

12. Problem. By [9], for *every* monoid M there exist a metrizable space X and a compact Hausdorff space Y such that all the nonconstant continuous selfmaps of X form a monoid isomorphic to M and the same statement is true for Y.

Question. For what monoids M such space X (or Y) can be also co-connected?

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