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THREE SPACES PROBLEM FOR LYAPUNOV THEOREM ON VECTOR MEASURE

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Abstract. It is proved that a Banach space X has the Lyapunov property if its subspace Y and the quotient space X/Y have it.

Let Σ be a σ-algebra of subsets of a set Ω, X be a Banach space, $\mu : \Sigma \to X$ be a countably additive measure. The famous theorem of A. A. Lyapunov ([9], [11, Theorem 5.5], [2, p. 264]) states that the range $\mu(\Sigma)$ of an arbitrary X-valued nonatomic measure μ is convex if X is a finite-dimensional space. Some generalizations of this theorem to infinite-dimensional case have been obtained by G. Knowles [7], I. I. Uhl [2], D. Pecherskii [10], V. Kadets [4], J. Elton and Th. P. Hill [3], V. Kadets and M. Popov [5] under various additional assumptions on the measure and the space. Most of these generalizations consider the closure of the measure range. Without additional restrictions the theorem is valid only in finite-dimensional case [2, p. 256, Corollary 6].

Definition 1. *A Banach space X is said to have the* Lyapunov property *($X \in LPr$) if for every nonatomic measure μ valued in X the closure of its range $cl\mu(\Sigma)$ is convex.*

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The spaces l_p $(1 \le p < \infty;\ p \ne 2)$ and c_0 [6] have the Lyapunov property. Other examples have not been constructed till now. The other known generalizations have been proved with additional restrictions on the measure (for example, about the measures of bounded variations).

"Three spaces problem" appears for every Banach space property. Namely, do a subspace Y and the quotient space X/Y have the property if X does and does X have the property if Y and X/Y do? Some of these problems can be solved easily in the case of the Lyapunov property. Indeed, if $X \in LPr$ and $Y \subset X$, then $Y \in LPr$, but X/Y do not need to have the Lyapunov property (for instance, $X = l_1$ because the set of its quotient spaces contains all separable Banach spaces [8, p. 108]). In the same time the answer to the last problem runs into difficulties. The purpose of this paper is the positive solution of the mentioned problem:

Theorem 1. *Let Y be a subspace of a Banach space X. If $Y, X/Y \in LPr$, then $X \in LPr$.*

Let us remark that some new examples of spaces possessing the Lyapunov property can be constructed due to this theorem (for instance $l_{p_1} \oplus l_{p_2} \oplus \cdots \oplus l_{p_n}$, where $1 \le p_k < \infty;\ p_k \ne 2$). To prove the theorem we need some lemmas.

Lemma 1. *Let $X \in LPr$ and $\mu : \Sigma \to X$ be a nonatomic measure. Then there is a nonatomic nonnegative measure $\lambda : \Sigma \to \mathbb{R}$ such that for every $A \in \Sigma$ and $n \in \mathbb{N}$ there exists $B_n \in \Sigma|_A$ satisfying the following inequalities:*

$$\left\| \mu\left(B_n\right) - \frac{1}{2}\mu\left(A\right) \right\| \le \frac{1}{2^n} \quad \text{and} \quad \left| \lambda\left(B_n\right) - \frac{1}{2}\lambda\left(A\right) \right| \le \frac{1}{2^n}. \tag{1}$$

P r o o f. By the Rybakov theorem [2, p. 267] there is a functional $x^* \in X^*$ with $\|x^*\| = 1$ for which $\mu \ll |x^*\mu|$. Put $\lambda = |x^*\mu|$. Let $A \in \Sigma$. In accordance with the Hahn decomposition theorem let us denote by Ω^+ and Ω^- the positivity and negativity sets for $x^*\mu$ respectively. Then $\lambda(C) = x^*\mu\left(C \bigcap \Omega^+\right) - x^*\mu\left(C \bigcap \Omega^-\right)$ for any $C \in \Sigma$. Put $A^+ = A \bigcap \Omega^+$ and $A^- = A \bigcap \Omega^-$. Since $X \in LPr$, we can choose $B_n^+ \in \Sigma|_{A^+}$ and $B_n^- \in \Sigma|_{A^-}$ such that $\left\| \mu\left(B_n^+\right) - \frac{1}{2}\mu\left(A^+\right) \right\| \le \frac{1}{2^{n+1}}$ and $\left\| \mu\left(B_n^-\right) - \frac{1}{2}\mu\left(A^-\right) \right\| \le \frac{1}{2^{n+1}}$. Then $\left| x^*\mu\left(B_n^+\right) - \frac{1}{2}x^*\mu\left(A^+\right) \right| \le \frac{1}{2^{n+1}}$ and $\left| x^*\mu\left(B_n^-\right) - \frac{1}{2}x^*\mu\left(A^-\right) \right| \le \frac{1}{2^{n+1}}$. Define B_n as $B_n^+ \bigcup B_n^-$. It is easy to check that for B_n the inequalities (1) are true. □

Lemma 2. *Let $X \in LPr$, $\mu : \Sigma \to X$ be a nonatomic measure, λ be the measure from Lemma* 1. *Then for every $A \in \Sigma$ with $\lambda(A) \neq 0$ and $\varepsilon > 0$ there exist $G' \in \Sigma|_A$, $G' = A \backslash G'$ such that*

(*i*) $\lambda(G') = \lambda(G') = \frac{1}{2}\lambda(A),$

(*ii*) $\left\| \mu(G') - \frac{1}{2}\mu(A) \right\| < \varepsilon.$

Proof. Let us choose B_n as in Lemma 1. We can select $C_n \in \Sigma|_{B_n}$ if $\lambda(B_n) \geq \frac{1}{2}\lambda(A)$ or $C_n \in \Sigma|_{A \backslash B_n}$ if $\lambda(B_n) < \frac{1}{2}\lambda(A)$ for which $\lambda(C_n) = \left| \frac{1}{2}\lambda(A) - \lambda(B_n) \right|$ (because λ is a nonatomic real-valued measure). By (1), $\lambda(C_n) \underset{n\to\infty}{\longrightarrow} 0$. Put $G'_n = B_n \triangle C_n, G''_n = A \backslash G'_n$. Then $\lambda(G'_n) = \lambda(G''_n) = \frac{1}{2}\lambda(A)$, $\lambda(G'_n \backslash B_n) \underset{n\to\infty}{\longrightarrow} 0$ and $\lambda(B_n \backslash G'_n) \underset{n\to\infty}{\longrightarrow} 0$. Since $\lambda \gg \mu$ the last condition implies that $\mu(B_n \backslash G'_n) \underset{n\to\infty}{\longrightarrow} 0$ and $\mu(G'_n \backslash B_n) \underset{n\to\infty}{\longrightarrow} 0$. Together with inequality (1) it gives us

$$\left\| \mu(G'_n) - \frac{1}{2}\mu(A) \right\| \underset{n\to\infty}{\longrightarrow} 0.$$

So for sufficiently large n the sets $G' = G'_n$ and $G'' = G''_n$ will satisfy the conditions (*i*) and (*ii*). □

Lemma 3. *Under the conditions of Lemma* 2 *for every $A \in \Sigma$ with $\lambda(A) \neq 0$ and $\varepsilon > 0$ there exists a σ-algebra $\Sigma' \subset \Sigma|_A$ such that for every $B \in \Sigma'$ we have*

$$\left\| \mu(B) - \lambda(B)\frac{\mu(A)}{\lambda(A)} \right\| \leq \varepsilon\lambda(B) \tag{2}$$

and measure λ is nonatomic on Σ'.

Proof. Take $A \in \Sigma$ and $\varepsilon > 0$. Employing Lemma 2, we choose sets $A_1 \in \Sigma|_A$, $A_2 = A \backslash A_1$ with $\lambda(A_1) = \lambda(A_2) = \frac{1}{2}\lambda(A)$ and $\left\| \mu(A_1) - \frac{1}{2}\mu(A) \right\| \leq \frac{1}{4}\varepsilon$ (note that $\left\| \mu(A_1) - \frac{1}{2}\mu(A) \right\| = \frac{1}{2}\left\| \mu(A_1) - \mu(A_2) \right\| = \left\| \mu(A_2) - \frac{1}{2}\mu(A) \right\| \leq \frac{1}{4}\varepsilon$).

Employing Lemma 2 twice (for $A = A_1$ and $A = A_2$) we obtain $A_{1,1} \in \Sigma|_{A_1}, A_{1,2} = A_1 \backslash A_{1,1}; A_{2,1} \in \Sigma|_{A_2}, A_{2,2} = A_2 \backslash A_{2,1}$ with $\lambda(A_{1,1}) = \lambda(A_{1,2}) = \lambda(A_{2,1}) = \lambda(A_{2,2}) = \frac{1}{4}\lambda(A)$ and $\left\| \mu(A_{1,1}) - \frac{1}{2}\mu(A_1) \right\| \leq \frac{\varepsilon}{16}$, $\left\| \mu(A_{2,1}) - \frac{1}{2}\mu(A_2) \right\|$

$\leq \dfrac{\varepsilon}{16}$. When we continue this process we receive a tree of sets $A_{i_1,i_2,\ldots,i_n}$, $i_k \in \{1,2\}$, $n \in \mathbb{N}$ with

$$A_{i_1,i_2,\ldots,i_{n+1}} \subset A_{i_1,i_2,\ldots,i_n},\ A_{i_1,i_2,\ldots,i_n,2} = A_{i_1,i_2,\ldots,i_n} \backslash A_{i_1,i_2,\ldots,i_n,1},$$

$$\left\| \mu\left(A_{i_1,i_2,\ldots,i_n}\right) - \frac{1}{2}\mu\left(A_{i_1,i_2,\ldots,i_{n-1}}\right) \right\| \leq \frac{1}{4^n}\varepsilon,\ \lambda\left(A_{i_1,i_2,\ldots,i_n}\right) = \frac{1}{2^n}\lambda\left(A\right)$$

Let Σ' be a σ-algebra generated by the sets $A_{i_1,i_2,\ldots,i_n}$. We are going to show that algebra Σ' has the required property.

Let $B = A_{i_1,i_2,\ldots,i_n}$. Then

$$\begin{aligned}
\left\| \mu\left(B\right) - \frac{\lambda\left(B\right)}{\lambda\left(A\right)}\mu\left(A\right) \right\| &= \left\| \mu\left(A_{i_1,i_2,\ldots,i_n}\right) - \frac{1}{2^n}\mu\left(A\right) \right\| \\
&\leq \left\| \mu\left(A_{i_1,i_2,\ldots,i_n}\right) - \frac{1}{2}\mu\left(A_{i_1,i_2,\ldots,i_{n-1}}\right) \right\| \\
&\quad + \frac{1}{2}\left\| \mu\left(A_{i_1,i_2,\ldots,i_{n-1}}\right) - \frac{1}{2}\mu\left(A_{i_1,i_2,\ldots,i_{n-2}}\right) \right\| + \cdots \\
&\quad + \frac{1}{2^{n-1}}\left\| \mu\left(A_{i_1}\right) - \frac{1}{2}\mu\left(A\right) \right\| \\
&\leq \frac{1}{4^n}\varepsilon + \frac{1}{2}\frac{1}{4^{n-1}}\varepsilon + \ldots + \frac{1}{2^{n-1}}\frac{1}{4}\varepsilon \\
&= \varepsilon\left(\frac{1}{2^{2n}} + \frac{1}{2^{2(n-1)}} + \cdots + \frac{1}{2^{n+1}}\right) \leq \frac{1}{2^n}\varepsilon \\
&= \varepsilon\lambda\left(A_{i_1,i_2,\ldots,i_n}\right).
\end{aligned}$$

Hence by triangle inequality and σ-additivity of μ and λ we get (2) for $B = \bigcup\limits_{n=1}^{\infty} B_n$, where B_n are disjoint sets of the form $A_{i_1,i_2,\ldots,i_k}$. Then we obtain (2) for every $B \in \Sigma'$ using approximation of $\lambda\left(B\right)$ by bigger sets of the form $B = \bigcup\limits_{n=1}^{\infty} B_n$. $\square$

Lemma 4. *The statements of Lemmas* 2 *and* 3 *are valid for arbitrary nonatomic measure* λ.

P r o o f. Let $\mu : \Sigma \to X$, $\lambda : \Sigma \to \mathbb{R}_+$ be nonatomic measures. Let ν be the measure which played the role of λ in Lemma 1. Consider two cases.

Case 1: $\lambda \ll \nu$. Take $A \in \Sigma$, $n \in \mathbb{N}$. In view of Lemma 3, there is $\Sigma' \subset \Sigma|_A$ such that for any $B \in \Sigma'$

$$\left\|\mu(B) - \frac{\nu(B)}{\nu(A)}\mu(A)\right\| \leq \frac{1}{2^n}\nu(B), \tag{3}$$

ν is nonatomic measure with respect to Σ'. Applying the Lyapunov theorem to the measure $\sigma : \Sigma' \to \mathbb{R}^2 : \sigma(A) = (\nu(A), \lambda(A))$, we obtain sets $G', G'' \in \Sigma'$ such that $\lambda(G') = \lambda(G'') = \frac{1}{2}\lambda(A)$ and $\nu(G') = \nu(G'') = \frac{1}{2}\nu(A)$. Then (3) implies inequality (ii) which we need.

Case 2: $\lambda \not\ll \nu$. We decompose λ into the sum of absolutely continuous and strictly singular measures with respect to $\nu : \lambda = \lambda_1 + \lambda_2$. Then λ_2 is concentrated on a ν-negligible set S. Now we consider $A \backslash S$ and $S \bigcap A$ separately. By the case 1 chose $G_1' \subset A \backslash S$ so that $\lambda_1(G_1') = \frac{1}{2}\lambda_1(A)$, $\left\|\mu(G_1') - \frac{1}{2}\mu(A)\right\| \leq \frac{1}{2^n}$ and $G_2' \subset S \bigcap A$ with $\lambda_2(G_2') = \frac{1}{2}\lambda(S \bigcap A)$. Because $\lambda_1(G_2') = 0$, $\mu(G_2') = 0$ and $\lambda_2(G_1') = 0$ it is clear that $G' = G_1' \bigcup G_2'$ satisfies (ii). Obviously, if Lemma 2 is valid for arbitrary λ then Lemma 3 is valid too. □

The following statement is evident.

Lemma 5. *If X is a Banach space, Y is a subspace of X, $\mu : \Sigma \to X$ is a nonatomic measure, then $\overline{\mu} : \Sigma \to X/Y$ ($\overline{\mu}(A) = \overline{\mu(A)}$-the equivalence class of $\mu(A)$) is a nonatomic measure too.*

Let X, Y be from the theorem, $\mu : \Sigma \to X$, $\lambda : \Sigma \to \mathbb{R}_+$ be nonatomic measures, $\lambda(\Omega) = 1$. Fix $A \in \Sigma$, $\lambda(A) \neq 0$ and $\varepsilon > 0$. By lemmas 3 and 4 there is a σ-algebra $\Sigma' \subset \Sigma|_A$ such that

$$\left\|\overline{\mu}(B) - \frac{\lambda(B)}{\lambda(A)}\overline{\mu}(A)\right\| < \frac{1}{2}\varepsilon\lambda(B) \tag{4}$$

for all $B \in \Sigma'$ and λ is nonatomic on Σ'. Define $\sigma : \Sigma' \to X$ by the rule $\sigma(B) = \mu(B) - \frac{\lambda(B)}{\lambda(A)}\mu(A)$.

Lemma 6. *There exist a σ-algebra $\widetilde{\Sigma} \subset \Sigma'$ and a nonatomic measure $\beta : \widetilde{\Sigma} \to Y$ such that*

$$\|\sigma(B) - \beta(B)\| < 2\varepsilon$$

for all $B \in \widetilde{\Sigma}$.

P r o o f. By inequality (4) and nonatomicity of λ we can choose sets $A_1, A_2 \in \Sigma'$ such that $A = A_1 \bigcup A_2, \lambda(A_1) = \lambda(A_2) = \frac{1}{2}\lambda(A)$, and $\|\overline{\sigma}(A_1)\| < \varepsilon\frac{1}{2}\lambda(A)$. Clearly, there is $x \in \overline{\sigma}(A_1)$ such that $\|x\| \le \varepsilon\frac{1}{2}\lambda(A)$. Denote

$$\alpha(A_1) = x,\ \alpha(A_2) = -x,$$

Applying step by step Lemma 2 and Lemma 4, we get sets $A_{i_1,\ldots,i_\kappa} \in \Sigma'$, $k = 2, 3, \ldots;\ \ i_1, \ldots, i_k = 1, 2$, such that $A_{i_1,\ldots,i_{k-1}} = A_{i_1,\ldots,i_{k-1},1} \bigcup A_{i_1,\ldots,i_k,2}$, $\lambda(A_{i_1,\ldots,i_k}) = \frac{1}{2}\lambda\left(A_{i_1,\ldots,i_{k-1}}\right) = \frac{1}{2^k}\lambda(A)$, and $\left\|\overline{\sigma}\left(A_{i_1,\ldots,i_{k-1},1}\right) - \frac{1}{2}\overline{\sigma}\left(A_{i_1,\ldots,i_{k-1}}\right)\right\| < \varepsilon\frac{1}{2^{2k}}\lambda(A)$. It is readily seen that in every equivalence class $\overline{\sigma}\left(A_{i_1,\ldots,i_{k-1},1}\right) - \frac{1}{2}\overline{\sigma}\left(A_{i_1,\ldots,i_{k-1}}\right) \subset X$ there exists an element $x_{i_1,\ldots,i_{k-1}}$ such that $\left\|x_{i_1,\ldots,i_{k-1}}\right\| \le \varepsilon\frac{1}{2^{2k}}\lambda(A)$. Put

$$\alpha\left(A_{i_1,\ldots,i_{k-1},1}\right) = \frac{1}{2}\alpha\left(A_{i_1,\ldots,i_{k-1}}\right) + x_{i_1,\ldots,i_{k-1}},$$
$$\alpha\left(A_{i_1,\ldots,i_{k-1},2}\right) = \frac{1}{2}\alpha\left(A_{i_1,\ldots,i_{k-1}}\right) - x_{i_1,\ldots,i_{k-1}}.$$

Evidently, $\alpha\left(A_{i_1,\ldots,i_{k-1},1}\right) \in \overline{\sigma}\left(A_{i_1,\ldots,i_{k-1},1}\right)$ and $\alpha\left(A_{i_1,\ldots,i_{k-1},2}\right) \in \overline{\sigma}\left(A_{i_1,\ldots,i_{k-1},2}\right)$. By $\widetilde{\Sigma}$ denote σ-algebra generated by the sets $A_{i_1,\ldots,i_k}$. Iterating the inequality $\|\alpha(A_{i_1,\ldots,i_k})\| \le \frac{1}{2}\left\|\alpha\left(A_{i_1,\ldots,i_{k-1}}\right)\right\| + \varepsilon\frac{1}{2^{2k}}\lambda(A)$, we obtain $\|\alpha(A_{i_1,\ldots,i_k})\| \le \varepsilon\frac{1}{2^{k-1}}\lambda(A) = 2\varepsilon\lambda(A_{i_1,\ldots,i_k})$. Let us extend α to $\widetilde{\Sigma}$ and show that

$$\|\alpha(B)\| \le 2\varepsilon\lambda(B) \tag{5}$$

for any $B \in \widetilde{\Sigma}$. For this purpose we take $B = \bigcup\limits_I A_{i_1,\ldots,i_k}$, where I is a finite set of indices and $A_{i_1,\ldots,i_k}$ are mutually disjoint sets. Put $\alpha(B) = \sum\limits_I \alpha(A_{i_1,\ldots,i_k})$. Clearly,

$$\|\alpha(B)\| \le \sum_I \|\alpha(A_{i_1,\ldots,i_k})\| \le 2\varepsilon\sum_I \lambda(A_{i_1,\ldots,i_k}) = 2\varepsilon\lambda(B).$$

This proves that inequality (5) is valid for elements of algebra S, generated by the sets $A_{i_1,\ldots,i_k}$. Now applying the Kluvanek-Uhl extension theorem [1] we obtain the extension of α to $\widetilde{\Sigma}$. Thus, we have constructed the measure $\alpha : \widetilde{\Sigma} \to X$ such that $\alpha(B) \in \overline{\sigma}(B)$ for any $B \in \widetilde{\Sigma}$ and $\|\alpha(B)\| \leq 2\varepsilon\lambda(B)$. α is a nonatomic measure because λ is nonatomic by construction. It is clear that $\beta = \alpha - \sigma$ has the required property. The lemma is proved. □

Let us complete the proof of the theorem. Since $\beta : \widetilde{\Sigma} \to Y$ is nonatomic measure and $Y \in LPr$, we see that by Lemma 2 there is $B \in \widetilde{\Sigma}$ such that $\lambda(B) = \frac{1}{2}\lambda(A)$ and

$$\left\|\beta(B) - \frac{1}{2}\beta(A)\right\| \leq \varepsilon.$$

Thus we have

$$\begin{aligned}\left\|\sigma(B) - \frac{1}{2}\sigma(A)\right\| &\leq \|\sigma(B) - \beta(B)\| + \left\|\frac{1}{2}\sigma(A) - \frac{1}{2}\beta(A)\right\| + \left\|\beta(B) - \frac{1}{2}\beta(A)\right\| \\ &\leq 3\varepsilon.\end{aligned}$$

Since $\sigma(B) - \frac{1}{2}\sigma(A) = \mu(B) - \frac{1}{2}\mu(A)$, the proof is completed. □

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