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CONTINUOUS DEPENDENCE OF SOLUTIONS OF QUASIDIFFERENTIAL EQUATIONS WITH NON-FIXED TIME OF IMPULSES

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Abstract. In this article on quasidifferential equation with non-fixed time of impulses we consider the continuous dependence of the solutions on the initial conditions as well as the mappings defined by these equations. We prove general theorems for quasidifferential equations from which follows corresponding results for differential equations, differential inclusion and equations with Hukuhara derivative.

1. Introduction. In work [1] has been defined the concept of quasidifferential equation, which is a generalization of the concept of R-solution [2, 3] of differential inclusion. In works [1, 4] were proved theorems for existence and uniqueness of the solution for quasidifferential equations and was showed, that these equations define an irreversible dynamical system in metric space.

A lot of researches has been carried out recently in differential equations with impulses. Some basic results and reference points can be found in [5]–[7].

In [8] were considered quasidifferential equations with fixed time of impulses. The present article is on quasidifferential equations with non-fixed time of impulses.

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2. Statement of the problem. Let X be a local compact metric space with distance function $\delta(\cdot,\cdot)$.

Assume that

$$\varphi\colon [o,\tau)\times[t_0,t_0+T)\times X\to X$$

defines a local quasimovement, i.e. the following conditions are satisfied:

Condition D:

1) an axiom of initial conditions: $\varphi(0,t_0,x_0)=x_0$;

2) an axiom of quasifitting:

$$\delta(\varphi(\tau_1+\tau_2,t,x),\varphi(\tau_2,t+\tau_1,\varphi(\tau_1,t,x)))=o(\tau_1+\tau_2);$$

3) an axiom of continuity, i.e. for every $\varepsilon>0$ there exist $\delta_1>0$ and $\delta_2>0$ such that $\delta(\varphi(\tau_1,t,x_1),\varphi(\tau_2,t,x_2))<\epsilon$ when $\delta(x_1,x_2)<\delta_1,\quad |\tau_1-\tau_2|<\delta_2$.

Definition 1. *A quasidifferential equation in a metric space is called the equation*

$$\delta(x(t+\Delta),\varphi(\Delta,t,x(t)))=o(\Delta). \tag{1}$$

A solution of the equation (1) *is called the continuity map* $x\colon[t_0,T]\to X$*, which satisfies* (1) *for* $t\in[t_0,T]$.

We consider in the domain $Q=\{\Delta\in[0,\tau),t\in[t_0,t_0+T],P\subset X\}$ a quasidifferential equation with impulses

$$\delta(x(t+\Delta),\varphi(\Delta,t,x(t)))=o(\Delta),\quad t\neq\tau_i(x), \tag{2}$$

$$x(\tau_i+0)=\psi_i(x(\tau_i)),$$

where $\psi_i\colon X\to X,\quad x(\tau_i)=x(\tau_i-0)$.

3. Main results.

Lemma 1. *Let in the domain* Q *the map* $\varphi(\Delta,t,x)$ *satisfy condition* D*, Lipschitz condition in* Δ *with constant* λ*, and in* x *the condition*

$$|\delta(x,y)-\delta(\varphi(\Delta,t,x),\varphi(\Delta,t,y))|\leq\Delta\gamma\delta(x,y). \tag{3}$$

Then for the solutions $x(t)$ *and* $y(t)$ *of quasidifferential equation* (1) *the following estimate is correct:*

$$\delta(x(t),y(t))\leq e^{\gamma(t-t_0)}\delta_0, \tag{4}$$

where $x(t_0)=x_0,\quad y(t_0)=y_0,\quad \delta_0=\delta(x_0,y_0)$.

P r o o f. Let us divide the interval $[t_0,t_0+T]$ into m parts. In the moment $t\in[t_k,t_{k+1}]\subset[t_0,t_0+T]$, the estimate of the error due to initial conditions is:

$$\begin{aligned}\delta(x(t),y(t)) &\leq \delta(\varphi(t-t_k,t_k,x(t_k)),\varphi(t-t_k,t_k,y(t_k)))+o(\Delta)\\ &\leq (1+\Delta.\gamma)\delta(x(t_k),y(t_k))+o(\Delta)\end{aligned}$$

$$
\begin{aligned}
&\leq (1+\Delta.\gamma)\delta(\varphi(\Delta,t_{k-1},x(t_{k-1})),\varphi(\Delta,t_{k-1},y(t_{k-1})))\\
&\quad +(1+\Delta\gamma)o(\Delta)+o(\Delta)\\
&\leq (1+\Delta\gamma)^2\delta(x(t_{k-1}),y(t_{k-1}))+o(\Delta)(1+\Delta\gamma)+o(\Delta).
\end{aligned}
$$

It is easy to check that:

$$
\begin{aligned}
\delta(x(t),y(t)) &\leq (1+\Delta\gamma)^{k+1}\delta_0+[(1+\Delta\gamma)^k+\ldots+(1+\Delta\gamma)+1]o(\Delta)\\
&\leq (1+\frac{(t-t_0)}{m}\gamma)^m\delta_0+\frac{(1+\Delta\gamma)^{k+1}-1}{\Delta\gamma}o(\Delta)\\
&\leq e^{\gamma(t-t_0)}\delta_0+\frac{o(\Delta)}{\Delta}\frac{e^{\gamma(\tau_1-t_0)}}{\gamma}. \qquad (5)
\end{aligned}
$$

From (5) when $\Delta\to 0$ we obtain (4). □

Theorem 1. *Let in the domain Q be fulfilled the conditions of the Lemma 1 and:*

1) *the maps $\psi_i(x)$ are continuous;*

2) *the functions $\tau_i(x)$ satisfy Lipschitz condition with constant μ;*

3) *for every $x\in P$ is correct the following inequality*

$$\tau_i(x)\geq\tau_i(\psi_i(x)). \qquad (6)$$

Then under the condition $\mu\lambda<1$ every solution $x(t,t_0,x_0)$ of equation (2), belonging to domain P in $t_0<t\leq t_0+T$, intersects every hypersurface $t=\tau_i(x)$ in the interval $[t_0,t_0+T]$ just once.

Proof. We assume the contrary. Let for some solution $x(t)$, go out of point $\psi_i(x_0)$ when $t=\tau_i(x_0)+0$, intersects the surface $t=\tau_i(x)$ in some point $(t_i^\star,x^\star)$, $t_i^\star=\tau_i(x^\star)$. It is obvious, that $t^\star>\tau_i(x_0)$ and the interval $\tau_i(x_0)<t<t_i^\star$ is a interval of continuity of solution $x(t,\tau_i(x)+0,\psi_i(x_0))$ and therefore

$$
\begin{aligned}
t_i^\star-\tau_i(x_0) &= \tau_i(x^\star)-\tau_i(x_0)=\tau_i(x^\star)-\tau_i(\psi_i(x_0))+\tau_i(\psi_i(x_0))-\tau_i(x_0)\\
&\leq \mu\delta(x^\star,\psi_i(x_0))+\tau_i(\psi_i(x_0))-\tau_i(x_0)\\
&\leq \mu\delta(x(t_i^\star,\tau_i(x_0)+0,\psi_i(x_0)),x(\tau_i(x_0)+0,\tau_i(x_0),\psi_i(x_0)))\\
&\quad +\tau_i(\psi_i(x_0))-\tau_i(x_0)\\
&\leq \mu\lambda(t_i^\star-\tau_i(x_0))+\tau_i(\psi_i(x_0))-\tau_i(x_0). \qquad (7)
\end{aligned}
$$

From (7) we obtain

$$(1-\mu\lambda)(t_i^\star-\tau_i(x_0))\leq\tau_i(\psi_i(x_0))-\tau_i(x_0). \qquad (8)$$

Since $(1-\mu\lambda) > 0$, $\quad t_i^\star - \tau_i(x_0) > 0$ we obtain that (8) contradict to (6). $\square$

Let $x(t, x_0)$ and $x(t, y_0)$ be two solutions of equation (2), belonging to domain P for every $t \in [t_0, t_0 + T]$. We suppose, that each of these solutions intersects every hypersurface $t = \tau_i(x)$ just once, and let us denote with $\tau_i^{x_0}$, $\tau_i^{y_0}$ correspondingly moments of intersection of these solutions with surfaces $t = \tau_i(x)$.

Theorem 2. *Let the conditions of Theorem* 1 *be fulfilled and*

$$\delta(\psi_i(x), \psi_i(y)) \le \nu\delta(x, y).$$

Then

$$\delta(x(t, t_0, x_0), x(t, t_0, y_0)) \le \Big(\frac{\lambda\mu + \nu}{1 - \lambda\mu}\Big)^p e^{\gamma T}\delta(x_0, y_0) \tag{9}$$

for every $t \in [\tau_i^+, \tau_{i+1}^-]$, *where* $\tau_i^- = \min\{\tau_i^{x_0}, \tau_i^{y_0}\}$, $\tau_i^+ = \max\{\tau_i^{x_0}, \tau_i^{y_0}\}$, p *is the number of the impulses in the interval* $[t_0, t_0 + T]$.

P r o o f. Let us denote $\delta_i^- = \delta(x(\tau_i^-, t_0, x_0), x(\tau_i^-, t_0, y_0))$. We suppose that $\tau_i^- = \tau_i^{y_0}, \tau_i^+ = \tau_i^{x_0}$. Then

$x(\tau_i^+ + 0, t_0, x_0) = \psi_i(x(\tau_i^+, t_0, x_0))$, $x(\tau_i^+ + 0, t_0, y_0) = x(\tau_i^+, \tau_i^-, \psi_i(x(\tau_i^-, t_0, y_0)))$

and

$$\begin{aligned}\delta(\psi_i(x(\tau_i^-, t_0, x_0)), \psi_i(x(\tau_i^+, t_0, x_0)) &\le \nu\delta(x(\tau_i^-, t_0, x_0), x(\tau_i^+, t_0, x_0)) \\ &\le \nu\lambda|\tau_i^+ - \tau_i^-|,\end{aligned}$$

$$\delta(x(\tau_i^+, \tau_i^-, \psi_i(x(\tau_i^-, t_0, y_0)), \psi_i(x(\tau_i^-, t_0, y_0))) \le \lambda|\tau_i^+ - \tau_i^-|,$$

$$\delta(\psi_i(x(\tau_i^-, t_0, y_0)), \psi_i(x(\tau_i^-, t_0, x_0))) \le \nu\delta(x(\tau_i^-, t_0, x_0), x(\tau_i^-, t_0, y_0)) = \nu\delta_i^-,$$

$$\begin{aligned}\delta_i^+ &= \delta(x(\tau_i^+ + 0, t_0, x_0), x(\tau_i^+ + 0, t_0, y_0)) \\ &= \delta(\psi_i(x(\tau_i^+, t_0, x_0)), x(\tau_i^+, \tau_i^-, \psi_i(x(\tau_i^-, t_0, y_0)) \\ &\le \delta(\psi_i(x(\tau_i^+, t_0, x_0)), \psi_i(x(\tau_i^-, t_0, x_0))) \\ &\quad + \delta(\psi_i(x(\tau_i^-, t_0, x_0)), \psi_i(x(\tau_i^-, t_0, y_0))) \\ &\quad + \delta(\psi_i(x(\tau_i^-, t_0, y_0), x(\tau_i^+, \tau_i^-, \psi_i(x(\tau_i^-, t_0, y_0)))) \\ &\le \lambda\nu|\tau_i^+ - \tau_i^-| + \nu\delta_i^- + \lambda|\tau_i^+ - \tau_i^-|. \qquad (10)\end{aligned}$$

Let $z_0 = x_0$, if $\tau_i^- = \tau_i^{y_0}$, and $z_0 = y_0$, if $\tau_i^- = \tau_i^{x_0}$ then

$$\begin{aligned} |\tau_i^{x_0} - \tau_i^{y_0}| &= |\tau_i(x(\tau_i^{x_0}, t_0, x_0)) - \tau_i(x(\tau_i^{y_o}, t_0, y_0))| \\ &\leq \mu\delta(x(\tau_i^{x_0}, t_0, x_0), x(\tau_i^{y_0}, t_0, y_0)) \\ &\leq \mu\delta(x(\tau_i^-, t_0, x_0), x(\tau_i^-, t_0, y_0)) + \mu\delta(x(\tau_i^{x_0}, t_0, z_0), x(\tau_i^{y_0}, t_0, z_0)) \\ &\leq \mu\delta(x(\tau_i^-, t_0, x_0), x(\tau_i^-, t_0, y_0)) + \mu\lambda|\tau_i^{x_0} - \tau_i^{y_0}|. \end{aligned} \tag{11}$$

From (11) we have

$$|\tau_i^{x_0} - \tau_i^{y_0}| \leq \frac{\mu\delta_i^-}{1-\lambda\mu}. \tag{12}$$

If we substitute (12) in (10) we obtain

$$\delta_i^+ \leq \frac{\lambda\mu + \nu}{1-\lambda\mu}\delta_i^-. \tag{13}$$

It is obviously that (13) is true also for $\tau_i^- = \tau_i^{x_0}, \tau_i^+ = \tau_i^{y_0}$. From Lemma 1 it follows that

$$\delta_1^- \leq e^{\gamma(\tau_1 - t_0)}\delta_0, \quad \delta_{i+1}^- \leq e^{\gamma(\tau_{i+1}^- - \tau_i^+)}\delta_i^+,$$

$$\delta(x(t_0+T, t_0, x_0), x(t_0+T, t_0, y_0)) \leq \delta_p^+ e^{\gamma(t_0+T-\tau_p)}. \tag{14}$$

From (13), (14) we obtain that (9) is correct. □

Theorem 3. *Let in the domain Q be fulfilled conditions of Theorem 2. If solution $x(t, t_0, x_0)$ is defined in $t \in [t_0, t_0+T]$, then this solution continuously depends from initial value x_0 in the following sense: for every $\varepsilon > 0$ there exists such $\delta_0 = \delta(\varepsilon) > 0$ that for every other solution $x(t, t_0, y_0)$ of equation (2) satisfying the inequality $\delta(x_0, y_0) < \delta_0$ we have*

$$\delta(x(t, t_0, x_0), x(t, t_0, y_0)) < \varepsilon$$

for all $t \in [t_0, t_0+T]$, which satisfy inequalities $|t - \tau_i^{x_0}| > \varepsilon$, where $\tau_i^{x_0}$ are the moments of intersection of solution $x(t, t_0, x_0)$ with hypersurfaces $t = \tau_i(x)$.

The conclusion of this Theorem follows directly from Theorems 1 and 2.

Theorem 4. *Let in the domain Q be given quasidifferential equations:*

$$\delta(x(t+\Delta), \varphi_1(\Delta, t, x(t))) = o(\Delta), \; t \neq \tau_i^1(x), \quad x(t_0) = x_0, \tag{15}$$

$$x(\tau_i + 0) = \psi_i^1(x(\tau_i)),$$

$$\delta(y(t+\Delta), \varphi_2(\Delta, t, x))) = o(\Delta), \; t \neq \tau_i^2(x), \quad y(t_0) = y_0, \tag{16}$$

$$y(\tau_i + 0) = \psi_i^2(y(\tau_i)).$$

Let us suppose, that the maps $\varphi_1(\Delta, t, x), \varphi_2(\Delta, t, x), \psi_i^1(x), \psi_i^2(x), \tau_i^1(x), \tau_i^2(x)$

satisfy condition of Theorem 2 and in addition they satisfy the conditions

$$|\tau_i^1(x) - \tau_i^2(x)| \le \eta, \;\; \delta(\varphi_1(\Delta, t, x), \varphi_2(\Delta, t, x)) \le \eta\Delta, \;\; \delta(\psi_i^1(x), \psi_i^2(x)) \le \eta. \tag{17}$$

If solution $x(t, t_0, x_0)$ of equation (15) is defined in $[t_0, t_0 + T]$, then this solution is continuous in the following sense: for every $\varepsilon > 0$ there exists such $\delta_0 = \delta(\varepsilon) > 0$ and $\eta_0 = \eta(\varepsilon) > 0$, that for every solution $y(t, t_0, y_0)$ of equation (16) satisfying the inequality $\delta(x_0, y_0) < \delta_0$ and inequalities (17) when $\eta < \eta_0$, we have

$$\delta(x(t, t_0, x_0), y(t, t_0, y_0)) < \varepsilon$$

for all $t \in [t_0, t_0 + T]$, which satisfy inequalities $|t - \tau_i^{x_0}| > \varepsilon$, where $\tau_i^{x_0}$ are the moments of intersection of solution $x(t, t_0, x_0)$ with hypersurfaces $t = \tau_i^1(x)$.

Proof. Let $x(t, t_0, x_0)$ and $y(t, t_0, y_0)$ be solutions of the equations (15) and (16) respectively ,which belong to the domain P for every $t \in [t_0, t_0 + T]$. Each of these solutions intersects every hypersurface $t = \tau_i^1(x)$ and $t = \tau_i^2(y)$ just once, and let us denote with $\tau_i^{x_0}, \tau_i^{y_0}$ correspondingly moments of intersection of these solutions with surfaces $t = \tau_i^1(x)$ and $t = \tau_i^2(y)$.

Let us denote $\tau_i^- = \min\{\tau_i^{x_0}, \tau_i^{y_0}\}$, $\tau_i^+ = \max\{\tau_i^{x_0}, \tau_i^{y_0}\}$, $\delta_i^- = \delta(x(\tau_i^-, t_0, x_0), y(\tau_i^-, t_0, y_0))$, $\delta_i^+ = \delta(x(\tau_i^+ + 0, t_0, x_0), y(\tau_i^+ + 0, t_0, y_0))$.

Then if $\tau_i^- = \tau_i^{y_0}, \tau_i^+ = \tau_i^{x_0}$ we have

$$x(\tau_i^+ + 0, t_0, x_0) = \psi_i^1(x(\tau_i^+, t_0, x_0)),$$

$$y(\tau_i^+ + 0, t_0, y_0) = y(\tau_i^+, \tau_i^-, \psi_i^2(y(\tau_i^-, t_0, y_0)))$$

and

$$\begin{aligned} \delta(\psi_i^1(x(\tau_i^-, t_0, x_0)), \psi_i^1(x(\tau_i^+, t_0, x_0))) &\le \nu\delta(x(\tau_i^-, t_0, x_0), x(\tau_i^+, t_0, x_0)) \\ &\le \nu\lambda|\tau_i^+ - \tau_i^-|, \end{aligned}$$

$$\delta(y(\tau_i^+, \tau_i^-, \psi_i^2(y(\tau_i^-, t_0, y_0))), \psi_i^2(y(\tau_i^-, t_0, y_0))) \le \lambda|\tau_i^+ - \tau_i^-|,$$

$$\delta(\psi_i^2(y(\tau_i^-, t_0, y_0), \psi_i^1(x(\tau_i^-, t_0, x_0))) \le \delta(\psi_i^2(y(\tau_i^-, t_0, y_0)), \psi_i^2(x(\tau_i^-, t_0, x_0)))+$$
$$+\delta(\psi_i^2(x(\tau_i^-, t_0, x_0)), \psi_i^1(x(\tau_i^-, t_0, x_0))) \le \nu\delta_i^- + \eta,$$

$$\begin{aligned} \delta_i^+ &\le \delta(\psi_i^1(x(\tau_i^+, t_0, x_0)), \psi_i^1(x(\tau_i^-, t_0, x_0))) \\ &= +\delta(\psi_i^1(x(\tau_i^-, t_0, x_0)), \psi_i^2(y(\tau_i^-, t_0, y_0))) \\ &\quad +\delta(\psi_i^2(y(\tau_i^-, t_0, y_0), y(\tau_i^+, \tau_i^-, \psi_i^2(y(\tau_i^-, t_0, y_0)))) \\ &\le \nu\lambda|\tau_i^+ - \tau_i^-| + \nu\delta_i^- + \lambda|\tau_i^+ - \tau_i^-|. \end{aligned} \tag{18}$$

Let $z_0 = x_0$, if $\tau_i^- = \tau_i^{y_0}$, and $z_0 = y_0$,if $\tau_i^- = \tau_i^{x_0}$, then

$$\begin{aligned} |\tau_i^{x_0} - \tau_i^{y_0}| &= |\tau_i^1(x(\tau_i^{x_0}, t_0, x_0)) - \tau_i^2(y(\tau_i^{y_0}, t_0, y_o))| \\ &\leq |\tau_i^1(x(\tau_i^{x_0}, t_0, x_0)) - \tau_i^1(y(\tau_i^{y_0}, t_0, y_0))| + \end{aligned}$$

$$+|\tau_i^1(y(\tau_i^{y_0}, t_0, y_0)) - \tau_i^2(y(\tau_i^{y_0}, t_0, y_0))| \leq \mu\delta_i^- + \mu\lambda|\tau_i^{x_0} - \tau_i^{y_0}| + \eta. \tag{19}$$

From (19) we have

$$|\tau_i^{x_0} - \tau_i^{y_0}| \leq \frac{\mu\delta_i^- + \eta}{1 - \mu\lambda}. \tag{20}$$

If we substitute (20) in (18) we obtain

$$\delta_i^+ \leq c_1\delta_i^- + c_2\eta, \tag{21}$$

where $c_1 = (\nu + \lambda\mu)/(1 - \mu\lambda), c_2 = (\nu\lambda + \lambda + 1 - \mu\lambda)/(1 - \mu\lambda)$.

It is obviously that (21) is true also for $\tau_i^- = \tau_i^{x_0}, \tau_i^+ = \tau_i^{y_0}$.

From Theorem 2 [8] it follows

$$\delta_1^- \leq e^{\gamma(\tau_1^- - t_0)}\delta_0 + \frac{e^{\gamma(\tau_1^- - t_0)} - 1}{\gamma}\eta,$$

$$\delta_{i+1}^- \leq e^{\gamma(\tau_{i+1}^- - \tau_i^+)}\delta_i^+ + \frac{e^{\gamma(\tau_{i+1}^- - \tau_i^+)} - 1}{\gamma}\eta, \tag{22}$$

$$\delta(x(t_0 + T, t_0, x_0), y(t_0 + T, t_0, y_0)) \leq e^{\gamma(t_0 + T - \tau_p)}\delta_p^+ + \frac{e^{\gamma(t_0 + T - \tau_p)} - 1}{\gamma}\eta.$$

From (21), (22) we obtain

$$\delta(x(t, t_0, x_0), y(t, t_0, y_0)) \leq \Big(\frac{\nu + \lambda\mu}{1 - \mu\lambda}\Big)^p e^{\gamma T}\delta_0 + C\eta, \tag{23}$$

where C is independent constant from δ_0 and η.

From (23) and (20) it follows the conclusion of the Theorem. □

4. Examples.

Example 1. Let $X = \mathbb{R}^n$ and

$$\varphi_1(\Delta, t, x) = x + \Delta \cdot f(t, x),$$

$$\varphi_2(\Delta, t, x) = x + \Delta \cdot (f(t, x) + R(t, x)),$$

$$\psi_{1i}(x) = x + I_i(x), \quad \psi_{2i}(x) = x + I_i(x) + R_i(x). \tag{24}$$

Then from Theorem 4 we can obtain the corresponding theorem for differential equation with impulses:

$$\dot{x} = f(t, x), \quad t \neq \tau_i^1(x), \quad x(t_0) = x_0,$$

$$\Delta x|_{t=\tau_i^1(x)} = I_i(x),$$

$$\dot{y} = f(t,y) + R(t,y), \quad t \neq \tau_i^2(y), \quad y(t_0) = y_0,$$

$$\Delta y|_{t=\tau_i^2(y)} = I_i(y) + R_i(y).$$

Example 2. Let $X = \mathrm{comp}(\mathbb{R}^n)$ where this space is the space of all nonempty compact sets within $\mathbb{R}^n$.

Let $\mathrm{conv}(R^n)$ be the space of nonempty convex and compact sets within $\mathbb{R}^n$,

$$\varphi_1(\Delta, t, Y) = \bigcup_{z \in Y} \Big(z + \int_t^{t+\Delta} F(t,z)dt \Big),$$

$$\varphi_2(\Delta, t, Y) = \bigcup_{z \in Y} \Big(z + \int_t^{t+\Delta} (F(t,z) + R(t,z))dt \Big),$$

$$\psi_i^1(Y) = \bigcup_{z \in Y} (z + I_i(z)),$$

$$\psi_i^2(Y) = \bigcup_{z \in Y} (z + I_i(x) + R_i(z)),$$

where

$$F\colon \mathbb{R}^1 \times \mathbb{R}^n \to \mathrm{conv}(\mathbb{R}^n), \quad \mathbb{R}\colon \mathbb{R}^1 \times \mathbb{R}^n \to \mathrm{conv}(\mathbb{R}^n),$$

$$I_i\colon \mathbb{R}^n \to \mathrm{conv}(\mathbb{R}^n), \quad \mathbb{R}_i\colon \mathbb{R}^n \to \mathrm{conv}(\mathbb{R}^n), \quad \tau_i^1(x) = \tau_i^2(x) \equiv t_i. \tag{25}$$

From Theorem 4 we obtain corresponding theorem for differential inclusions with impulses:

$$\dot{x} \in F(t,x), \quad t \neq t_i, \quad x(t_0) = x_0, \quad \Delta x|_{t=t_i} \in I_i(x), \tag{26}$$

$$\dot{y} \in F(t,y) + R(t,y), \; t \neq t_i, \quad y(t_0) = y_0, \quad \Delta y|_{t=t_i} \in I_i(y) + R_i(y).$$

The multivalued solutions $Y(t, t_0, x_0)$ of differential inclusion (26) should be understood as R – solution [2, 3].

If $\tau_i^j(x) \not\equiv \mathrm{const}$, then $\tau_i^j\colon \mathrm{comp}(\mathbb{R}^n) \to \mathbb{R}^1$.

In this case if

$$\tau_i(Y) = \min_{z \in Y} \rho_i(z) \quad \mathit{or} \quad \tau_i(Y) = \max_{z \in Y} \rho_i(z), \tag{27}$$

from Theorem 4 we obtain corresponding theorem for differential inclusions with impulses:

$$\dot{z} \in F(t,z), \quad , t \neq \rho_i(z), \quad z(t_0) = z_0,$$

$$\Delta z|_{t=\rho_i} \in I_i(z). \tag{28}$$

The multivalued solution $Y(t, t_0, z_0)$ of differential inclusion (28) should be understood as R-solution and on hypersurfaces $t = \rho_i(z)$ all solution's bundle undergo impulse when the first (or the last) solution $z(t, t_0, x_0)$ belonging to this bundle reaches the surface $t = \rho_i(z)$. Analogous meaning gets condition (6). If we define functions $\tau_i(Y)$ by different way from (27), we can obtain different conditions for impulses of all bundle of trajectories.

If $X = \operatorname{comp}(\mathbb{R}^n)$ then from Theorem 4 for quasidifferential equation (2),(24) we can obtain the corresponding theorem for differential equation with Hukuhara derivative [9] with impulses:

$$DY(t) = F(t, Y(t)), \quad t \neq \tau_i^1(Y),$$
$$Y(\tau_i^1 + 0) = \psi(Y(\tau_i^1)),$$

where $DY(t)$ is Hukuhara derivative [10] for a multivalued function $Y(t)$, $F\colon \mathbb{R}^1 \times \operatorname{conv}(\mathbb{R}^n) \to \operatorname{conv}(\mathbb{R}^n)$.

Remark 1. The proof of Theorem 4 is completely the same for quasidifferential equation with variable structure

$$\delta(x(t+\Delta), \varphi_i(\Delta, t, x(t))) = o(\Delta), \quad \tau_i(x) < t \leq \tau_{i+1}(x), \quad x(t_0) = x_0,$$
$$x(\tau_i + 0) = \psi_i(x(\tau_i)).$$

Impulsive differential equations with variable structure have been considered in [11]–[13].

REFERENCES

[1] A. I. Panasyuk. Quasidifferential equation in metric space. *Differential Equation* **21**, *8* (1985), 1344-1353 (in Russian).

[2] A. I. Panasyuk, V. I. Panasyuk. Asymptotic Optimization of Nonlinear Systems and Control. Minsk, BSU – Edition, 1977 (in Russian).

[3] A. I. Panasyuk. On an equation which is generated by the differential inclusion. *Maths. Zametki* **27**, *3* (1980), 429-437 (in Russian).

[4] V. A. Plotnikov, L. I. Plotnikova. Asymptotic Method for Quasidifferential Equations in the Metric Space. Deposited G. N. T. B. Ukraine, No 825 – Uk, 1994 (in Russian).

[5] V. Lakshmikantham, D. D. Bainov, P. S. Simeonov. Theory of impulsive Differential Equations. World Scientific Publishers, Singapore, 1990.

[6] A. M. Samoylenko, N. A. Perestyuk. Differential Equations With Impulses. Kiev, Vischa Shkola, 1987 (in Russian).

[7] S. T. Zavalishin, A. N. Sesekin. Impulsive Processes. Models and Applications. Moscow, Nauka, 1991 (in Russian).

[8] V. A. Plotnikov, P. M. Kitanov. Continuous dependence of solutions of quasidifferential equations with impulses. *Mathematca Balkanica, New Series* **11**, *1–2* (1997), 79-85.

[9] F. S. De Blasi, F. Ervolino. Equazioni differentiali con soluzioni a valore compatto convesso. *Boll. Unione Mat. Ital.* **2**, *4-5* (1969), 491-501.

[10] M. Hukuhara. Integration des applications measurable dont la valeur est un compact convexe. *Func. Ekvacioj* **10** (1967), 205-223.

[11] D. D. Bainov, S. D. Milusheva. Investigation of partially-multiplicative averaging for a class of functional equations with variable structure and impulses. *Tsukuba J. Math.* **9**, *1* (1985), 1-19.

[12] A. B. Dishliev, D. D. Bainov. Continuous dependance on the initial condition of the solution of a system of differential equations with variable structure and with impulses. Publ. RIMS, Kyoto Univ. vol. **23** (1987), 923-936.

[13] V. A. Plotnikov. The Method of Averaging in Control Problems. Kiev-Odessa, Lebid, 1992 (in Russian).

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