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DENSE CONTINUITY AND SELECTIONS OF SET-VALUED MAPPINGS

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Abstract. A theorem proved by Fort in 1951 says that an upper or lower semi-continuous set-valued mapping from a Baire space A into non-empty compact subsets of a metric space is both lower and upper semi-continuous at the points of a dense G_δ-subset of A.

In this paper we show that the conclusion of Fort's theorem holds under the weaker hypothesis of either upper or lower quasi-continuity. The existence of densely defined continuous selections for lower quasi-continuous mappings is also proved.

1. Introduction and preliminaries. This paper is devoted to the study of some generalizations of the following two well-known notions of continuity for set-valued mappings.

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A set-valued mapping Φ from a topological space A, into subsets of a topological space X is said to be *upper semi-continuous* at $t_0 \in A$ if, for each open subset W of X containing $\Phi(t_0)$, there exists an open neighbourhood U of t_0 such that $\Phi(U) \subset W$, and *lower semi-continuous* at $t_0 \in A$ if for each open subset W of X with $W \cap \Phi(t_0) \neq \emptyset$, there exists an open neighbourhood U of t_0 such that $\Phi(t) \cap W \neq \emptyset$ for each $t \in U$. If Φ is both upper and lower semi-continuous at a point $t_0 \in A$, then we simply say that Φ is *continuous* at $t_0 \in A$.

Given a set-valued mapping $\Phi : A \to X$ defined in the Baire space A the following two questions are of some interest:

1) When does there exist a dense G_δ-subset A_1 of A such that Φ is upper (lower) semi-continuous at the points of A_1?

2) When do there exist a dense G_δ-subset A_1 of A and a continuous (single-valued) mapping $\phi : A_1 \to X$ such that ϕ is a selection of Φ on A_1, i.e. $\phi(t) \in \Phi(t)$ for every $t \in A_1$?

Starting with the papers of Hill [17] and Kuratowski [25], the first question has been given a lot of attention in many papers-see e.g. [2, 3, 4, 9, 10, 11, 12, 15, 31, 18, 19, 20, 21, 26, 27, 30] just to mention a few. We focus here our attention on the original results of Fort [12] which assert that a lower (resp. upper) semi-continuous set-valued mapping Φ from the Baire space A to the metric space X which is compact-valued must be upper (resp. lower) semi-continuous outside some subset of the first Baire category in A.

Our aim here is to investigate what kind of weaker notions of continuity-like properties of set-valued mappings could lead to the same conclusions as in Fort's theorems. More precisely, we focus our attention on the following relaxation of semi-continuity: A set-valued mapping Φ from a topological space A into subsets of a topological space X is *upper quasi-continuous* at $t_0 \in A$ if for each open subset W of X, containing $\Phi(t_0)$, and each open neighbourhood U of t_0, there exists a non-empty open subset V of U such that $\Phi(V) \subset W$, and Φ is *lower quasi-continuous* at $t_0 \in A$ if for each open subset W of X with $\Phi(t_0) \cap W \neq \emptyset$ and each open neighbourhood U of t_0 there exists a non-empty open subset V of U such that $\Phi(t) \cap W \neq \emptyset$ for every $t \in V$. The origins of this notion (for real-valued functions) go back to Voleterra (see [1]). The corresponding notion for set-valued mappings (sometimes under different name) has been studied in many papers, see e.g. [32, 33, 10, 26, 27, 28, 13] and reference therein.

These notions can equivalently be defined as follows: $\Phi : A \to X$ is upper (resp. lower) quasi-continuous at $t_0 \in A$ if for every open $W \subset X$ with $\Phi(t_0) \subset W$ (resp. $\Phi(t_0) \cap W \neq \emptyset$) there exists an open U in A such that $t_0 \in \overline{U}$ and $\Phi(t) \subset W$ (resp. $\Phi(t) \cap W \neq \emptyset$) whenever $t \in U$. Here $\overline{U}$ is the closure of U in A.

For a set-valued mapping which is single-valued everywhere on its domain, we see that the notions of upper and lower quasi-continuity coincide. In this case, we simply say that the set-valued mapping is *quasi-continuous* at t_0. Simple examples show that a quasi-continuous single-valued mapping need not be continuous.

Of course, every upper (resp. lower) semi-continuous mapping is upper (resp. lower) quasi-continuous. We mention also that another class of mappings which share these properties and which have prominent role in the study of differentiability of convex functions in Banach spaces is the class of the so called minimal mappings (see Section 3 for the definitions).

As it was mentioned above our goal is to see whether the conclusions of Fort's theorem remain valid under these weaker continuity-like properties of the set-valued mapping Φ. The following theorem can be derived from [26], Theorem 2.1, Theorem 2.2 (see also [13]).

Theorem 1.1. *An upper (resp. lower) quasi-continuous set-valued mapping Φ from a Baire space A into non-empty (resp. non-empty compact) subsets of a separable metric space X is lower (resp. upper) semi-continuous at the points of some dense G_δ-subset of A.*

For the case when Φ is compact-valued the same conclusions as above can be derived without assuming the separability of X. Namely, the following result holds which is a consequence from [27], Theorem 1 and 2.

Theorem 1.2. *An upper (resp. lower) quasi-continuous set-valued mapping Φ from a Baire space A into non-empty compact subsets of a metric space X is lower (resp. upper) semi-continuous at the points of some dense G_δ-subset of A.*

The "non-brackets" part of Theorem 1.1 is no longer valid without the separability of X as it is seen from Example 4.1. The same example shows that, in general, Theorem 1.2 is not true if the values of Φ are assumed only closed. However, as it will be seen later the above theorems remain true even for non-separable metric spaces under some additional (but weaker than those in Theorem 1.2) assumptions on the images. Moreover, sometimes upper quasi-continuity (or lower quasi-continuity) implies both upper and lower semi-continuity generically, i.e. at the points of a dense G_δ-subset of the domain (see Corollary 2.3 and 2.9 below).

Now, we turn to the second question posed at the beginning of this Section, namely to the question of existence of densely defined selections of a given set-valued mapping. This question was paid much attention in a series of papers [5, 6, 7, 8, 24, 29]. Giles and Bartlett used in [13] the notion of lower quasi-continuity to prove the following theorem.

Theorem 1.3. *Let Φ be a lower quasi-continuous non-empty closed-valued mapping between the Baire space A and the complete metric space X. Then a dense G_δ-subset A_1 of A and a continuous single-valued mapping $\phi : A_1 \to X$ exist such that ϕ is a selection of Φ on A_1.*

We show here that this result is true also in many cases (some of them rather important) when X is not completely metrizable (Theorem 3.3).

The remainder of the paper is organized as follows: Section 2 is devoted to the extension of the results from Theorem 1.1 and Theorem 1.2 to the case when the range space is an arbitrary metric space and the set-valued mapping is not necessarily everywhere compact-valued. In Section 3 we discuss the existence of densely defined selections for lower quasi-continuous mappings and their possible applications. Finally, in Section 4 we give several counter examples and applications.

Notations

In a topological space (X, τ) we shall denote by $\mathrm{int}(A)$ the interior of a set A and by $\overline{A}$ the closure of A in X.

In a metric space (X, d) we denote by $B[x, r]$ the closed ball $\{y \in X : d(x, y) \leq r\}$ and by $B(x, r)$ the open ball $\{y \in X : d(x, y) < r\}$.

For a non-empty subset C of a metric space (X, d), $B(C, \varepsilon)$ designates the set $\cup\{B(y, \varepsilon) : y \in C\}$ and $\mathrm{diam}(C) = \sup\{d(x, y) : x, y \in C\}$ stands for the diameter of the set C .

2. Metric versions of upper and lower quasi-continuity of set-valued mappings. In this section we extend the results contained in [12, 26, 27] (see also [13]) by showing (as a consequence from more general results) that an upper (lower) quasi-continuous set-valued mapping from a Baire space into non-empty subsets of a metric space which is generically compact-valued is continuous on a dense G_δ-subset of its domain.

Before stating the first result we recall two notions of semi-continuity already considered in [18, 19] and then we give their quasi-versions. A set-valued mapping Φ from a topological space A into non-empty subsets of a metric space (X, d) is *metric upper semi-continuous* at $t_0 \in A$ if for each $\varepsilon > 0$ there exists an open neighbourhood U of t_0 such that $\Phi(U) \subset B(\Phi(t_0), \varepsilon)$. It is easily verified that if $\Phi(t_0)$ is compact, then Φ is metric upper semi-continuous at t_0 if and only if Φ is upper semi-continuous at t_0. However, we see from Example 4.3 that in general metric upper semi-continuity is a weaker notion than upper semi-continuity. Generalizing the definition above we say that Φ is *metric upper quasi-continuous* at $t_0 \in A$, if for each $\varepsilon > 0$ and each open neighbourhood U of t_0, there exists a non-empty open subset V of U such that $\Phi(V) \subset B(\Phi(t_0), \varepsilon)$. On the other hand, the set-valued mapping Φ is said to be *metric lower semi-continuous* at $t_0 \in A$ if for each $\varepsilon > 0$ there exists an open neighbourhood U of t_0 such that $\Phi(t_0) \subset B(\Phi(t), \varepsilon)$ for every $t \in U$. We note that, if $\Phi(t_0)$ is totally bounded, then Φ is lower semi-continuous at t_0 if and only if Φ is metric lower semi-continuous at t_0. However, Example 4.2 shows that in general lower semi-continuity is a weaker notion than metric lower semi-continuity. If Φ is both metric upper semi-continuous and metric lower semi-continuous at some $t_0 \in A$ then we say that Φ is *metric continuous* at t_0. We mention also the quasi version of metric lower semi-continuity: the set-valued mapping Φ is said to be *metric lower quasi-continuous* at $t_0 \in A$ if for each $\varepsilon > 0$ and each open neighbourhood U of t_0 there exists a non-empty open $V \subset U$ such that $\Phi(t_0) \subset B(\Phi(t), \varepsilon)$ for every $t \in V$.

Below are two diagrams showing the relations between the introduced notions. Let us point out (as mentioned above; see also the examples in Section 4) that neither of the implications in the diagrams is reversible without additional assumptions.

upper semi-continuous	$\Rightarrow$	metric upper semi-continuous	metric lower semi-continuous	$\Rightarrow$	lower semi-continuous
$\Downarrow$		$\Downarrow$	$\Downarrow$		$\Downarrow$
upper quasi-continuous	$\Rightarrow$	metric upper quasi-continuous	metric lower quasi-continuous	$\Rightarrow$	lower quasi-continuous

We give here equivalent definitions of the above metric notions. For a subset C of the metric space (X, d) we shall denote as usual by $d(x, C) := \inf\{d(x, y) : y \in C\}$, $x \in X$, the distance function generated by C. Given non-empty subsets C, D of X the *excess* of C to D is denoted by $e(C,D) := \sup\{d(x,D) : x \in C\}$. Observe that e is not symmetric and that the following triangle inequality holds: for each non-empty subsets C, D, H of X it is true that $e(C,D) \leq e(C,H) + e(H,D)$. The following is obvious: the mapping $\Phi : A \to X$ is metric upper semi-continuous at $t_0 \in A$ if, and only if, for every $\varepsilon > 0$ there exists an open U such that $t_0 \in U$ and $e(\Phi(t), \Phi(t_0)) < \varepsilon$ for every $t \in U$; Φ is metric upper quasi-continuous at $t_0 \in A$ if, and only if, for every open U containing t_0 and $\varepsilon > 0$ there exists an open $V \subset U$ such that $e(\Phi(t), \Phi(t_0)) < \varepsilon$ for every $t \in V$; the mapping $\Phi : A \to X$ is metric lower semi-continuous at $t_0 \in A$ if, and only if, for every $\varepsilon > 0$ there exists an open U such that $t_0 \in U$ and $e(\Phi(t_0), \Phi(t)) < \varepsilon$ for every $t \in U$; $\Phi : A \to X$ is metric lower quasi-continuous at $t_0 \in A$ if, and only if, for every $\varepsilon > 0$ and every open set U containing t_0 there exists a non-empty open $V \subset U$ such that $e(\Phi(t_0), \Phi(t)) < \varepsilon$ for every $t \in V$. We mention that another notion of upper semi-continuity of a set-valued mapping when the range space is metric is considered in [14].

Now, we are ready to formulate one of our main results. Its proof is close in spirit to the proof of Theorem 1 from [19] (see also Theorem 3 from [18]) where it is proved that every metric upper semi-continuous (resp. metric lower semi-continuous) set-valued mapping from a complete metric space A into X must be metric lower semi-continuous (resp. metric upper semi-continuous) at the points of a dense G_δ-subset of A. The "quasi" versions of these results are not so symmetric.

Theorem 2.1. *A metric upper quasi-continuous set-valued mapping Φ from a Baire space A into non-empty subsets of a metric space X, whose images on a dense subset D of A are totally bounded, is both metric upper semi-continuous and metric lower semi-continuous (i.e. metric continuous) at the points of a dense G_δ-subset of A.*

P r o o f. For each $\varepsilon > 0$ consider the set

$$O_\varepsilon := \cup\{U : U \text{ is an open subset of } A \text{ and } e(\Phi(t'), \Phi(t'')) < \varepsilon \text{ for every couple } t', t'' \in U\}.$$

Clearly O_ε is open. We shall show that it is also dense in A. To this end, suppose the contrary and let V_0 be a non-empty open subset of A such that $O_\varepsilon \cap V_0 = \emptyset$. Let $\delta > 0$ be such that $\delta < \varepsilon/4$ and choose $t_0 \in D \cap V_0$. Since Φ is metric upper quasi-continuous at t_0 there is some non-empty open $V_1 \subset V_0$

such that $e(\Phi(t), \Phi(t_0)) < \delta$ for every $t \in V_1$. It is not difficult to see that for some $t_1 \in V_1$ we have $e(\Phi(t_0), \Phi(t_1)) > \varepsilon - \delta$. Indeed, if for all $t' \in V_1$ we have $e(\Phi(t_0), \Phi(t')) \leq \varepsilon - \delta$ then for all $t', t'' \in V_1 \subset V_0$

$$e(\Phi(t'), \Phi(t'')) < e(\Phi(t'), \Phi(t_0)) + e(\Phi(t_0), \Phi(t'')) < \delta + \varepsilon - \delta = \varepsilon$$

and we would get that $V_1 \subset O_\varepsilon$. This contradiction shows that there exists $t_1 \in V_1$ with $e(\Phi(t_0), \Phi(t_1)) > \varepsilon - \delta$. Since Φ is metric upper quasi-continuous at t_1 there is a non-empty open $V_2 \subset V_1$ so that $e(\Phi(t), \Phi(t_1)) < \delta$ for every $t \in V_2$. As above one sees that there exists $t_2 \in V_2$ with $e(\Phi(t_1), \Phi(t_2)) > \varepsilon - \delta$. Proceeding by induction we construct sequences of open sets $(V_i)_{i \geq 0}$ and of points $(t_i)_{i \geq 0}$ such that for all $i \geq 0$ one has:

(i) $V_{i+1} \subset V_i$;

(ii) $t_i \in V_i$;

(iii) $e(\Phi(t), \Phi(t_i)) < \delta$ for every $t \in V_{i+1}$;

(iv) $e(\Phi(t_i), \Phi(t_{i+1})) > \varepsilon - \delta$.

The last inequality implies that for every $i = 1, 2, \ldots$ there exists some $y_i \in \Phi(t_i)$ such that $d(y_i, \Phi(t_{i+1})) > \varepsilon - \delta$. Moreover, by **(i)-(iii)**, for every $j \geq i+1$ we have $\Phi(t) \subset B(\Phi(t_i), \delta)$ for every $t \in V_j$. Hence, for every $j \geq i+1$ we get $d(y_i, y_j) \geq d(y_i, \Phi(t_j)) \geq \varepsilon - 2\delta$. On the other hand, $t_i \in V_i \subset V_1$ and therefore $y_i \in \Phi(t_i) \subset B(\Phi(t_0), \delta)$. For each y_i, $i = 1, 2, \ldots$, select some $x_i \in \Phi(t_0)$ so that $d(x_i, y_i) < \delta$. Then, for every $j \geq i+1$ we have

$$d(x_i, x_j) \geq d(y_i, y_j) - d(x_i, y_i) - d(x_j, y_j) > \varepsilon - 4\delta > 0$$

This contradicts total boundedness of $\Phi(t_0)$. Hence, O_ε is dense in A.

Put now $G := \cap_{n \geq 1} O_{1/n}$. The set G is dense G_δ in A. By the remarks before the theorem, Φ is both metric upper semi-continuous and metric lower semi-continuous at the points of G. The proof is completed. $\square$

Remark 2.2. We note that, in general, metric upper quasi-continuity does not by itself necessarily imply metric upper semi-continuity anywhere (see Example 4.1). We note also that the conclusion of Theorem 2.1 cannot be strengthened to " Φ is continuous at the points of a dense G_δ-subset of A" (see Example 4.3).

The next corollary generalizes several results (see [12], [15]) previously known for upper semi-continuous mappings. The partial case when Φ is upper semi-continuous and compact-valued generalizes one of the well-known theorems of Fort (see [12]). In the special situation when X is a separable metric space and Φ is everywhere compact-valued the same conclusion as below is a consequence from results in [10], Theorems 15, 16, and Theorem 1.1 above from [26].

Corollary 2.3. *A metric upper quasi-continuous set-valued mapping Φ from a Baire space A into non-empty subsets of a metric space X, whose images on a dense G_δ-subset of A are compact, is continuous at the points of some dense G_δ-subset of A. In particular, if Φ is upper quasi-continuous and compact-valued at every point of A, then Φ is continuous at the points of some dense G_δ-subset of A.*

Proof. The result follows from the observation mentioned above that, if $\Phi(t_0)$ is compact and Φ is metric upper semi-continuous at t_0, then Φ is upper semi-continuous at t_0. $\square$

Remark 2.4. We see from Example 4.3 that a set-valued mapping Φ from a Baire space A into non-empty (not necessarily compact) subsets of a separable metric space X, which is upper quasi-continuous on A may be nowhere upper semi-continuous on A.

We would like to present here one way in which the result of Theorem 2.1 may be partially improved. In the next result we relax the requirement on the images of the set-valued mapping and deduce only generic lower semi-continuity (not metric lower semi-continuity).

Theorem 2.5. *A metric upper quasi-continuous set-valued mapping Φ from a Baire space A into non-empty subsets of a metric space X, whose images on a dense subset D of A are separable, is lower semi-continuous at the points of a dense G_δ-subset of A.*

Proof. For each $n \in \mathbb{N}$ let $\gamma_n := \{(O, \mathcal{B})\}$ be a maximal family of couples $(O, \mathcal{B})$ with the following properties:

(a) for every $(O, \mathcal{B}) \in \gamma_n$, O is a non-empty open subset of A and $\mathcal{B}$ is a countable family of closed balls in X of radius $1/n$;

(b) for every $(O, \mathcal{B}) \in \gamma_n$ one has $\Phi(O) \subset \cup\{B : B \in \mathcal{B}\}$;

(c) the family $\{O : (O, \mathcal{B}) \in \gamma_n \text{ for some } \mathcal{B}\}$ is pair-wise disjoint.

We claim that the open set $O_n := \cup\{O : (O, \mathcal{B}) \in \gamma_n \text{ for some } \mathcal{B}\}$ is dense in A. Indeed, suppose the contrary. Then for some non-empty open $U \subset A$ we have $O_n \cap U = \emptyset$. Take some $t_0 \in D \cap U$ and let $\{x_k : k \in \mathbb{N}\}$, be a countable dense subset of $\Phi(t_0)$. By metric upper quasi-continuity of Φ we get some open $O \subset U$ with $\Phi(O) \subset \cup_{k \geq 1} B[x_k, 1/n]$. Now, the family $\gamma_n' := \gamma_n \cup \{(O, \mathcal{B})\}$ where $\mathcal{B} := \{B[x_k, 1/n] : k \in \mathbb{N}\}$ is strictly larger than γ_n and satisfies **(a)-(c)** above. This contradicts the maximality of γ_n and shows that the set O_n is dense in A.

Now fix some $n \in \mathbb{N}$ and $(O, \mathcal{B}) \in \gamma_n$. Let $\mathcal{B} = \{B_k : k \in \mathbb{N}\}$, B_k-closed balls with radius $1/n$. For each $k \in \mathbb{N}$ let $H_k := \{t \in O : \Phi(t) \cap B_k \neq \emptyset\}$ and put $F_k := H_k \setminus \mathrm{int}\overline{H_k}$. Obviously, for every $k \in \mathbb{N}$ the set F_k is nowhere dense in O and hence the set $\cup_{k \geq 1} F_k$ is of the first Baire category in O. Now we designate $G(O, \mathcal{B}) := O \setminus \cup_{k \geq 1} F_k$ and let $G_n := \cup\{G(O, \mathcal{B}) : (O, \mathcal{B}) \in \gamma_n\}$. Because of **(c)** above and the fact that O_n is dense in A we get that the set G_n is residual in A and hence $G := \cap_{n \geq 1} G_n$ is residual in A as well.

We claim that Φ is lower semi-continuous at the points of G. To prove this, consider $t_0 \in G$ and let W be an open subset of X such that $\Phi(t_0) \cap W \neq \emptyset$. Take some $y \in \Phi(t_0)$ and $n \in \mathbb{N}$ such that $B[y, 3/n] \subset W$. Since $t_0 \in G$ we have some uniquely determined $(O, \mathcal{B}) \in \gamma_n$ with $t_0 \in G(O, \mathcal{B})$. Hence for some $B_k \in \mathcal{B}$ we have $y \in B_k \subset B[y, 2/n]$. On the other hand, $t_0 \in \mathrm{int}\overline{H_k}$. We will show that for every $t \in \mathrm{int}\overline{H_k}$ we have $\Phi(t) \cap B[y, 3/n] \neq \emptyset$. Indeed, suppose that for some $t' \in \mathrm{int}\overline{H_k}$ we have $\Phi(t') \cap B[y, 3/n] = \emptyset$. This means that $B(\Phi(t'), 1/n) \cap B[y, 2/n] = \emptyset$. In particular $B(\Phi(t'), 1/n) \cap B_k = \emptyset$. By metric upper quasi-continuity of Φ there exists a non-empty open $U \subset \mathrm{int}\overline{H_k}$ such that $\Phi(U) \subset B(\Phi(t'), 1/n)$. This entails $\Phi(U) \cap B_k = \emptyset$. This is a contradiction since $U \cap H_k \neq \emptyset$. Hence Φ is lower semi-continuous at t_0. The proof is completed. $\square$

Remark 2.6. In example 4.2, an upper quasi-continuous set-valued mapping Φ is given from a Baire space into a metric space whose images are everywhere separable, but which is nowhere metric upper semi-continuous. The same example shows that the conclusion of Theorem 2.5 cannot be strengthened to get generic metric lower semi-continuity. Therefore, we see that Theorem 2.5 cannot be extended to give a true improvement of Theorem 2.1.

The next part of this section is devoted to showing that a lower quasi-continuous set-valued mapping Φ from a Baire space A into non-empty compact subsets of a metric space X is continuous on a dense G_δ-subset of A. We begin by considering a function defined on the subsets of a metric space (X, d) (see e.g. [12]).

For each $\varepsilon > 0$ consider the function m_ε defined on the subsets C of X by:

$$m_\varepsilon(C) := \min\{n \in \mathbb{N} : C \subset \cup_{k=1}^{n} B(x_k, \varepsilon) \text{ and } \{x_1, x_2, \dots, x_n\} \subset C\}$$

when C can be covered by a finite family of balls of radius ε, otherwise $m_\varepsilon(C) := \infty$.

Observe that if C is totally bounded then $m_\varepsilon(C)$ is finite for every $\varepsilon > 0$.

Theorem 2.7. *A lower quasi-continuous set-valued mapping Φ from a Baire space A into non-empty subsets of a metric space X, whose images on an everywhere second category subset D of A are totally bounded, is both metric upper semi-continuous and metric lower semi-continuous (i.e. metric continuous) at the points of a dense G_δ-subset of A.*

Proof. Given $\varepsilon > 0$ consider the set

$$\begin{aligned} O_\varepsilon \quad := \quad & \cup\{U : U \text{ is an open subsets of } A \text{ and } e(\Phi(t'), \Phi(t'')) < \varepsilon \\ & \text{for every } t', t'' \in U\}. \end{aligned}$$

Clearly O_ε is open; we will show that it is also dense in A. This will complete the proof of the theorem since, as it was already mentioned, Φ is metric continuous at the points of $\cap_{n\geq 1} O_{1/n}$.

For the purposes of obtaining a contradiction, suppose O_ε is not dense in A, that is, suppose that there exists a non-empty open subset W of A such that $O_\varepsilon \cap W = \emptyset$. Consider the following decomposition of the set of points in A where Φ has totally bounded images: $D := \cup_{k\geq 1} D_k$ where

$$D_k := \{t \in D : m_{\varepsilon/12}(\Phi(t)) = k\}, \quad k := 1, 2, \dots$$

I.e., D_k consists of those points t of D whose images under Φ can be covered by k balls with centers at $\Phi(t)$ and radii $\varepsilon/12$ and this k is minimal with this property.

By assumption D is of the second category in W. Hence there exists $k_0 \in \mathbb{N}$ such that $U_1 := W \cap \operatorname{int}\overline{D_{k_0}} \neq \emptyset$. Take $t_1 \in U_1$ and $x_1 \in \Phi(t_1)$. Now, since Φ is lower quasi-continuous at t_1 there exists a non-empty open subset V_1 of U_1 such that $x_1 \in B(\Phi(t), \varepsilon/12)$ (equivalently $\Phi(t) \cap B(x_1, \varepsilon/12) \neq \emptyset$) for each $t \in V_1$. On the other hand, $\Phi(V_1) \not\subset B[x_1, \varepsilon/3]$ since otherwise we would have $e(\Phi(t'), \Phi(t'')) < \varepsilon$ for every pair $t', t'' \in V_1$ which contradicts $W \cap O_\varepsilon = \emptyset$. Hence for some $t' \in V_1$ we have

$$\Phi(t') \cap (X \setminus B[x_1, \varepsilon/3]) \neq \emptyset.$$

Again by lower quasi-continuity of Φ there exists some non-empty open subset $U_2 \subset V_1$ such that

$$\Phi(t) \cap (X \setminus B[x_1, \varepsilon/3]) \neq \emptyset$$

for every $t \in U_2$. Fix some $t_2 \in U_2$ and some $x_2 \in \Phi(t_2)$ such that $x_2 \notin B[x_1, \varepsilon/3]$.

The mapping Φ is lower quasi-continuous at t_2, so there exists a non-empty open subset V_2 of U_2 such that $x_2 \in B(\Phi(t), \varepsilon/12)$ for each $t \in V_2$. Observe that, in fact, we have that $\{x_1, x_2\} \subset B(\Phi(t), \varepsilon/12)$ for each $t \in V_2$. Now, we claim that $\Phi(V_2) \not\subset \cup_{k=1}^{2} B[x_k, \varepsilon/3]$. Indeed, suppose the contrary and let $t', t'' \in V_2$. Let $x \in \Phi(t')$. Then $x \in B[x_i, \varepsilon/3]$ for some $i = 1, 2$. On the other hand, $\Phi(t'') \cap B(x_i, \varepsilon/12) \neq \emptyset$. This entails $d(x, \Phi(t'')) < (2/3)\varepsilon$. Hence, $e(\Phi(t'), \Phi(t'')) < \varepsilon$ which contradicts the assumption $W \cap O_\varepsilon = \emptyset$. Consequently, $\Phi(t') \setminus \cup_{i=1}^{2} B[x_i, \varepsilon/3] \neq \emptyset$. The set $\cup_{i=1}^{2} B[x_i, \varepsilon/3]$ is closed, therefore by lower quasi-continuity of Φ there exists a non-empty open $U_3 \subset V_2$ such that

$$\Phi(t) \setminus \cup_{i=1}^{2} B[x_i, \varepsilon/3] \neq \emptyset$$

for every $t \in U_3$. Let $t_3 \in U_3$ and $x_3 \in \Phi(t_3)$ be such that $x_3 \notin \cup_{i=1}^{2} B[x_i, \varepsilon/3]$.

Proceeding inductively, on the $k_0 + 1$ step we get a finite sequence of open sets of A, $U_{k_0+1} \subset V_{k_0} \subset U_{k_0} \subset \cdots \subset V_1 \subset U_1 = W \cap \operatorname{int}\overline{D_{k_0}}$ and a finite sequence of points in X $x_1, x_2, \ldots, x_{k_0}$ so that

(i) $d(x_n, x_m) > \varepsilon/3$ provided $m \neq n$, $m, n = 1, 2, \ldots, k_0$;

(ii) $\{x_1, x_2, \ldots, x_{k_0}\} \subset B(\Phi(t), \varepsilon/12)$ for each $t \in V_{k_0}$;

(iii) $\Phi(t) \setminus \cup_{i=1}^{k_0} B[x_i, \varepsilon/3] \neq \emptyset$ for every $t \in U_{k_0+1}$.

Since $U_{k_0+1} \subset \operatorname{int}\overline{D_{k_0}}$ there exists $t_0 \in U_{k_0+1} \cap D_{k_0}$. Let $\Phi(t_0) \subset \cup_{i=1}^{k_0} B(y_i, \varepsilon/12)$ for some $y_1, y_2, \ldots, y_{k_0} \in \Phi(t_0)$.

By **(ii)** above, for every $n = 1, 2, \ldots, k_0$ there is $z_n \in \Phi(t_0)$ such that $d(z_n, x_n) < \varepsilon/12$. On the other hand, for every such n there exists $y_{i(n)}$ with $d(z_n, y_{i(n)}) < \varepsilon/12$. Hence, $d(x_n, y_{i(n)}) \leq d(x_n, z_n) + d(z_n, y_{i(n)}) \leq \varepsilon/6$. This automatically implies that if $n \neq m$, $n, m = 1, 2, \ldots, k_0$, then $y_{i(n)} \neq y_{i(m)}$ because otherwise $d(x_n, x_m) \leq \varepsilon/3$ contradicting **(i)**. Hence, without loss of generality we may assume that $y_{i(n)} = y_n$, $n = 1, 2, \ldots, k_0$. Therefore, for every such n we have $B(y_n, \varepsilon/12) \subset B[x_n, \varepsilon/3]$. This implies $\Phi(t_0) \subset \cup_{n=1}^{k_0} B[x_n, \varepsilon/3]$ and the last inclusion contradicts **(iii)** above. This contradiction completes the proof. $\square$

Remark 2.8. We remark here that (as Example 4.5 shows) only the density of the set D of points where the mapping Φ is totally bounded is not enough to deduce the conclusion of Theorem 2.7.

We conclude this section by the following obvious corollary of the previous theorem. The partial case when Φ is lower semi-continuous and compact-valued is again (as in Corollary 2.3) one of Fort's theorems [12]. As before Corollary 2.3, we mention also that in the special situation when X is a separable metric space and Φ is everywhere compact-valued the same conclusion as below is a consequence from Theorems 15, 16 in [10] and Theorem 1.1 above (from [26]).

Corollary 2.9. *A lower quasi-continuous set-valued mapping Φ from a Baire space A into non-empty subsets of a metric space X, whose images on a dense G_δ-subset of A are compact, is continuous at the points of a dense G_δ-subset of A. In particular, if Φ is lower quasi-continuous and compact-valued at every point of A, then it is continuous at the points of some dense G_δ-subset of A.*

3. Densely defined selections. In [8, 24] a general approach was developed to assure the existence of densely defined (set-valued or single-valued) continuous selections for a given set-valued mapping $\Phi : A \to X$ where A was a Baire space and X a topological space (not necessarily metrizable). Implicit results of similar type had been previously obtained in [5, 6, 7] and the special case when Φ is the inverse of a usual single-valued mapping had been considered by E. Michael [29].

We briefly sketch the general setting in [8] and give one sample result. The mapping $\Phi : A \to X$ between the topological spaces A and X is allowed to have empty values and its *domain* is $\mathrm{Dom}(\Phi) := \{t \in A : \Phi(t) \neq \emptyset\}$. The graph of Φ is the set $\mathrm{Gr}(\Phi) := \{(t, x) \in A \times X : x \in \Phi(t)\}$. The mapping Φ is said to be *lower demi-continuous* in A (see [8]) if for every open set $V \subset X$ the set $\mathrm{int}\overline{\Phi^{-1}(V)}$ is dense in $\overline{\Phi^{-1}(V)}$. Here $\Phi^{-1}(V) := \{t \in A : \Phi(t) \cap V \neq \emptyset\}$. An equivalent "local definition" is the following: Φ is lower demi-continuous in A if, and only if, it is lower demi-continuous at any $t_0 \in A$ by which we mean that for every open $W \subset X$ with $\Phi(t_0) \cap W \neq \emptyset$ there exists an open U of A such that $t_0 \in \overline{U}$ and the set $\{t \in U : \Phi(t) \cap W \neq \emptyset\}$ is dense in U. Obviously, every lower quasi-continuous mapping is lower demi-continuous. The converse is not true.

Finally, a relation between the mapping Φ and a subspace $X_1 \subset X$ was also introduced in [8]. Namely, Φ was said to *embrace* X_1 if for every open set

$W \subset X$ which contains X_1 the set $\{(t,x) \in \mathrm{Gr}(\Phi) : x \in W\}$ is dense in $\mathrm{Gr}(\Phi)$. The following is a result from [8] (Proposition 3.3):

Proposition 3.1. *Let $\Phi : A \to X$ be a set-valued mapping and X be regular. Then Φ embraces $X_1 \subset X$ if, and only if, for all open sets $V \subset X$ and $V_\lambda \subset X, \lambda \in \Lambda$, for which $V \cap X_1 = \cup\{V_\lambda \cap X_1 : \lambda \in \Lambda\}$, the set $\cup\{\Phi^{-1}(V_\lambda) : \lambda \in \Lambda\}$ is dense in $\Phi^{-1}(V)$.*

Two sufficient conditions for the mapping Φ to embrace X_1 are the following (see [8], Proposition 3.5):

i) $\Phi(A) \subset X_1$; and
ii) X_1 is dense in X and the mapping Φ is demi-open.

The mapping Φ is *demi-open* in A (see [16]) if for every open set $U \subset A$ the set $\mathrm{int}\overline{\Phi(U)}$ is dense in $\overline{\Phi(U)}$. Φ is demi-open if, and only if, the mapping $\Phi^{-1} : \overline{\Phi(A)}^X \to A$ is lower demi-continuous ([8], Proposition 3.2).

The following is a sample result from [8]. For similar result when X is a separable metric space and Φ is compact-valued see [28].

Theorem 3.2 ([8], Theorem 4.7)**.** *Let $\Phi : A \to X$ be a lower demi-continuous mapping with closed graph and dense domain from the Baire space A into the regular space X. Suppose in addition that X contains a completely metrizable subspace X_1 which is embraced by Φ. Then there exist a dense G_δ-subset A_1 of A and a continuous single-valued mapping $\phi : A_1 \to X_1$ such that $A_1 \subset \mathrm{Dom}(\Phi)$ and ϕ is a selection of Φ on A_1.*

The formulated Theorem 1.3 of Giles and Bartlett [13] and Theorem 3.2 have identical conclusions but different assumptions. In Theorem 1.3 no requirements are imposed on the graph of Φ. On the other hand, lower quasi-continuity of Φ is a stronger assumption than the lower demi-continuity of Φ in Theorem 3.2. Therefore, neither of the two results can be derived from the other. Our next result shows that Theorem 1.3 remains valid for a class of spaces X larger than the class of complete metric spaces.

Theorem 3.3. *Let Φ be a lower quasi-continuous non-empty closed-valued mapping between the Baire space A and the regular space X. Suppose X contains a completely metrizable subspace X_1 which is embraced by Φ. Then there exist a dense G_δ-subset A_1 of A and a continuous single-valued mapping $\phi : A_1 \to X_1$ such that ϕ is a selection of Φ on A_1.*

As we mentioned above, two important cases when Φ embraces X_1 are when $\Phi(A) \subset X_1$ or when X_1 is dense in X and Φ is demi-open.

Proof. Let d be a complete metric in X_1 which is compatible with the inherited topology from X. Since Φ embraces X_1 then $\Phi(A) \subset \overline{X_1}$ (see Proposition 3.4 in [8]). Hence, it is no loss of generality to assume that X_1 is dense in X.

The pair (U, V) will be called *admissible* if:

1) $U \subset A$ and $V \subset X$ are non-empty open subsets of A and X respectively;
2) $\Phi(t) \cap V \neq \emptyset$ for every $t \in U$.

Let $\{\gamma_n\}_{n \geq 0}$, where $\gamma_0 = \{(A, X)\}$, be a sequence of families of admissible pairs which is maximal with respect to the following properties:

a) for every n the family $\{U : (U, V) \in \gamma_n \text{ for some } V\}$ is pair-wise disjoint;
b) if $(U, V) \in \gamma_n$ then $\mathrm{diam}(V \cap X_1) < 1/n$;
c) for every $(U, V) \in \gamma_{n+1}$ there exists $(U', V') \in \gamma_n$ such that $U \subset U'$ and $\overline{V}^X \subset V'$.

We claim that for every n the set $H_n := \cup\{U : (U, V) \in \gamma_n \text{ for some } V\}$ is dense (and open) in A. To prove this we proceed by induction. For $n = 0$ this is obviously true. Suppose this is true for some $k \geq 0$ but H_{k+1} is not dense in A. Hence, there is an open set $U_0 \subset A$ such that $U_0 \cap H_{k+1} = \emptyset$. On the other hand, $U_0 \cap H_k \neq \emptyset$. Therefore, there exists some $(U_k, V_k) \in \gamma_k$ such that $U_0 \cap U_k \neq \emptyset$.

Consider the family $\Delta := \{V \subset X : V \text{ is open}, \mathrm{diam}(V \cap X_1) < 1/(k+1)$ and $\overline{V}^X \subset V_k\}$. It is easily seen that $V_k \cap X_1 = \cup\{V \cap X_1 : V \in \Delta\}$. Hence, by Proposition 3.1, we have that $\cup\{\Phi^{-1}(V) : V \in \Delta\}$ is dense in $\Phi^{-1}(V_k)$. Consequently, $U_0 \cap U_k \cap \Phi^{-1}(V_{k+1}) \neq \emptyset$ for some $V_{k+1} \in \Delta$. By the fact that Φ is lower quasi-continuous it follows that for some $U_{k+1} \subset U_0 \cap U_k$ the pair (U_{k+1}, V_{k+1}) is admissible. Now, the family $\gamma_{k+1} \cup \{(U_{k+1}, V_{k+1})\}$ is strictly larger than γ_{k+1} and still satisfies a)-c). This is a contradiction showing that the sets H_n are open and dense subsets of A.

Put now $A_1 := \cap_{n=0}^{\infty} H_n$. Since A is a Baire space then A_1 is a dense G_δ-subset of A. By a) above, each $t \in A_1$ uniquely determines a sequence of admissible pairs $\{(U_n(t), V_n(t))\}_{n=0}^{\infty}$ such that $(U_n(t), V_n(t)) \in \gamma_n$ for every n and $t \in \cap_{n=0}^{\infty} U_n(t)$. Hence, the following mapping (which will turn out to be single-valued) $\phi : A_1 \to X$:

$$\phi(t) := \cap_{n=0}^{\infty} V_n(t), \ \ t \in A_1,$$

is well-defined.

Fix $t \in A_1$. By b) and c) above and the fact that (X_1, d) is a complete metric space, it follows that $\cap_{n=0}^{\infty} V_n(t) \cap X_1$ is a one-point set in X_1, say x, and that the family $\{V_n(t) \cap X_1\}_{n=0}^{\infty}$ is a local base for x in X_1. Since X_1 is dense in X and X is regular, routine considerations show that $\cap_{n=0}^{\infty} V_n(t) = \{x\}$ and that again $\{V_n(t)\}_{n=0}^{\infty}$ is a local base, this time in X, for x. Hence the mapping ϕ is single-valued and takes its values in X_1. Moreover, ϕ is continuous. To this end, let $t_0 \in A_1$ and V be an open subset of X with $\phi(t_0) \in V$. Since $\{V_n(t_0)\}_{n=0}^{\infty}$ is a local base for $\phi(t_0)$ in X we have $V_n(t_0) \subset V$ for some n. Let now $t \in U_n(t_0)$. Then by a) above, $U_n(t) = U_n(t_0)$, and hence $V_n(t) = V_n(t_0)$. Therefore, $\phi(t) \in V_n(t) = V_n(t_0) \subset V$. Consequently, ϕ is continuous in A_1.

We show finally that $\phi(t) \in \Phi(t)$ for every $t \in A_1$. Suppose the contrary and let $t_0 \in A_1$ be such that $\phi(t_0) \notin \Phi(t_0)$. Since Φ is closed-valued there are some open sets V and W of X such that $\Phi(t_0) \subset W$, $\phi(t_0) \in V$ and $V \cap W = \emptyset$. As above, we have $V_n(t_0) \subset V$ for some n. But the couple $(U_n(t_0), V_n(t_0))$ is admissible, hence, in particular $\Phi(t_0) \cap V_n(t_0) \neq \emptyset$. This is a contradiction. Therefore, $\phi(t) \in \Phi(t)$ for every $t \in A_1$. The proof of the theorem is completed. $\square$

The above result could be sharpened if we consider a smaller class of mappings. A set-valued mapping $\Phi : A \to X$ is usco if it is upper semi-continuous and non-empty compact-valued. An usco $\Phi : A \to X$ is *minimal* if its graph $\mathrm{Gr}(\Phi)$ does not contain properly the graph of any other usco from A into X. Zorn's lemma implies that every usco mapping contains a minimal usco one. The following is a well-known characterization of the minimal usco maps (see e.g. [4]): Φ is minimal if, and only if, for every open $W \subset X$ and every open $U \subset A$ with $\Phi(U) \cap W \neq \emptyset$ there is a non-empty open $V \subset U$ such that $\Phi(V) \subset W$. Sometimes mappings that have this property are called *minimal* (even in the case when they are not usco (see [23])). It is a routine matter to see that every mapping $\Phi' : A \to X$ which is contained in some minimal mapping $\Phi : A \to X$ is both upper and lower quasi-continuous. Observe that, an equivalent way to say that $\Phi : A \to X$ is a minimal mapping is the following: if $\Phi(t_0) \cap V \neq \emptyset$ for some $t_0 \in A$ and some open $V \subset X$, it follows that there exists a non-empty open U in A such that $t_0 \in \overline{U}$ and $\Phi(U) \subset V$. We have now the following result:

Theorem 3.4. *Let Φ be a minimal closed-valued mapping between the Baire space A and the regular space X. Suppose X contains a completely metrizable space X_1 which is embraced by Φ. Then there exist a dense G_δ-subset A_1 of A at the points of which Φ is single-valued and upper semi-continuous. Moreover,*

$\Phi(t) \in X_1$ *whenever* $t \in A_1$*.*

P r o o f. By Theorem 3.3 there are a dense G_δ-subset A_1 of A and a continuous single-valued mapping $\phi : A_1 \to X_1$ such that ϕ is a selection of Φ on A_1. Using the minimality of Φ we will show next that $\Phi(t) = \phi(t)$ for every $t \in A_1$.

Indeed, suppose for some $t_0 \in A_1$ there exists $x_0 \in \Phi(t_0)$ with $x_0 \neq \phi(t_0)$. Take non-empty open subsets V_1, V_2 of X such that $x_0 \in V_1$, $\phi(t_0) \in V_2$ but $V_1 \cap V_2 = \emptyset$. By the minimality of Φ there exists an open $U_1 \subset A$ such that $t_0 \in \overline{U}_1$ and $\Phi(U_1) \subset V_1$. On the other hand, the continuity of ϕ gives the existence of a non-empty open U_2 such that $t_0 \in U_2$ and $\phi(U_2 \cap A_1) \subset V_2$. Obviously, for $t^* \in U_1 \cap U_2 \cap A_1 \neq \emptyset$ we have $\phi(t^*) \in \Phi(t^*) \cap V_2 \subset V_1 \cap V_2 = \emptyset$. This is a contradiction.

To prove that Φ is upper semi-continuous at the points of A_1 take some arbitrary $t_0 \in A_1$ and let V be an open subset of X such that $\Phi(t_0) = \phi(t_0) \in V$. Since X is regular there is an open $W \subset X$ with $\phi(t_0) \in W \in \overline{W} \subset V$. By the continuity of ϕ there exists an open U which contains t_0 and $\phi(U \cap A_1) \subset W$. We claim that $\Phi(U) \subset \overline{W}$ (i.e. Φ is upper semi-continuous at t_0). To see this we assume that there exists $x_0 \in \Phi(U) \setminus \overline{W}$ and proceed as above to get a contradiction. The proof is completed. $\square$

We will show now that Theorem 3.3 (and Theorem 3.4) can be applied to a larger class of situations than Theorem 1.3.

Let X be a compact Hausdorff space and $C(X)$ be the space of all continuous real-valued functions in X equipped with the usual sup-norm $\|f\| := \max\{|f(x)| : x \in X\}$, $f \in C(X)$. Consider further the map $M : C(X) \to X$ assigning to each $f \in C(X)$ the set of its maximizers $M(f) := \{x \in X : f(x) = \max\{f(x) : x \in X\}\}$. It is known (see [5, 6]) that M is upper semi-continuous, open and minimal. This implies that M is lower quasi-continuous (and hence lower demi-continuous). The inverse mapping $M^{-1} : X \to C(X)$ is defined by $M^{-1}(x) := \{f \in C(X) : x \in M(f)\}$. The map M^{-1} is lower semi-continuous (since M is open) and demi-open (since M is lower demi-continuous). In particular M and M^{-1} embrace every dense subset of X and $C(X)$ correspondingly, and we can apply Theorem 3.4 to M and Theorem 3.3 to M^{-1}.

Theorem 3.5 ([5, 6])**.** *For the compact Hausdorff space X the following statements are equivalent:*

a) X contains a dense completely metrizable subset X_1;

b) $C(X)$ contains a dense G_δ-subset A_1 such that every function $f \in A_1$

attains its maximum in X at just one point.

P r o o f. Suppose that a) is fulfilled, i.e. there exists a dense completely metrizable subspace X_1 of X. By Theorem 3.4 (applied to $A := C(X)$ and $\Phi := M$) we find some dense G_δ-subset A_1 of $C(X)$ at the points of which M is single-valued. Hence b) takes place.

Conversely, suppose now that b) is fulfilled. The set A_1 being G_δ in a complete metric space is completely metrizable. So apply Theorem 3.3 to the map $\Phi := M^{-1} : X \to C(X)$. This yields the existence of a dense G_δ-subset X_1 of X and a continuous single-valued mapping $\phi : X_1 \to A_1$ such that $\phi(x) \in M^{-1}(x)$ for every $x \in X_1$. Since, by assumption, the restriction of M on A_1 is single-valued, ϕ is a one-to-one mapping. Moreover, taking into account that M is upper semi-continuous we see that the inverse map ϕ^{-1} is also continuous. Thus, ϕ homeomorphically embeds X_1 into a complete metric space. In particular, X_1 is metrizable. As a G_δ-subset of the compact space X the space X_1 is a Čech complete metrizable space. Thus, X_1 is completely metrizable. □

For the similar result when X is not necessarily compact see [7].

4. Counter examples and some more applications.

Some counter examples. In this section we present some examples which illustrate the erratic behaviour which may be possessed by upper or lower quasi-continuous mappings. In what follows, for a normed linear space $(X, \|\cdot\|)$, X^* will mean its dual space, $B(X)$ the unit ball $\{x \in X : \|x\| \le 1\}$ and $S(X)$ its unit sphere $\{x \in X : \|x\| = 1\}$.

Example 4.1. Consider the Hilbert space $\ell_2(\mathbb{R})$ and the set-valued mapping Φ from $\mathbb{R} \setminus \{0\}$ into non-empty closed subsets of $\ell_2(\mathbb{R})$, defined by

$$\Phi(t) = \cup\{\delta_y : y \in [-t, t]\}, t \in \mathbb{R} \setminus \{0\},$$

where the function $\delta_y : \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$\delta_y(s) = \begin{cases} 0 & \text{for } s \neq y \\ 1 & \text{for } s = y. \end{cases}$$

Clearly, Φ is both upper and lower quasi-continuous on $\mathbb{R}$, but nowhere metric upper semi-continuous or lower semi-continuous.

Example 4.2. Consider the Hilbert space $\ell_2(\mathbb{Q})$ and the set-valued mapping Φ from $\mathbb{R} \setminus \{0\}$ into non-empty closed separable subsets of $\ell_2(\mathbb{Q})$ defined by:

$$\Phi(t) = \cup\{\delta_y : y \in [-t, t] \cap \mathbb{Q}\}, t \in \mathbb{R}.$$

As in Example 4.1 we see that Φ is both upper and lower quasi-continuous, and nowhere metric upper semi-continuous. However, in contrast to Example 4.1, for each $t \in \mathbb{R} \setminus \mathbb{Q}$ Φ is lower semi-continuous (but not metric lower semi-continuous).

Example 4.3. Let X be a non-reflexive Banach space such that X^* is separable. We note that since X^* is separable, there exists a metric on $B(X)$ which generates the relative weak topology on $B(X)$, and for which $(B(X), weak)$ is totally bounded. Consider the set-valued mapping Φ from the open interval $(0,1)$ into non-empty closed, totally bounded, subsets of $(B(X), weak)$ defined by $\Phi(t) = tB(X)$ for each $t \in (0,1)$.

It is readily seen that Φ is upper quasi-continuous and lower semi-continuous on $(0,1)$. We will now show that Φ is nowhere upper semi-continuous on $(0,1)$.

By James' characterization of reflexivity there exists an $f \in S(X^*)$ such that $|f(x)| < 1$ for each $x \in B(X)$. For $t \in (0,1)$ consider the weak open subset $W_t := \{x \in B(X) : |f(x)| < t\}$. Then $\Phi(t) \subset W_t$, however, for each $\varepsilon > 0$ $\Phi((t-\varepsilon, t+\varepsilon)) \not\subset W_t$, and so Φ is not upper semi-continuous at t.

In our next example we describe single-valued mapping from a complete metric space into a separable compact Hausdorff topological space, which is everywhere quasi-continuous but nowhere continuous. Before that we recall another class of minimal mappings. A set-valued mapping $\Phi : A \to X$ from the topological space A into the non-empty subsets of a Hausdorff locally convex linear topological space X is called *cusco* if it is usco and convex-valued. Φ is minimal cusco if it is cusco and its graph does not contain properly the graph of any other cusco. By Zorn's lemma every cusco contains a minimal cusco. Moreover, the minimal cuscos have a similar characterization as minimal uscos: Φ is minimal cusco if, and only if, for every open $U \subset A$ and every open half space $W \subset X$ such that $\Phi(U) \cap W \neq \emptyset$ there exists a non-empty open $U' \subset U$ such that $\Phi(U') \subset W$ (see e.g. [9]).

Example 4.4. Consider the Banach space $X := \ell_\infty(\mathbb{N})$ and the continuous gauge functional $p : X \to \mathbb{R}$ defined by $p(x) := \limsup\{|\lambda_n| : x = \{\lambda_n\}_{n=1}^\infty\}$. The subdifferential mapping of p, $x \to \partial p(x)$ is a minimal weak* cusco from X into subsets of $B(X^*)$. It is known that $x \to \partial p(x)$ is nowhere single-valued, [34], p.13. The graph of the mapping $x \to \partial p(x)$ contains the graph of a minimal weak* usco, Φ say. Let $\phi : X \to B(X^*)$ be a selection of Φ, i.e. for each $t \in X$ $\phi(t) \in \Phi(t)$. Then, as it was mentioned, ϕ is weak* quasi-continuous on

X. But it is nowhere continuous, because if it were continuous at a point $t_0 \in X$ then $\partial p(t_0)$ would be a singleton, [34], p.19.

Finally, we give an example showing that even lower semi-continuity (not only lower quasi-continuity) does not imply upper semi-continuity almost everywhere, provided that the set D where the given set-valued mapping is totally bounded is only dense in its domain.

Example 4.5. Let $D_0 := \mathbb{Z}$, $D_n := \{x \in \mathbb{R} : x = p/2^n, p \text{ an odd integer}\}$, $n \geq 1$. Let further $D := \cup_{n=0}^{\infty} D_n$. Observe that D is dense in $\mathbb{R}$. Consider now a set-valued mapping $\Phi : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ from the usual topology in $\mathbb{R} \setminus \{0\}$ into the discrete topology in $\mathbb{R}$ defined as follows:

$$\Phi(t) := \begin{cases} \cup_{j=0}^{n} D_j \cap (-|t|, |t|) & \text{if } t \in D_n \text{ for some } n \\ D \cap (-|t|, |t|) & \text{if } t \in \mathbb{R} \setminus D \end{cases}$$

Now Φ has compact values on D and is lower semi-continuous on $\mathbb{R}$. However, Φ is nowhere upper semi-continuous, not even metric upper semi-continuous. (Note that the images are everywhere countable and closed).

Some more applications and results. Let A be a topological space and let $\mathcal{A} = \{A_\gamma : \gamma \in \Gamma\}$ be a family of subsets of A such that $A = \cup\{A_\gamma : \gamma \in \Gamma\}$.

We say that $\mathcal{A}$ is *point-finite*, if for each $t \in A$ $\{\gamma \in \Gamma : t \in A_\gamma\}$ is finite, and we say that $\mathcal{A}$ is *densely locally finite* if there exists a dense subset D of A such that for each $t \in D$ there exists an open neighbourhood U of t such that $\{\gamma \in \Gamma : A_\gamma \cap U \neq \emptyset\}$ is finite. It is well-known that if A is a Baire space and $\{A_\gamma : \gamma \in \Gamma\}$ is a point-finite open cover of A, then it is densely locally finite. We extend this result as follows:

Proposition 4.6. *Let A be a Baire space and $\mathcal{A} = \{A_\gamma : \gamma \in \Gamma\}$ be a point-finite cover of A, with the property that for each $\gamma \in \Gamma$ $A_\gamma \subset \overline{\operatorname{int}(A_\gamma)}$. Then $\mathcal{A}$ is densely locally finite.*

P r o o f. Consider the metric space (Γ, d), where d is the discrete metric on Γ, and the set-valued mapping Φ from A into non-empty compact subsets of Γ, defined by $\Phi(t) = \{\gamma \in \Gamma : t \in A_\gamma\}$. Clearly Φ is compact-valued on A, however, it is not too difficult to see that Φ is also lower quasi-continuous on A. Hence, by Corollary 2.9 Φ is continuous at the points of a dense G_δ-subset G of A. It is now only a routine matter to check that $\mathcal{A}$ is locally finite at each point of G. $\square$

Similar idea as above can be applied to the study of a class of Banach spaces. Call a Banach space X *generic continuity space* (briefly GC space) (see [22]) if for every minimal mapping $\Phi : A \to (X, weak)$ with non-empty images acting from a Baire space A into X equipped with the weak topology, there exists a dense G_δ-subset A_1 of A such that at the points of A_1 the mapping Φ is single-valued and norm-upper semi-continuous. Note that (as it was mentioned in Section 3) every single-valued selection of a minimal mapping is quasi-continuous. On the other hand, it is a known fact that if a minimal mapping $\Phi : A \to (X, weak)$ has a densely defined selection f which is norm-continuous at some point $t \in A$ then Φ is single-valued and norm-upper semi-continuous at t. Hence, an equivalent definition is: X is a GC space if every quasi-continuous single-valued mapping $f : A \to (X, weak)$ from a Baire space A into $(X, weak)$ is norm-continuous at the points of a dense G_δ-subset of A.

Under c_0-product of a family of Banach spaces $\{(X_\gamma, \|\cdot\|_\gamma) : \gamma \in \Gamma\}$ we mean the space

$$X := \{x \in \prod_{\gamma\in\Gamma} X_\gamma : \text{ the set } \{\gamma \in \Gamma : \|x(\gamma)\|_\gamma > \varepsilon\} \text{ is finite for every } \varepsilon > 0\}$$

equipped with the sup-norm $\|x\|_\infty := \max\{\|x(\gamma)\|_\gamma : \gamma \in \Gamma\}$, $x \in X$.

Theorem 4.7. *Let $\{(X_\gamma, \|\cdot\|_\gamma) : \gamma \in \Gamma\}$ be a family of* GC *spaces. Then its c_0-product X is also a* GC *space.*

Proof. Take a quasi-continuous single-valued mapping $f{:}A{\to}(X, weak)$ defined in the Baire space A. We have to show that the mapping f is norm-continuous at the points of a dense G_δ-subset of A.

Let $\varepsilon > 0$ and

$$O_\varepsilon := \cup\{U : U \subset A \text{ is open and } \|f(t') - f(t'')\|_\infty < \varepsilon \text{ for every } t', t'' \in U\}.$$

Having in mind what was mentioned several times in Section 3, it is enough to show that the open set O_ε is dense in A. This will complete the proof. To see this, take some r, $0 < r < \varepsilon/2$, and consider the sets $A' := \{t \in A : \|f(t)\|_\infty \le r\}$ and $A'' := \{t \in A : \|f(t)\|_\infty > r\}$. Obviously, $A = A' \cup A''$ and $\mathrm{int}A' \subset O_\varepsilon$. If $A'' = \emptyset$ we are done. Suppose $A'' \neq \emptyset$. Using the fact that the sup-norm $\|\cdot\|_\infty$ is lower semi-continuous (as a real-valued function) from the weak topology of X into $\mathbb{R}$ and that $f : A \to (X, weak)$ is quasi-continuous, we easily get that $\mathrm{int}A'' \neq \emptyset$. So, in order to show that O_ε is dense in A it is enough to show that O_ε is dense in $A_1 := \mathrm{int}A''$. Observe that A_1 is again a Baire space.

Let further Φ be a set-valued mapping between A_1 and Γ equipped with the discrete topology, defined by

$$\Phi(t) := \{\gamma \in \Gamma : \|f(t)(\gamma)\|_\gamma > r\}, \quad t \in A_1.$$

Obviously, Φ is non-empty and finite valued at any $t \in A_1$. Also, observe that the composition of the usual projection of X on any X_γ and the norm $\|\cdot\|_\gamma$ is continuous (with respect to the norm in X) and convex and hence it is lower semi-continuous (as a real-valued function) when X is considered with the weak topology. This allows to be easily seen (using quasi-continuity of $f : A \to (X, weak)$) that the mapping Φ is lower quasi-continuous. Hence, by Theorem 2.7 the mapping Φ is both upper and lower semi-continuous at the points of a dense G_δ-subset A_2 of A_1. Since we consider Γ with the discrete topology, it easily then follows that the set A_2 can be considered to be also open in A_1 (and hence, also in A) and that the mapping Φ is locally constant at the points of A_2. The latter means that for every $t \in A_2$ there exists an open subset $W \subset A_2$ so that $t \in W$ and for every $t' \in W$ we have $\Phi(t) = \Phi(t')$.

Take now an open subset U of A such that $U \cap A_1 \neq \emptyset$. Hence, $U \cap A_2 \neq \emptyset$. Let $t_0 \in U \cap A_2$. Then for some open set W of A we have $t_0 \in W \subset U \cap A_2$ and $\Phi(t_0) = \Phi(t)$ for every $t \in W$. Further, it can be seen that for each $\gamma \in \Gamma$ the usual projection of X on X_γ is continuous from the weak topology in X into the weak topology in X_γ. Hence, for each $\gamma \in \Gamma$ the composition $f(\cdot)(\gamma)$ is quasi-continuous in the weak topology in X_γ. By the fact that each X_γ is a GC space it follows that for every $\gamma \in \Phi(t_0)$ there exists a dense G_δ-subset G_γ of W so that $f(\cdot)(\gamma) : A_1 \to (X_\gamma, \|\cdot\|_\gamma)$ is norm-continuous at the points of G_γ. Put $G := \cap\{G_\gamma : \gamma \in \Phi(t_0)\}$. The latter is again a dense subset of W since $\Phi(t_0)$ was finite. Let $t_1 \in W \cap G$. Because of the continuity at t_1 of each $f(\cdot)(\gamma)$, $\gamma \in \Phi(t_0)$, with respect to the norm in X_γ, and because of the fact that $\Phi(t_1) = \Phi(t_0)$ is finite, we get an open subset $W_1 \subset W$ such that $t_1 \in W_1$ and

$$(*) \quad \|f(t_1)(\gamma) - f(t)(\gamma)\|_\gamma < \varepsilon/2 \text{ for each } t \in W_1 \text{ and each } \gamma \in \Phi(t_1) = \Phi(t_0).$$

Now, we claim that $W_1 \subset O_\varepsilon$. Indeed, take $t', t'' \in W_1$. Then, if $\gamma \notin \Phi(t_0) = \Phi(t') = \Phi(t'')$ we have, having in mind the definition of Φ, that

$$\|f(t')(\gamma) - f(t'')(\gamma)\|_\gamma \leq 2r < \varepsilon.$$

If $\gamma \in \Phi(t_0)$ we get the same inequality by $(*)$ above. Hence, $\|f(t') - f(t'')\|_\infty < \varepsilon$. The proof is completed. $\square$

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