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NEW SYMMETRIC (61,16,4) DESIGNS INVARIANT UNDER THE DIHEDRAL GROUP OF ORDER 10*

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Abstract. In this note we construct five new symmetric 2-(61,16,4) designs invariant under the dihedral group of order 10. As a by-product we obtain 25 new residual 2-(45,12,4) designs. The automorphism groups of all new designs are computed.

1. Introduction. Let $\mathcal{V} = \{0, 1, \ldots, v-1\}$ be a finite set of elements called points, and let $\mathcal{B} = \{B_0, B_1, \ldots, B_{b-1}\}$ be a collection of k-element subsets of $\mathcal{V}$ called blocks. The incidence structure $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ is a $2-(v, k, \lambda)$ design (or BIB (v, k, λ) design) if every unordered pair of points is contained in exactly λ blocks of $\mathcal{B}$. Each point from the point set of a 2-design is contained in a constant number of blocks. This number is usually denoted by r. Obviously,

$$\lambda(v-1) = r(k-1),$$

$$\lambda v(v-1) = bk(k-1).$$

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Every design $\mathcal{D}$ is determined by its incidence matrix $\mathbf{A}(\mathcal{D}) = (a_{ij})_{v\times b}$, where

$$a_{ij} = \begin{cases} 1 & \text{if } \ i \in B_j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}_1 = (\mathcal{V}, \mathcal{B}_1)$ and $\mathcal{D}_2 = (\mathcal{V}, \mathcal{B}_2)$ be two 2-designs with the same parameters. They are said to be isomorphic if there exists a permutation $\varphi \in S_v$ which maps the blocks of $\mathcal{B}_1$ onto the blocks of $\mathcal{B}_2$, i.e. $\{x_1, x_2, \ldots, x_k\} \in \mathcal{B}_1$ implies $\{x_1^\varphi, x_2^\varphi, \ldots, x_k^\varphi\} \in \mathcal{B}_2$. If $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}$ the permutation φ is called an automorphism of $\mathcal{D}$. The set of all automorphisms of $\mathcal{D}$ forms a group – the so-called full automorphism group of $\mathcal{D}$. We denote it by $Aut\,\mathcal{D}$. Every subgroup of $Aut\,\mathcal{D}$ is referred to as an automorphism group of $\mathcal{D}$.

It is well-known that for every design with $k < v$, one has $b \geq v$ [4]. Designs with $b = v$ are called symmetric. Let $\mathcal{D}$ be a symmetric $2-(v, k, \lambda)$ design with incidence matrix $\mathbf{A}(\mathcal{D})$. Then the matrix $\mathbf{A}(\mathcal{D})^t$ is incidence matrix of a symmetric design with the same parameters, which is called the dual $\overline{\mathcal{D}}$ of $\mathcal{D}$. $\mathcal{D}$ and $\overline{\mathcal{D}}$ are not necessarily isomorphic.

Let $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ be a symmetric $2-(v, k, \lambda)$ design and let B be an arbitrary block from $\mathcal{B}$. Then the incidence structure $\mathcal{D}' = (\mathcal{V} \setminus B, \mathcal{B}' = \{B_i \setminus B\}_{i=1}^{b-1}$)is a $2-(v-k, k-\lambda, \lambda)$ design. It is called the residual of $\mathcal{D}$ with respect to B. For further notions and results on 2-designs we refer to [2], [10].

In this note we consider symmetric 2-(61,16,4) designs and the residual 2-(45,12,4) designs. Only one 2-(61,16,4) design is known to exist. It has been constructed by Mitchell [8] as a member of an infinite family of symmetric designs (see also [9]). Using a different method we produce here five new 2-(61,16,4) designs and 25 new 2-(45,12,4) designs. Using a computer, we were able to compute the full automorphism group of all new designs.

2. Possible automorphism groups of 2-(61,16,4) designs. In order to construct new symmetric 2-(61,16,4) designs we assume that they possess a nice group of automorphisms. We want it to be as large as possible. It turns out that the largest prime dividing the order of the full automorphism group of a hypothetical 2-(61,16,4) design is 5. Moreover, an automorphism of order 5 fixes exactly one point and one block.

Let $\mathcal{D}$ be a symmetric $2-(v, k, \lambda)$ design with an automorphism φ of prime order p. It is well-known that an automorphism of a symmetric design fixes the same number of points and blocks. Denote by f the number of fixed points (blocks) and by $l = (v-f)/p$ the number of nontrivial point (block) orbits of $\mathcal{D}$ under $\langle\varphi\rangle$. The following lemma is due to Aschbacher [1].

Lemma 2.1. *If p is a prime which is an order of an automorphism of a $2-(v,k,\lambda)$ design with $v > k$ then either p divides v or else $p \leq r$.*

Lemma 2.2. *Let $\mathcal{D}$ be a symmetric $2-(v,k,\lambda)$ design and let p divide $|Aut\ \mathcal{D}|, p > \lambda$. Then*

(a) $l \geq f$;

(b) $l \geq \lceil \frac{k-f}{p} \rceil f$.

P r o o f. (a) Each fixed point is contained in a nontrivial block orbit. If not, there would be a point orbit contained in two different fixed blocks and thus there would be two blocks intersecting in more than λ points. On the other hand, each block from a nontrivial block orbit contains at most one fixed point. This proves (a).

(b) Each fixed block contains at least $\lceil (k-f)/p \rceil$ nontrivial point orbits. Furthermore, a nontrivial point orbit is contained in at most one fixed block. This implies (b). $\square$

Theorem 2.3. *The only primes which might be orders of automorphisms of a 2-(61,16,4) design $\mathcal{D}$ are 2, 3 and 5. An automorphism of order 5 of a hypothetical 2-(61,16,4) design fixes exactly one point (and one block).*

P r o o f. By Lemma 2.1 the primes $p = 2, 3, 5, 7, 11, 13,$ and 61 are admissible orders of automorphisms of a 2-(61,16,4) design. It is known that a (61,16,4) difference set in the cyclic group of order 61 does not exist [6]. Hence $p = 61$ is ruled out. By Lemma 2.2(a) the only possibilities for p, f and l are $p = 7, f = 5, l = 8$; $p = 5, f = 6, l = 11$; $p = 5, f = 1, l = 12$. The first two are ruled out by Lemma 2.2(b). $\square$

Using similar arguments we can prove that the largest prime dividing the order of the full automorphism group of a 2-(45,12,4) design is 5. Moreover, an automorphism of order 5 fixes no points and no blocks.

3. New symmetric 2-(61,16,4) designs. In what follows, we consider symmetric 2-(61,16,4) designs with an automorphism φ of order 5. Without loss of generality we may assume that

$$\varphi = (0)(1\,2\,\ldots\,5)(6\,7\,\ldots\,10)\;\ldots\;(56\,57\,\ldots\,60).$$

Let us note that such a design (if it exists) will be different from the one constructed by Mitchell. The Mitchell design has $C_3 \times C_3$ as a full group of automorphisms.

Suppose there exists a 2-(61,16,4) design $\mathcal{D}$ with an automorphism group $G = \langle\varphi\rangle$. The orbit matrix $\mathbf{M} = (m_{ij})_{i,j=0}^{12}$ of $\mathcal{D}$ with respect to G is defined as a matrix whose rows and columns are indexed by the point and block orbits of $\mathcal{D}$, respectively, where m_{ij} is the number of points from the i-th point orbit contained in a block from the j-th block orbit. Here we assume that the row (resp. column) indexed by 0 corresponds to the fixed point (resp. block). In this notation, an orbit matrix $\mathbf{M}$ satisfies the following equations:

$$\sum_{i=1}^{12} m_{ij} = 15,\;\; \sum_{i=1}^{12} m_{ij}^2 = 27, i = 1,2,3; \tag{3.1}$$

$$\sum_{i=1}^{12} m_{ij} = 16,\;\; \sum_{i=1}^{12} m_{ij}^2 = 32, i = 4,5,\ldots,12; \tag{3.2}$$

$$\sum_{j=1}^{12} m_{\alpha j} m_{\beta j} = 15, 1 \le \alpha < \beta \le 3; \tag{3.3}$$

$$\sum_{j=1}^{12} m_{\alpha j} m_{\beta j} = 20, 1 <= \alpha < \beta, \beta \ge 4. \tag{3.4}$$

Using a computer we have found 2913 different matrices satisfying (3.1-3.4). To get a design we have to replace the entries m_{ij}, $i,j = 1,2,\ldots,12$, in every matrix $\mathbf{M}$ of this list by a (0,1)-circulant of order 5 having m_{ij} ones per row (column). This has to be done in such a way that the resulting matrix is the incidence matrix of a 2-(61,16,4) design. In order to make the task of extending the 2913 (hypothetical) orbit matrices tractable, we assume an additional automorphism of order 2:

$$\psi = (0)(1)\,(2\,5)\,(3\,4)\,(6)\,(7\,10)\,(8\,9)\;\ldots\;(56)\,(57\,60)\,(58\,59),$$

in other words, we assume that a hypothetical 2-(61,16,4) design is invariant under the dihedral group

$$D_{10} \;=\; \langle \varphi, \psi \mid \varphi^5 = \psi^2 = id, \psi^{-1}\varphi\psi = \varphi^{-1} \rangle.$$

It turns out that just one of the matrices satisfying (3.1-3.4) yields designs. In fact, it gives five nonisomorphic designs, which we denote by $\mathcal{D}_i, i = 1, 2, 3, 4, 5$. Their incidence matrices are denoted by $\mathbf{A}(\mathcal{D}_i)$. It can be checked that $\mathbf{A}(\mathcal{D}_4) = \mathbf{A}(\mathcal{D}_1)^t$ and $\mathbf{A}(\mathcal{D}_5) = \mathbf{A}(\mathcal{D}_2)^t$. The design $\mathcal{D}_3$ is self-dual. The incidence matrices of $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ are given below.

$$\mathbf{A}(\mathcal{D}_1) = \begin{pmatrix} 1 & \mathbf{e} & \mathbf{e} & \mathbf{e} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} \\ \mathbf{e}^t & I & I & I & 0 & 0 & 0 & B & B & B & A & A & A \\ \mathbf{e}^t & I & I & I & A & A & A & 0 & 0 & 0 & B & B & B \\ \mathbf{e}^t & I & I & I & B & B & B & A & A & A & 0 & 0 & 0 \\ \mathbf{o}^t & 0 & B & A & 0 & B & A & 0 & B & A & 0 & B & A \\ \mathbf{o}^t & 0 & B & A & B & A & 0 & A & 0 & B & A & 0 & B \\ \mathbf{o}^t & 0 & B & A & A & 0 & B & B & A & 0 & B & A & 0 \\ \mathbf{o}^t & A & 0 & B & 0 & B & A & A & 0 & B & B & A & 0 \\ \mathbf{o}^t & A & 0 & B & A & 0 & B & 0 & B & A & A & 0 & B \\ \mathbf{o}^t & A & 0 & B & B & A & 0 & B & A & 0 & 0 & B & A \\ \mathbf{o}^t & B & A & 0 & 0 & B & A & B & A & 0 & A & 0 & B \\ \mathbf{o}^t & B & A & 0 & A & 0 & B & A & 0 & B & 0 & B & A \\ \mathbf{o}^t & B & A & 0 & B & A & 0 & 0 & B & A & B & A & 0 \end{pmatrix},$$

$$\mathbf{A}(\mathcal{D}_2) = \begin{pmatrix} 1 & \mathbf{e} & \mathbf{e} & \mathbf{e} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} \\ \mathbf{e}^t & I & I & I & 0 & 0 & 0 & B & B & B & A & A & A \\ \mathbf{e}^t & I & I & I & A & A & A & 0 & 0 & 0 & B & B & B \\ \mathbf{e}^t & I & I & I & B & B & B & A & A & A & 0 & 0 & 0 \\ \mathbf{o}^t & 0 & B & A & 0 & B & A & 0 & B & A & 0 & B & A \\ \mathbf{o}^t & 0 & B & A & B & A & 0 & A & 0 & B & A & 0 & B \\ \mathbf{o}^t & 0 & A & B & B & 0 & A & B & A & 0 & A & B & 0 \\ \mathbf{o}^t & A & 0 & B & 0 & B & A & A & 0 & B & B & A & 0 \\ \mathbf{o}^t & A & 0 & B & A & 0 & B & 0 & B & A & A & 0 & B \\ \mathbf{o}^t & B & 0 & A & A & B & 0 & B & A & 0 & 0 & A & B \\ \mathbf{o}^t & A & B & 0 & 0 & A & B & B & A & 0 & B & 0 & A \\ \mathbf{o}^t & B & A & 0 & A & 0 & B & A & 0 & B & 0 & B & A \\ \mathbf{o}^t & B & A & 0 & B & A & 0 & 0 & B & A & B & A & 0 \end{pmatrix},$$

$$\mathbf{A}(\mathcal{D}_3) = \begin{pmatrix} 1 & \mathbf{e} & \mathbf{e} & \mathbf{e} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} & \mathbf{o} \\ \mathbf{e}^t & I & I & I & 0 & 0 & 0 & B & B & B & A & A & A \\ \mathbf{e}^t & I & I & I & A & A & A & 0 & 0 & 0 & B & B & B \\ \mathbf{e}^t & I & I & I & B & B & B & A & A & A & 0 & 0 & 0 \\ \mathbf{o}^t & 0 & B & A & 0 & B & A & 0 & A & B & 0 & A & B \\ \mathbf{o}^t & 0 & B & A & B & A & 0 & B & 0 & A & B & 0 & A \\ \mathbf{o}^t & 0 & B & A & A & 0 & B & A & B & 0 & A & B & 0 \\ \mathbf{o}^t & A & 0 & B & 0 & B & A & B & 0 & A & A & B & 0 \\ \mathbf{o}^t & A & 0 & B & A & 0 & B & 0 & A & B & B & 0 & A \\ \mathbf{o}^t & A & 0 & B & B & A & 0 & A & B & 0 & 0 & A & B \\ \mathbf{o}^t & B & A & 0 & 0 & B & A & A & B & 0 & B & 0 & A \\ \mathbf{o}^t & B & A & 0 & A & 0 & B & B & 0 & A & 0 & A & B \\ \mathbf{o}^t & B & A & 0 & B & A & 0 & 0 & A & B & A & B & 0 \end{pmatrix}.$$

Here I is the identity matrix of order 5, 0 – the all-zero matrix of size 5-by-5, A is the circulant of order five with first row (01001), B is the circulant with first row (00110), $\mathbf{e} = (11111)$, and $\mathbf{o} = (00000)$. Generators of the full automorphism groups of $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ are given in Table 1.

Table 1.

Design	Generators	$\|Aut\,\mathcal{D}_i\|$
$\mathcal{D}_1$	φ, ψ (16 21 26)...(20 25 30)(31 41 36)...(35 45 40)(46 56 51)...(50 60 55) (16 31 46)...(20 35 50)(21 41 56)...(25 45 60)(26 36 51)...(30 40 55)	90
$\mathcal{D}_2$	φ, ψ (16 36 56)...(20 40 60)(21 31 51)...(25 35 55)(26 41 46)...(30 45 50)	30
$\mathcal{D}_3$	φ, ψ (1 6 11)...(5 10 15)(31 36 41)...(35 40 45)(46 56 51)...(50 60 55) (16 21 26)...(20 25 30)(31 41 36)...(35 45 40)(46 56 51)...(50 60 55) (16 31 46)...(20 35 50)(21 41 56)...(25 45 60)(26 36 51)...(30 40 55)	270

4. The residual 2-(45,12,4) designs. Deleting blocks from the constructed 2-(61,16,4) designs, we obtain 25 nonisomorphic 2-(45,12,4) designs. They are denoted by $\mathcal{D}'_i, i = 1, 2, \ldots, 25$. In Table 2 we list the way of obtaining each one of them along with the order of its full automorphism group.

For each point x we calculated the number $m^{(x)}$ of unordered pairs (y, z), $y \neq x$, $z \neq x$, such that x, y and z occur together in exactly λ blocks. Let $\mathcal{D}$ be a block design. The number of points C_i with a given $m^{(x)} = i$ is an invariant for

$\mathcal{D}$. It turns out that this invariant distinguishes all residual designs in Table 2 with one exception – the designs $\mathcal{D}'_1$ and $\mathcal{D}'_6$. These two designs are distinguished by the orders of their full automorphism groups.

Table 2.

	$\mathcal{D}'_1$	$\mathcal{D}'_2$	$\mathcal{D}'_3$	$\mathcal{D}'_4$	$\mathcal{D}'_5$	$\mathcal{D}'_6$	$\mathcal{D}'_7$	$\mathcal{D}'_8$	$\mathcal{D}'_9$
Obtained from	$\mathcal{D}_1$	$\mathcal{D}_1$	$\mathcal{D}_1$	$\mathcal{D}_1$	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_2$	$\mathcal{D}_2$	$\mathcal{D}_2$
Deleted block	0	1	16	31	46	0	1	16	31
$\lvert Aut\, \mathcal{D}'_i \rvert$	540	6	6	6	6	180	2	2	6

	$\mathcal{D}'_{10}$	$\mathcal{D}'_{11}$	$\mathcal{D}'_{12}$	$\mathcal{D}'_{13}$	$\mathcal{D}'_{14}$	$\mathcal{D}'_{15}$	$\mathcal{D}'_{16}$	$\mathcal{D}'_{17}$
Obtained from	$\mathcal{D}_2$	$\mathcal{D}_2$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{D}_4$	$\mathcal{D}_4$
Deleted block	36	41	46	1	16	1	6	11
$\lvert Aut\, \mathcal{D}'_i \rvert$	6	6	2	18	6	18	18	18

	$\mathcal{D}'_{18}$	$\mathcal{D}'_{19}$	$\mathcal{D}'_{20}$	$\mathcal{D}'_{21}$	$\mathcal{D}'_{22}$	$\mathcal{D}'_{23}$	$\mathcal{D}'_{24}$	$\mathcal{D}'_{25}$
Obtained from	$\mathcal{D}_4$	$\mathcal{D}_5$	$\mathcal{D}_5$	$\mathcal{D}_5$	$\mathcal{D}_5$	$\mathcal{D}_5$	$\mathcal{D}_5$	$\mathcal{D}_5$
Deleted block	16	0	1	6	11	16	21	26
$\lvert Aut\, \mathcal{D}'_i \rvert$	2	30	6	6	6	2	2	2

5. Concluding remarks. Designs with parameters 2-(45,12,4) might be of interest in connection with the problem of finding new extremal self-orthogonal codes of length 60. The incidence matrix of a 2-(45,12,4) design can be considered as a generator matrix of a binary self-orthogonal code with parameters $[60, k], k \leq 30$. There is a special interest in such codes of dimension $k = 30$ and minimum distance $d = 12$ [5][3]. It has been proved in [5] that the possible weight enumerators of an extremal self orthogonal $[60, 30, 12]$ code are either

$$W(z) = 1 + (2555 + 64\beta)z^{12} + (33600 - 384\beta)z^{14} + (278865 + 576\beta)z^{16} \dots,$$

where $0 \leq \beta \leq 10$, or

$$W(z) = 1 + 3451z^{12} + 24128z^{14} + 336081z^{16} \dots.$$

It is still unknown whether there exist extremal self-orthogonal singly-even codes for the weight enumerators with $\beta = 2, 3, \dots, 9$. Unfortunately, all codes obtained from $\mathcal{D}'_1$–$\mathcal{D}'_{25}$ have dimension less than 30. There is some hope that such an approach may work for $[45, 12, 4]$ designs invariant under the cyclic group of order 5 fixing no points or blocks.

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