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DUBROVIN TYPE EQUATIONS FOR COMPLETELY INTEGRABLE SYSTEMS ASSOCIATED WITH A POLYNOMIAL PENCIL

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Abstract. Dubrovin type equations for the N-gap solution of a completely integrable system associated with a polynomial pencil is constructed and then integrated to a system of functional equations. The approach used to derive those results is a generalization of the familiar process of finding the 1-soliton (1-gap) solution by integrating the ODE obtained from the soliton equation via the substitution $u = u(x + \lambda t)$.

1. Introduction. Back in 1967 Gardner, Green, Kruskal and Miura [1] solved the Cauchy problem for the Korteweg-de Vries (KdV) equation

$$u_t = 6uu_x - u_{xxx} \tag{1.1}$$

with an initial condition $u(x, t = 0) = u_0(x)$ decreasing sufficiently fast at infinity thus starting a new branch in mathematics, Soliton Theory (or Inverse Scattering

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Method, ISM) [2]–[4]. What they did was establishing a connection between Eq. (1.1) and the time-independent Schrödinger equation

$$-f_{xx} + uf = \lambda f, \qquad \lambda = \text{const} \tag{1.2}$$

in the sense that a time evolution of the potential u according to (1.1) leads to a simple linear time evolution for the respective scattering data S of (1.2) resulting in an exponential dependence of S on time. In that way, the Cauchy problem for (1.1) is reduced to solving the inverse problem for Eq. (1.2), i.e., finding $u(x,t)$ from a given set of scattering data $S(t,\lambda)$ for any fixed time t.

Later on, many other nonlinear evolution equations (NEEs) like (1.1) were discovered for which ISM can be applied due to the existence of respective linear spectral problems associated with them.

The inverse problem for the latter is standardly solved by using a Gelfand-Levitan-Marchenko (GLM) equation. For reflectionless potentials u corresponding to a finite number of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ in the spectrum, the GLM equation is reduced to a linear algebraic $N \times N$ system of equations which can be solved explicitly yielding the so called N-soliton solution. In the case of KdV, that solution has the form

$$u(x,t) = -2\frac{d^2}{dx^2}\ln(\det V(x,t)), \qquad v_{ij} = \delta_{ij} - m_j e^{-8p_j^3 t}\left(\frac{e^{(p_i+p_j)x}}{p_i+p_j}\right) \tag{1.3}$$

with arbitrary constants $m_j > 0$, $p_j > 0$, $p_i \neq p_j$ for $i \neq j$.

The N-gap solution is a generalization of the N-soliton solution and corresponds to periodic boundary conditions. It is generated by N functions $Q_1, \ldots, Q_N$ which satisfy two (compatible) systems of first-order ODEs, one in space (x) and one in time (t), called Dubrovin equations [5]. The link between $Q_1, \ldots, Q_N$ and the potential u is provided by the trace formula which, for the KdV equation, has the form

$$u(x,t) = \mu_0 + \sum_{k=1}^{N}\left[\mu_{2k-1} + \mu_{2k} - 2Q_k(x,t)\right]$$

(see [6]). By using techniques from algebraic geometry, Its and Matveev [7] expressed the N-gap solution $u(x,t)$ of KdV explicitly in terms of θ-functions,

$$u(x,t) = -2\frac{d^2}{dx^2}\ln\theta(x + 4\lambda_1 t + a_1, \ldots, x + 4\lambda_N t + a_N) + \tilde{c}, \quad \tilde{c} = \text{const}\,. \tag{1.4}$$

At the present time, there are several direct methods (i.e., not using ISM) for deriving the N-soliton solution of completely integrable NEEs such as the Hirota method, the dressing method, use of Bäcklund transformations, etc., an overview of which can be found, for instance, in [2]–[4].

The present article introduces yet another approach for finding the N-soliton solution of KdV in its explicit form (1.3). In addition, the new approach yields the Dubrovin equations for the N-gap solution and integrates them to a system of N coupled functional equations for $Q_1, \ldots, Q_N$. The trace formula is also obtained.

As an application of the scheme developed in the paper, all results concerning the N-gap solution are extended to the case of the polynomial KdV equation (PKdV). The latter is a system of NEEs associated with the spectral problem

$$-f_{xx} + \left(\sum_{r=0}^{M-1} \lambda^r u_r \right) f = \lambda^M f, \qquad \lambda = \text{const} \tag{1.5}$$

known as Polynomial Pencil or Polynomial Schrödinger Equation [8]–[13], [15]. Eq. (1.5) is a generalization of (1.2) and has the peculiar property that its respective NEEs which are of the form [8]

$$u_t = \Omega(\Lambda) u_x, \qquad \Omega - \text{polynomial} \tag{1.6}$$

where

$$u = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{M-1} \end{pmatrix}, \qquad \Lambda = \left(\begin{array}{ccccc|c} 0 & 0 & \cdots & 0 & & (-\frac{1}{4}\partial_{xxx} + j(u_0))\partial_x^{-1} \\ 1 & 0 & \cdots & 0 & & j(u_1)\partial_x^{-1} \\ 0 & 1 & \ddots & \vdots & & j(u_2)\partial_x^{-1} \\ \vdots & \ddots & \ddots & 0 & & \vdots \\ 0 & \cdots & 0 & 1 & & j(u_{M-1})\partial_x^{-1} \end{array} \right) \tag{1.7}$$

$\Big($here we have $j(u_r) = u_r \partial_x + \frac{1}{2} u_{r,x}$ and $\partial_x^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right)$ in the case of $u_r \in L^1(-\infty, \infty)$, $r = 0, 1, \ldots, M-1\Big)$ possess some features characteristic of completely integrable systems but lack (for $M > 2$ only!) others. For instance, Eq. (1.6) has Lax representations [9, 12, 15] and a bi-Hamiltonian structure [8] while at the same time there are serious difficulties finding a GLM equation [10], asymptotics of the Jost solutions for $\lambda \to \infty$ [11], Bäcklund transformations [13], etc. Even 1-soliton solution does not exist in an explicit form for $M > 2$ (see Appendix A).

In light of all that, obtaining Dubrovin equations for the N-gap solution and integrating them is a step forward in the solution of that problem. The success of the new approach is due to its non-spectral nature which allows it to avoid the problems associated with the asymptotics for $\lambda \to \infty$, the GLM equation, etc. Rather, the proposed scheme is based on the hereditary symmetry property [16] of the recursion operator Λ in (1.7) [15] and the existing as a result of that Lax pair (Λ, B) [13]–[15].

The new approach also reveals a certain duality between the pairs (KdV, PKdV) and (KdV, GKdV) (GKdV being Generalized KdV, or the KdV hierarchy), namely, the fact that the time evolution of $Q_1, \ldots, Q_N$ (as well as $F_1, \ldots, F_N$ — see the scheme below) for KdV and PKdV is the same but the space evolution is different while for KdV and GKdV it is the other way around.

Here, by PKdV we denote the first nonlinear system of equations in the hierarchy (1.6) corresponding to $\Omega(\mu) = 4\mu$, i.e., $u_t = 4\Lambda u_x$ or

$$u_{r,t} = 4u_{r-1,x} + 4j(u_r)u_{M-1} - \delta_{r0}\, u_{M-1,xxx}, \quad r = 0, 1, \ldots, M-1, \tag{1.8}$$

$(u_{-1} = 0)$ so that for $M = 1$ it is reduced to the KdV equation (1.1). If $M = 2$ then the Jaulent-Miodek system of equations is obtained [17].

Let us briefly recall the procedure for finding the 1-soliton (1-gap) solution of KdV which serves as a foundation of our approach.

By looking for solutions of Eq. (1.1) in the form $u = u(x + \lambda t)$ we make the substitution

$$u_t = \lambda u_x \tag{1.9}$$

into (1.1) reducing it to a third-order ODE,

$$-u_{xxx} + 6uu_x = \lambda u_x, \tag{1.10}$$

which, after a multiplication by $2u$, can be integrated to

$$-2uu_{xx} + u_x^2 + 4u^3 = \lambda u^2 + c, \qquad c = \text{const}\,. \tag{1.11}$$

Now we get rid of the first derivative u_x by using the transformation $u = f^2$, and then we integrate once again:

$$-f_{xx} + f^3 = \frac{\lambda}{4} f + \frac{c}{4f^3}, \tag{1.12}$$

$$f_x^2 = \frac{f^4}{2} - \frac{\lambda}{4} f^2 - d + \frac{c}{4f^2}, \qquad d = \text{const}\,. \tag{1.13}$$

Finally, we take a square root on both sides and then integrate one more time to obtain $f(x)$ and the corresponding solution $u(x,t) = f^2(x + \lambda t)$ of Eq. (1.1).

For $c = d = 0$ we find

$$f^2(x) = \frac{\lambda}{2\,\mathrm{ch}^2\left[\frac{\sqrt{-\lambda}}{2}(x - x_0)\right]} \tag{1.14}$$

which yields the 1-soliton solution for negative values of λ. In the general case the solution of Eq. (1.13) can be expressed as (see, i.e. [3])

$$f^2(x) = \mu - \nu\, cn^2\left(\sqrt{\frac{\lambda + 4\nu - 6\mu}{4}}(x - x_0); \sqrt{\frac{2\nu}{\lambda + 4\nu - 6\mu}}\right)$$

where μ and ν are defined via the equalities

$$2\mu^3 - \lambda\mu^2 - 4d\mu + c = 0$$

and

$$\nu^2 + \frac{\lambda - 6\mu}{2}\nu + \left(3\mu^2 - \lambda\mu - 2d\right) = 0,$$

in agreement with the familiar form of the 1-gap solution as a cnoidal wave. (Eq. (1.14) is obtained when $\mu = 0$, $\nu = -\lambda/2$.)

The contents of the article are as follows.

In Sec. 2, the analogs of Eqs. (1.10) – (1.13) corresponding to the N-gap solution are obtained and the respective time evolution, compatible with those equations, is presented. It is shown that the constants of integration are time-independent. The analog of Eq. (1.13) is found to be separable into a few "independent" parts with a common structure.

Then, in Sec. 3, the N-soliton solution of KdV is obtained by viewing the respective system of equations as a linear system with a cubic perturbation term.

In Sec. 4, the analog of Eq. (1.13), being a system of equations with respect to N functions $F_1(x,t), \dots, F_N(x,t)$, is diagonalized in order to allow for an extraction of a square root as in (1.13). That naturally leads to a change of variables $\{F_1, \dots, F_N\} \to \{Q_1, \dots, Q_N\}$ where $Q_1, \dots, Q_N$ satisfy the Dubrovin equations for KdV. Then those equations are integrated to a system of functional equations by extending the N first integrals available with the N-soliton solution.

Finally, in Sec. 5, the entire procedure is applied to the case of PKdV leading to analogous results for the N-gap solution of PKdV.

2. First-order ODE system for the N–gap KdV solution. In order to find the right way of generalizing Eq. (1.9) we have to realize that its purpose is reducing Eq. (1.1) to an ODE in a meaningful way, i.e., Eqs. (1.1) and (1.9) have to be compatible. Indeed, they are, due to the apparent compatibility between (1.9) and (1.10).

All that naturally leads us to one of the Lax pairs associated with the KdV equation and, for reasons which will become clear in a moment, we choose the Lax pair (Λ, B) associated with the recursion operator Λ,

$$\Lambda_t = B\Lambda - \Lambda B, \tag{2.1}$$

$$\Lambda = -\frac{1}{4}\partial_{xx} + u + \frac{1}{2}u_x\partial_x^{-1}, \qquad B = -\partial_{xxx} + 6u\partial_x + 6u_x$$

which expresses the compatibility of the equations

$$G_t = BG \tag{2.2}$$

and

$$\Lambda G = \lambda G. \tag{2.3}$$

Here ∂_x^{-1} is a suitably defined operator, inverse to ∂_x. For our purpose, however, its specific form is not important because Eqs. (2.2) and (2.3) are transformed below into differential equations not containing ∂_x^{-1}.

Since Λ is an (integro-differential) operator of x only, Eqs. (2.2) and (2.3) should be analogs of Eqs. (1.1) and (1.10), respectively. Such analogy exists due to the fact that Eq. (2.2) is actually the linearized (perturbed) KdV equation and, therefore, u_x satisfies it. In other words, the substitution $G = u_x$ transforms Eqs. (2.2) and (2.3) into Eqs. (1.1) (differentiated in x) and (1.10) (with a different λ), respectively.

To find a wider variety of solutions to the KdV equation, we use the linearity of Eq. (2.2) and look for u_x as a linear combination of other solutions of that equation:

$$u_x = G_1 + G_2 + \ldots + G_N \tag{2.4}$$

where G_k is a solution to both Eqs. (2.2) and (2.3) (for $\lambda = \lambda_k$). We can do that because the two equations are compatible as already noted.

The conclusion is that for $N \geq 1$ Eq. (1.10) can be generalized to a system of (integro-)differential equations in x,

$$\Lambda G_k = \lambda_k G_k, \qquad k = 1, \ldots, N, \tag{2.5}$$

together with Eq. (2.4). We assume that $\lambda_i \neq \lambda_j$ for $i \neq j$, otherwise the functions G_i and G_j can be combined into one function.

The substitution $G_k = F_{k,x}$ makes the equations in (2.5) purely differential and provides us with the final version for a generalization of Eq. (1.10):

$$-\frac{1}{4}F_{k,xxx} + uF_{k,x} + \frac{1}{2}u_x F_k = \lambda_k F_{k,x}, \qquad k = 1, \ldots, N, \tag{2.6}$$

where

$$u = -b + F_1 + \ldots + F_N, \qquad b = b(t). \tag{2.7}$$

The system (2.6), (2.7) is already known to have a solution for $b = 0$ due to the fact that the N-soliton solution u of the KdV equation is a sum of N squares of eigenfunctions $F_k = f_k^2$ of the Schrödinger equation (1.2) (see, e.g., [1]).

Now, let us begin integrating (2.6) and (2.7), and track the respective changes in the time evolution equation (2.2).

Eq. (2.2) (with the notation $G = F_x$) is integrated to

$$F_{k,t} = -F_{k,xxx} + 6uF_{k,x} + e_k, \quad e_k = e_k(t), \quad k = 1, \ldots, N. \tag{2.8}$$

Here we find that $b'(t) = e_1(t) + \ldots + e_N(t)$ due to Eqs. (1.1), (2.7) and (2.8). Also, Eq. (2.8) yields

$$(\partial_t + \partial_{xxx} - 6u_x - 6u\partial_x)\left[-\frac{1}{4}F_{k,xxx} + uF_{k,x} + \frac{1}{2}u_x F_k - \lambda_k F_{k,x}\right] = \frac{u_x}{2}e_k \tag{2.9}$$

leading to

$$e_k(t) = 0 \qquad \text{and} \qquad b'(t) = \sum_{i=1}^{N} e_i(t) = 0. \tag{2.10}$$

Note that, in addition to providing the constants of integration $e_k(t)$, Eq. (2.9) actually represents the compatibility of (2.6) and (2.8) (in the form $(\partial_t - B)(\Lambda G - \lambda G) = 0$, cf. Eq. (2.1)).

The constant b in Eq. (2.7) may be dropped as well since otherwise we would apply the transformation $\tilde{u}(x,t) = u(x+6bt,t)+b$, $\tilde{F}_k(x,t) = F_k(x+6bt,t)$, $\tilde{\lambda}_k = \lambda_k + b$ and make that constant disappear.

In accordance with the case $N = 1$, we multiply Eq. (2.6) by $2F_k$ and integrate:

$$-\frac{1}{2}F_k F_{k,xx} + \frac{1}{4}F_{k,x}^2 + uF_k^2 = \lambda_k F_k^2 + c_k, \qquad c_k = c_k(t). \tag{2.11}$$

Then the transformation $F_k = f_k^2$ yields

$$-f_{k,xx} + uf_k = \lambda_k f_k + \frac{c_k}{f_k^3}, \qquad k = 1, \ldots, N, \tag{2.12}$$

and Eq. (2.8) becomes

$$f_{k,t} = -4f_{k,xxx} + 6uf_{k,x} + 3u_x f_k + 12c_k \frac{f_{k,x}}{f_k^4} \tag{2.13}$$

with the use of (2.12). Now Eq. (2.13) implies

$$\left(\partial_t + 4\partial_{xxx} - 6u\partial_x - 3u_x - 12\frac{c_k}{f_k^4}\partial_x\right)\left(f_{k,xx} - uf_k + \lambda_k f_k + \frac{c_k}{f_k^3}\right) = \frac{c_k'}{f_k^3}$$

which leads to $c_k'(t) = 0$ in view of Eq. (2.12), and expresses the compatibility of Eqs. (2.12) and (2.13).

In the case $c_k = 0$, Eq. (2.12) becomes the standard Schrödinger equation and, together with (2.13), provides the usual Lax pair $L = -\partial_{xx} + u$, $A = -4\partial_{xxx} + 6u\partial_x + 3u_x$ for the KdV equation.

Eq. (2.13) can be replaced by a simpler evolution equation as a result of (2.12):

$$f_{k,t} = 4\lambda_k f_{k,x} + 2uf_{k,x} - u_x f_k, \tag{2.14}$$

with a corresponding equation for F_k,

$$F_{k,t} = 4\lambda_k F_{k,x} + 2\sum_{i=1}^{N}(F_{k,x}F_i - F_k F_{i,x}). \tag{2.15}$$

After a multiplication by $2f_{k,x}$, Eq. (2.12) is integrated once again:

$$
\begin{gathered}
f_{k,x}^2 + \lambda_k f_k^2 - \frac{1}{2} f_k^2 \sum_{i=1}^{N} f_i^2 + \frac{1}{2} \sum_{i \neq k} \frac{(f_{k,x} f_i - f_k f_{i,x})^2}{\lambda_k - \lambda_i} - \\
- \frac{c_k}{f_k^2} - \frac{1}{2} \sum_{i \neq k} \frac{c_k f_k^{-2} f_i^2 + c_i f_i^{-2} f_k^2}{\lambda_k - \lambda_i} + d_k = 0
\end{gathered} \tag{2.16}
$$

with $d_k = d_k(t)$, $k = 1, \dots, N$ (cf. [18, Ch. IIIa]). The respective compatibility condition is

$$
(\partial_t - R\partial_x)\left(\hat{\mathcal{M}} + \hat{\mathcal{N}} - \hat{C} + \hat{D}\right) = \hat{D}_t \tag{2.17}
$$

where R is the matrix multiplication operator

$$
R = 4 \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} + 2\left(f_1^2 + \ldots + f_N^2\right) - 2 \begin{pmatrix} f_1^2 \\ \vdots \\ f_N^2 \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}
$$

and $\hat{\mathcal{M}}, \hat{\mathcal{N}}, \hat{C}, \hat{D}$ are vector functions representing the different parts in Eq. (2.16):

$$
\begin{aligned}
\mathcal{M}_k &= f_{k,x}^2 + \frac{1}{2} \sum_{i \neq k} \frac{(f_{k,x} f_i - f_k f_{i,x})^2}{\lambda_k - \lambda_i}, \quad \mathcal{N}_k = \lambda_k f_k^2 - \frac{1}{2} f_k^2 \sum_{i=1}^{N} f_i^2, \\
C_k &= \frac{c_k}{f_k^2} + \frac{1}{2} \sum_{i \neq k} \frac{c_k f_k^{-2} f_i^2 + c_i f_i^{-2} f_k^2}{\lambda_k - \lambda_i}, \quad D_k = d_k.
\end{aligned}
$$

(Here we have $\hat{\mathcal{M}} = (\mathcal{M}_1, \dots, \mathcal{M}_N)^\top$, etc.) Just as before, Eq. (2.17) is a result of the respective time evolution (2.13) for f_k and implies that $d_k'(t) = 0$, $k = 1, \dots, N$.

For the N-soliton solution we must have $c_i = d_i = 0$ $(i = 1, \dots, N)$ in Eqs. (2.11) and (2.16) due to the vanishing of the functions $F_k = f_k^2$ at $x \to \pm\infty$. Later on we will derive that solution in the form (1.3) by using Eqs. (2.12), (2.13) and (2.16) subjected to the above restrictions (i.e., $c_i = d_i = 0$).

It turns out that the compatibility condition (2.17) holds for the different parts of (2.16) as well, namely, the relation

$$
(\partial_t - R\partial_x)\hat{Q} = 0 \tag{2.18}
$$

takes place for $\hat{Q} = \hat{\mathcal{M}}$, $\hat{\mathcal{N}}$, $\hat{C}$ and $\hat{D}$. The reason for that can be found in the following lemma.

Lemma 2.1. *The vector-functions $\hat{\mathcal{M}}$, $\hat{\mathcal{N}}$, $\hat{C}$ and $\hat{D}$ are generated by two matrix multiplication operators V and W acting on $\hat{F} = (F_1, \dots, F_N)^\top$ and such that*

$$[\partial_t - R\partial_x, V] = [\partial_t - R\partial_x, W] = 0, \tag{2.19}$$

namely,

$$\begin{aligned} \hat{\mathcal{M}} &= W^2\hat{F}, & \hat{\mathcal{N}} &= V\hat{F}, \\ \hat{D} &= \sum_{k=1}^{N} \frac{d_k}{2}(\lambda_k - V)^{-1}\hat{F}, & \hat{C} &= \sum_{k=1}^{N} \frac{c_k}{4}(\lambda_k - V)^{-2}\hat{F}, \end{aligned} \tag{2.20}$$

where

$$V = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} - \frac{1}{2}\begin{pmatrix} f_1^2 \\ \vdots \\ f_N^2 \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \tag{2.21}$$

and $W = (w_{ki})_{k,i=1}^N$ is defined by

$$\begin{aligned} w_{kk} &= \frac{1}{f_k}\left[f_{k,x} + \frac{1}{2}\sum_{i\neq k} \frac{f_i(f_{k,x}f_i - f_k f_{i,x})}{\lambda_k - \lambda_i} \right], \\ w_{ki} &= -\frac{1}{f_i}\left[\frac{f_k(f_{k,x}f_i - f_k f_{i,x})}{2(\lambda_k - \lambda_i)} \right], \quad i \neq k. \end{aligned} \tag{2.22}$$

P r o o f. It is easy to see that $\hat{F}_x = 2W\hat{F}$ and $\hat{\mathcal{M}} = \frac{1}{2}W\hat{F}_x$. The expression for $\hat{D}$ in (2.20) follows from the equality

$$\hat{F} = 2(\lambda_k - V)\hat{e}_k, \qquad k = 1, \dots, N \tag{2.23}$$

where $\hat{e}_k$ is a unit vector in $\mathbb{R}^N$, $\hat{e}_k = (\delta_{k1}, \dots, \delta_{kN})^\top$. As for $\hat{C}$, we use the relation $c_k\hat{e}_k = 2(\lambda_k - V)\hat{C}^{(k)}$ where $\hat{C}^{(k)}$ is defined by $\hat{C}^{(k)} = \hat{C}_{|c_i=0, i\neq k}$, so that $\hat{C} = \hat{C}^{(1)} + \ldots + \hat{C}^{(N)}$. Now we apply again Eq. (2.23) to find the result for $\hat{C}$.

What concerns Eq. (2.19), we note that $[\partial_t - R\partial_x, V] = 0$ is equivalent to the pair of equations $[R, V] = 0$ and $V_t - RV_x = 0$. Those, in turn, follow from the relations $R = 4V + 2(F_1 + \ldots + F_N)$ and $\hat{F}_t = R\hat{F}_x$ (cf. (2.15)), respectively. (The latter also means that (2.18) holds for $\hat{Q} = \hat{F}$ as well. $\hat{F}$ would have appeared in Eq. (2.16) if we had kept b in (2.7).)

The commutation relation $[\partial_t - R\partial_x, W] = 0$ is proved in a similar way.

Now Eqs. (2.15) and (2.19) lead to $(\partial_t - R\partial_x)V^m\hat{F} = (\partial_t - R\partial_x)W^m\hat{F} = 0$ for $m = 0, \pm 1, \pm 2, \ldots$ and, therefore, (2.18) indeed takes place for $\hat{Q} = \hat{\mathcal{M}}$, $\hat{\mathcal{N}}$, $\hat{C}$ and $\hat{D}$ (as well as for $\hat{F}$ and $\hat{F}_x$). $\square$

So far we have proved the following

Theorem 2.1. *The system of N third-order equations* (2.6), (2.7) *(for $b = 0$) is integrated to the first-order system* (2.16). *The constants of that integration are time-independent as a result of* (2.15). *The system* (2.16) *has the matrix form*

$$W^2\hat{F} + V\hat{F} + \sum_{k=1}^{N} \frac{d_k}{2}(\lambda_k - V)^{-1}\hat{F} - \sum_{k=1}^{N} \frac{c_k}{4}(\lambda_k - V)^{-2}\hat{F} = 0 \tag{2.24}$$

and its structure is determined entirely by the time evolution equation (2.15) *via the commutation relations* (2.19) *(in the sense that* (2.24) *is just one of a whole hierarchy of ODE systems compatible with* (2.15) *and generated by V and W).*

In Sec. 4 we will continue with the integration of Eq. (2.24) to a system of functional equations. For the moment, however, we turn our attention to the N-soliton solution of the KdV equation.

3. Derivation of the N–soliton solution for KdV. It was noted in Sec. 2 that the N-soliton solution corresponds to $c_i = d_i = 0$, $i = 1, \ldots, N$. Then Eq. (2.12) becomes

$$-f_{k,xx} + \left(\sum_{i=1}^{N} f_i^2 \right) f_k = \lambda_k f_k \tag{3.1}$$

and for (2.16) we find

$$f_{k,x}^2 + \left(\lambda_k - \frac{1}{2} \sum_{i=1}^{N} f_i^2 \right) f_k^2 + \frac{1}{2} \sum_{i \neq k} \frac{(f_{k,x} f_i - f_k f_{i,x})^2}{\lambda_k - \lambda_i} = 0. \tag{3.2}$$

Eq. (3.1) leads to $f_{k,xx}f_i - f_k f_{i,xx} = (\lambda_i - \lambda_k)\, f_k f_i$, so that (3.2) can be written as

$$f_{k,x}^2 + \left(\lambda_k - \frac{1}{2}\sum_{i=1}^{N} f_i^2\right) f_k^2 - \frac{1}{2}\sum_{i=1}^{N} (f_{k,x}f_i - f_k f_{i,x})\left(\partial_x^{-1} f_k f_i\right) = 0. \tag{3.3}$$

Now we multiply Eq. (3.1) by f_k and add it to (3.3):

$$f_{k,x}\left[f_{k,x} - \frac{1}{2}\sum_{i=1}^{N} f_i\left(\partial_x^{-1} f_k f_i\right)\right] - f_k\left[f_{k,x} - \frac{1}{2}\sum_{i=1}^{N} f_i\left(\partial_x^{-1} f_k f_i\right)\right]_x = 0.$$

The conclusion is that

$$f_{k,x} - \frac{1}{2}\sum_{i=1}^{N} f_i\left(\partial_x^{-1} f_k f_i\right) = p_k f_k, \qquad p_k = p_k(t), \qquad k = 1, \ldots, N, \tag{3.4}$$

which, in a matrix notation, has the form

$$\hat{f}_x = P\hat{f} + \frac{1}{2}\left(\partial_x^{-1}\hat{f}\hat{f}^\top\right)\hat{f} \tag{3.5}$$

with

$$\hat{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}, \qquad P = \begin{pmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_N \end{pmatrix}.$$

Eq. (3.5) can be viewed as a linear system ($\hat{f}_x = P\hat{f}$) with a cubic perturbation term and one can solve it order by order.

The $(2n+1)$-order correction to Eq. (3.5) will have the form

$$\left(T^n\hat{f}\right)_x = PT^n\hat{f} + 2\sum_{i=1}^{n} T^i P T^{n-i}\hat{f} - 2\sum_{i=1}^{n+1} T^i P T^{n+1-i}\hat{f}, \; T = -\frac{1}{4}\left(\partial_x^{-1}\hat{f}\hat{f}^\top\right)P^{-1}, \tag{3.6}$$

and can be proved by differentiating $T^n\hat{f}$ in x and replacing everywhere $\hat{f}_x$ with the expression (3.5). Thus, by adding the corrections for all odd orders, we obtain

$$\left(\sum_{n=0}^{\infty} T^n\hat{f}\right)_x = P\sum_{n=0}^{\infty} T^n\hat{f} \tag{3.7}$$

leading to

$$\sum_{n=0}^{\infty} T^n \hat{f} = \hat{\varphi}, \qquad \hat{\varphi} = (q_1(t)e^{p_1 x}, q_2(t)e^{p_2 x}, \ldots, q_N(t)e^{p_N x})^\top . \tag{3.8}$$

Now we invert the series (3.8), i.e., express $\hat{f}$ as a function of $\hat{\varphi}$, by applying similar arguments concerning order as above. Equivalently, we have, due to (3.8),

$$\hat{\varphi} = \sum_{n=0}^{\infty} T^n \hat{f} = \hat{f} + T\left(\sum_{n=0}^{\infty} T^n \hat{f}\right) = \hat{f} + T\hat{\varphi} \tag{3.9}$$

resulting in $\hat{f} = (1-T)\hat{\varphi}$. Then, in order to express $\hat{f}$ entirely in terms of $\hat{\varphi}$, we use the relation $\hat{\varphi}^\top = \sum\limits_{n=0}^{\infty} \hat{f}^\top \left[-\frac{1}{4}P^{-1}\left(\partial_x^{-1}\hat{f}\hat{f}^\top\right)\right]^n$ (see (3.8)) to calculate $\hat{\varphi}\hat{\varphi}^\top$, or

$$-\frac{1}{4}\left(\partial_x^{-1}\hat{\varphi}\hat{\varphi}^\top\right)P^{-1} = \sum_{n=1}^{\infty}\left[-\frac{1}{4}\left(\partial_x^{-1}\hat{f}\hat{f}^\top\right)P^{-1}\right]^n = \sum_{n=1}^{\infty} T^n. \tag{3.10}$$

Now Eqs. (3.9) and (3.10) imply that

$$\hat{f} = (1-T)\hat{\varphi} \equiv \left(1+T+T^2+\ldots\right)^{-1}\hat{\varphi} = \left[1-\frac{1}{4}\left(\partial_x^{-1}\hat{\varphi}\hat{\varphi}^\top\right)P^{-1}\right]^{-1}\hat{\varphi}. \tag{3.11}$$

Eq. (3.11) represents the N-soliton solution since it is equivalent to the $N \times N$ linear system of equations

$$f_k(x,t) = \varphi_k(x,t) + \sum_{i=1}^{N} f_i(x,t)\int_{\pm\infty}^{x} \frac{\varphi_i(s,t)}{4p_i}\varphi_k(s,t)ds, \quad k = 1,\ldots,N, \tag{3.11$'$}$$

that one obtains when solving the GLM equation.

Now we need to find the dependence on time as well. The first step in that direction is proving the relation

$$p_k^2 = -\lambda_k. \tag{3.12}$$

which implies that the constants (in x) p_k are constants in time as well. Eq. (3.12) is a result of the next two lemmas.

Lemma 3.1. *If $\hat{f} = (f_1, \ldots, f_N)^\top$ is a solution of the matrix equation*

$$\tilde{T}\hat{f} = \hat{\varphi}, \qquad \tilde{T} = I - \left(\partial_x^{-1}\hat{\varphi}\hat{\varphi}^\top S\right), \qquad S = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_N \end{pmatrix} \tag{3.13}$$

where I is the $N \times N$ identity matrix and $s_1, \ldots, s_N$ are constants (cf. Eq. (3.11) for $S = (4P)^{-1}$) then $f_k(x)$ satisfies the Schrödinger equation (1.2) *with $\lambda_k = -p_k^2$ and*

$$u = 2\frac{d}{dx}\left(\sum_{r=1}^{N} s_r f_r \varphi_r\right) \tag{3.14}$$

(*see, e.g.,* [3, Ch. 3]).

P r o o f. We differentiate Eq. (3.13) twice and use $\hat{\varphi}_{xx} = P^2\hat{\varphi}$:

$$\hat{\varphi}_{xx} = \tilde{T}\hat{f}_{xx} - 2\left(\hat{\varphi}\hat{\varphi}^\top S\right)\hat{f}_x - \left(\hat{\varphi}\hat{\varphi}^\top S\right)_x \hat{f} \equiv \tilde{T}\hat{f}_{xx} - 2\hat{\varphi}\left(\hat{\varphi}^\top S\hat{f}\right)_x + \left(\hat{\varphi}\hat{\varphi}_x^\top - \hat{\varphi}_x\hat{\varphi}^\top\right) S\hat{f} =$$

$$= \tilde{T}\left[\hat{f}_{xx} - 2\hat{f}\left(\hat{\varphi}^\top S\hat{f}\right)_x\right] + \left[\partial_x^{-1}\left(\hat{\varphi}\hat{\varphi}_{xx}^\top - \hat{\varphi}_{xx}\hat{\varphi}^\top\right)\right] S\hat{f} \equiv$$

$$\equiv \tilde{T}\left[\hat{f}_{xx} - 2\hat{f}\left(\hat{\varphi}^\top S\hat{f}\right)_x\right] + \left[\partial_x^{-1}\left(\hat{\varphi}\hat{\varphi}^\top P^2 - P^2\hat{\varphi}\hat{\varphi}^\top\right)\right] S\hat{f} \equiv$$

$$\equiv \tilde{T}\left[\hat{f}_{xx} - 2\hat{f}\left(\hat{\varphi}^\top S\hat{f}\right)_x\right] + \left(-\tilde{T}P^2 + P^2\tilde{T}\right)\hat{f} = \tilde{T}\left[\hat{f}_{xx} - 2\hat{f}\left(\hat{\varphi}^\top S\hat{f}\right)_x - P^2\hat{f}\right] + P^2\hat{\varphi}.$$

Thus we find $\tilde{T}\left[\hat{f}_{xx} - 2\hat{f}\left(\hat{\varphi}^\top S\hat{f}\right)_x - P^2\hat{f}\right] = 0$ and then apply $\tilde{T}^{-1}$. $\square$

Lemma 3.2. *The potential u from Lemma* 3.1 *can be presented also as*

$$u = 4\sum_{r=1}^{N} p_r s_r f_r^2.$$

P r o o f. Again, we use Eq. (3.13) and $\hat{\varphi}_x = P\hat{\varphi}$ to obtain

$$u = 2\left(\hat{\varphi}^\top S\hat{f}\right)_x = 2\hat{\varphi}_x^\top S\hat{f} + 2\left(\tilde{T}\hat{f}\right)^\top S\hat{f}_x \equiv$$

$$\equiv 2\hat{\varphi}_x^\top S\hat{f} + 2\hat{f}^\top\left[I - S\left(\partial_x^{-1}\hat{\varphi}\hat{\varphi}^\top\right)\right] S\hat{f}_x \equiv 2\hat{\varphi}_x^\top S\hat{f} + 2\hat{f}^\top S\tilde{T}\hat{f}_x =$$

$$= 2\hat{\varphi}_x^\top S\hat{f} + 2\hat{f}^\top S\left(\hat{\varphi}_x - \tilde{T}_x\hat{f}\right) = 2\left(P\hat{\varphi}\right)^\top S\hat{f} + 2\hat{f}^\top S\left(P\hat{\varphi}\right) + 2\hat{f}^\top S\hat{\varphi}\hat{\varphi}^\top S\hat{f} =$$

$$= 2\left[\left(P\tilde{T}\hat{f}\right)^{\top} S\hat{f} + \hat{f}^{\top} SP\tilde{T}\hat{f}\right] + 2\hat{f}^{\top} S\hat{\varphi}\hat{\varphi}^{\top} S\hat{f} \equiv$$

$$\equiv 2\left[2\hat{f}^{\top} PS\hat{f} - \hat{f}^{\top}\left(S\partial_x^{-1}\hat{\varphi}\hat{\varphi}^{\top}\right) PS\hat{f} - \hat{f}^{\top} SP\left(\partial_x^{-1}\hat{\varphi}\hat{\varphi}^{\top} S\right)\hat{f}\right] + 2\hat{f}^{\top} S\hat{\varphi}\hat{\varphi}^{\top} S\hat{f} \equiv$$

$$\equiv 4\hat{f}^{\top} PS\hat{f} + 2\hat{f}^{\top} S\left[-\left(\partial_x^{-1}\hat{\varphi}\hat{\varphi}^{\top}\right) P - P\left(\partial_x^{-1}\hat{\varphi}\hat{\varphi}^{\top}\right) + \hat{\varphi}\hat{\varphi}^{\top}\right] S\hat{f} = 4\hat{f}^{\top} PS\hat{f}. \ \square$$

Now, by comparing that result (for $s_k = (4p_k)^{-1}$) with Eq. (3.1), we come to (3.12).

In addition to $\hat{\varphi}_x = P\hat{\varphi}$ (see (3.7)), the following relation takes place:

$$\hat{\varphi}_t = -4P^3\hat{\varphi}. \tag{3.15}$$

Its proof is based on the time analog of Eq. (3.5), namely,

$$\hat{f}_t = -4P^3\hat{f} - 2\left(\partial_x^{-1}\hat{f}\hat{f}^{\top}\right) P^2\hat{f}. \tag{3.16}$$

Eq. (3.16), in turn, follows from (2.14) and (3.5),

$$\hat{f}_t = -4P^2\hat{f}_x + 2\left(\hat{f}_x\hat{f}^{\top} - \hat{f}\hat{f}_x^{\top}\right)\hat{f} = -4P^3\hat{f} - 2P^2\left(\partial_x^{-1}\hat{f}\hat{f}^{\top}\right)\hat{f} +$$

$$+2\left(P\hat{f}\hat{f}^{\top} - \hat{f}\hat{f}^{\top}P\right)\hat{f} + \left[\left(\partial_x^{-1}\hat{f}\hat{f}^{\top}\right)\hat{f}\hat{f}^{\top} - \hat{f}\hat{f}^{\top}\left(\partial_x^{-1}\hat{f}\hat{f}^{\top}\right)\right]\hat{f} \equiv -4P^3\hat{f} -$$

$$-2P^2\left(\partial_x^{-1}\hat{f}\hat{f}^{\top}\right)\hat{f} + 2\left[P\left(\partial_x^{-1}(\hat{f}_x\hat{f}^{\top} + \hat{f}\hat{f}_x^{\top})\right) - \left(\partial_x^{-1}(\hat{f}_x\hat{f}^{\top} + \hat{f}\hat{f}_x^{\top})\right)P\right]\hat{f} +$$

$$+\left[\left(\partial_x^{-1}\hat{f}\hat{f}^{\top}\right)\left(\partial_x^{-1}(\hat{f}_x\hat{f}^{\top} + \hat{f}\hat{f}_x^{\top})\right) - \left(\partial_x^{-1}(\hat{f}_x\hat{f}^{\top} + \hat{f}\hat{f}_x^{\top})\right)\left(\partial_x^{-1}\hat{f}\hat{f}^{\top}\right)\right]\hat{f}.$$

(Here $\hat{f}_x$ is replaced by the expression (3.5) to yield (3.16).) Eq. (3.16) leads to

$$\left(T^n\hat{f}\right)_t = -4P^3\left(T^n\hat{f}\right) - 8\left[\sum_{i=1}^{n} T^i P^3 T^{n-i}\right]\hat{f} + 8\left[\sum_{i=1}^{n+1} T^i P^3 T^{n+1-i}\right]\hat{f} \tag{3.17}$$

which is analogous to (3.6) and results in Eq. (3.15). The conclusion is that

$$\varphi_k = q_k(t)e^{p_k x} = r_k e^{p_k x - 4p_k^3 t}. \tag{3.18}$$

Finally, from Eqs. (3.11′) and (3.14) it follows (see, e.g., [19]) that the potential u can be presented in the form (1.3) where

$$V(x,t) = 1 - \frac{1}{4}\left(\partial_x^{-1}\hat{\varphi}\hat{\varphi}^{\top}\right)P^{-1}, \qquad \text{i.e.,} \qquad v_{ij} = \delta_{ij} - \frac{q_i(t)q_j(t)e^{(p_i+p_j)x}}{4p_j(p_i+p_j)}.$$

The standard form for the entries of $V(x,t)$ (see (1.3)) is obtained after dividing the i-th row of $V(x,t)$ by $q_i(t)$ and then multiplying the j-th column by $q_j(t)$ for all $i,j=1,\dots,N$. Obviously, that does not change the determinant of $V(x,t)$.

4. Dubrovin equations for KdV. In this section we diagonalize the term $\mathcal{M}=W^2\hat{F}$ (the one containing the derivatives) in Eq. (2.24) which involves a change of variables $\{F_k\}_{k=1}^N \to \{Q_k\}_{k=1}^N$, and then derive the Dubrovin equations for the N-gap solution of KdV. Those equations are afterwards integrated to a system of functional equations.

We are essentially looking for functions $\alpha_k(f_1,f_2,\dots,f_N)$, $k=1,\dots,N$, such that $\alpha_1\mathcal{M}_1+\alpha_2\mathcal{M}_2+\ldots+\alpha_N\mathcal{M}_N$ is a complete square.

Lemma 4.1. *A necessary and sufficient condition for* $\alpha_1\mathcal{M}_1+\cdots+\alpha_N\mathcal{M}_N$ *to be a complete square is the set of equations*

$$\sum_{i=1}^N \alpha_i f_i^2 = -2\alpha_k\alpha_j\left(\frac{\lambda_k-\lambda_j}{\alpha_k-\alpha_j}\right) \qquad (k\neq j;\quad k,j=1,\dots,N) \tag{4.1}$$

to hold for the functions $\alpha_1,\dots,\alpha_N$.

Proof. Let $\sum\limits_{k=1}^N \alpha_k\mathcal{M}_k$ be a complete square, i.e., $\sum\limits_{k=1}^N \alpha_k\mathcal{M}_k = \left(\sum\limits_{k=1}^N \beta_k f_{k,x}\right)^2$. Then the matrices corresponding to those quadratic forms coincide too:

$$\alpha_k+\sum_{i\neq k}\frac{(\alpha_k-\alpha_i)f_i^2}{2(\lambda_k-\lambda_i)}=\beta_k^2,\ -\frac{(\alpha_k-\alpha_i)f_kf_i}{2(\lambda_k-\lambda_i)}=\beta_k\beta_i\ (i\neq k;\ i,k=1,\dots,N). \tag{4.2}$$

Eq. (4.2) leads to $\beta_k\sum\limits_{i=1}^N \beta_i f_i = \alpha_k f_k$ which means that the vector $(\beta_1,\dots,\beta_N)$ must be proportional to $(\alpha_1 f_1,\dots,\alpha_N f_N)$. From here one easily obtains (4.1).

The opposite follows from the relation

$$\sum_{k=1}^N \alpha_k\mathcal{M}_k \overset{(4.1)}{=} \frac{(\alpha_1 f_1 f_{1,x}+\ldots+\alpha_N f_N f_{N,x})^2}{\alpha_1 f_1^2+\ldots+\alpha_N f_N^2}. \tag{4.3}$$

$\square$

Lemma 4.1 can be used, at least in principle, to calculate the ratios between $\alpha_1, \alpha_2, \ldots, \alpha_N$. In order to do that, however, N-th degree algebraic equations have to be solved. A way out of that problem is by looking for a solution in terms of series. We assume that, for a fixed k,

$$\alpha_k = 1+\alpha_k^{(1)}+\alpha_k^{(2)}+\ldots, \qquad \alpha_i = \alpha_i^{(1)}+\alpha_i^{(2)}+\ldots \qquad (i \neq k; \quad i = 1, \ldots, N),$$

where $\alpha_s^{(m)}$, $s = 1, \ldots, N$, is a term of order $2m$ with respect to $f_1, \ldots, f_N$, and then solve Eq. (4.1) with the additional requirement that $2(\alpha_1 f_1 f_{1,x} + \ldots + \alpha_N f_N f_{N,x})$ (see (4.3)) be an x-derivative of a function $H_k(f_1, \ldots, f_N)$, i.e., the equality $\dfrac{\partial(\alpha_i f_i)}{\partial f_j} = \dfrac{\partial(\alpha_j f_j)}{\partial f_i}$ has to hold for $i, j = 1, \ldots, N$. Thus, at $N = 2$, for example, we obtain

$$\alpha_k = 1 + \left[-\frac{f_i^2}{2(\lambda_k-\lambda_i)} + \omega_1 f_k^2\right] + \left[\frac{-2f_k^2 f_i^2 + f_i^4}{4(\lambda_k-\lambda_i)^2} - \frac{2\omega_1 f_k^2 f_i^2}{2(\lambda_k-\lambda_i)} + \omega_2 f_k^4\right] + \ldots$$

$$\alpha_i = -\frac{f_k^2}{2(\lambda_k-\lambda_i)} + \left[\frac{-f_k^4 + 2f_k^2 f_i^2}{4(\lambda_k-\lambda_i)^2} - \frac{\omega_1 f_k^4}{2(\lambda_k-\lambda_i)}\right] + \ldots$$

($\omega_{1,2} = \text{const}$), and

$$\text{(4.4)} \quad H_k = f_k^2 + \left[-\frac{f_k^2 f_i^2}{2(\lambda_k-\lambda_i)} + \frac{\omega_1 f_k^4}{2}\right] + \left[\frac{-f_k^4 f_i^2 + f_k^2 f_i^4}{4(\lambda_k-\lambda_i)^2} - \frac{\omega_1 f_k^4 f_i^2}{2(\lambda_k-\lambda_i)} + \frac{\omega_2 f_k^6}{3}\right] + \cdots$$

It turns out that inverting Eq. (4.4) (i.e., expressing f_1^2 and f_2^2 as series in H_1 and H_2 which can be done order by order) for $\omega_1 = \omega_2 = 0$ results in finite series! We find that $f_k^2 = H_k\left(1 - \dfrac{H_i}{2(\lambda_i-\lambda_k)}\right)$, $k, i = 1, 2$; $k \neq i$.

For an arbitrary N, the series are also finite and the corresponding change of variables that diagonalizes $W^2\hat{F}$ is $f_k^2 = H_k \prod\limits_{i\neq k}\left(1 - \dfrac{H_i}{2(\lambda_i-\lambda_k)}\right)$, $k = 1, \ldots, N$. The proof of that can be found in Theorem 4.1 below. At first, however, we introduce a more convenient set of variables $\{Q_k\}_{k=1}^N$ by the formula $Q_k = \lambda_k - \dfrac{H_k}{2}$. Then the above change of variables becomes (see, e.g., [4, Ch. 2])

$$\text{(4.5)} \qquad F_k = f_k^2 = 2\frac{\prod\limits_{i=1}^{N}(\lambda_k - Q_i)}{\prod\limits_{i\neq k}(\lambda_k-\lambda_i)}, \qquad k = 1, \ldots, N.$$

Theorem 4.1. *Let the matrix* $\mathcal{A} = \{\alpha_{ij}\}_{i,j=1}^N$ *be defined by* $\alpha_{ij} = \dfrac{1}{2(\lambda_j - Q_i)}$. *Then the following relations hold:*

(*i*)
$$\mathcal{A}\hat{F} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix};$$

(*ii*)
$$\mathcal{A}V\mathcal{A}^{-1} = \tilde{V}, \qquad \tilde{V} = \begin{pmatrix} Q_1 & & 0 \\ & \ddots & \\ 0 & & Q_N \end{pmatrix};$$

(*iii*)
$$\mathcal{A}W\mathcal{A}^{-1} = \tilde{W}, \qquad \tilde{W} = \begin{pmatrix} \frac{Q_{1,x}\prod\limits_{i\neq 1}(Q_1-Q_i)}{2\prod\limits_{i=1}^N(Q_1-\lambda_i)} & & 0 \\ & \ddots & \\ 0 & & \frac{Q_{N,x}\prod\limits_{i\neq N}(Q_N-Q_i)}{2\prod\limits_{i=1}^N(Q_N-\lambda_i)} \end{pmatrix}.$$

As a result, in the variables Q_k *Eq.* (2.24) *takes the form*

$$\left[\tilde{W}^2 + \tilde{V} + \sum_{k=1}^N \frac{d_k}{2}(\lambda_k - \tilde{V})^{-1} - \sum_{k=1}^N \frac{c_k}{4}(\lambda_k - \tilde{V})^{-2}\right]\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 0. \tag{4.6}$$

Before proving the above theorem we need to find the inverse matrix $\mathcal{A}^{-1}$. We will show that $\mathcal{A}^{-1}$ has elements

$$\overline{\alpha}_{ij} = 2\frac{\prod\limits_{s\neq j}(\lambda_i - Q_s)\prod\limits_{s=1}^N(\lambda_s - Q_j)}{\prod\limits_{s\neq i}(\lambda_i - \lambda_s)\prod\limits_{s\neq j}(Q_s - Q_j)}, \qquad i,j = 1,\dots,N.$$

Indeed, we have

$$\sum_{r=1}^N \overline{\alpha}_{ir}\alpha_{rj} = \frac{\prod\limits_{s=1}^N(\lambda_i - Q_s)}{\prod\limits_{s\neq i}(\lambda_i - \lambda_s)}\sum_{r=1}^N \frac{\prod\limits_{s=1}^N(\lambda_s - Q_r)}{(\lambda_i - Q_r)(\lambda_j - Q_r)\prod\limits_{s\neq r}(Q_s - Q_r)} = \delta_{ij} \tag{4.7}$$

as a result of the next lemma.

Lemma 4.2. *a) The linear in λ_k ($k \neq i,j$) polynomial* $S_N^{(i,j)} = \sum_{r=1}^{N} \frac{\prod_{s\neq i,j}(\lambda_s - Q_r)}{\prod_{s\neq r}(Q_s - Q_r)}$ *where $i \neq j$ and $N \geq 2$ is identically equal to zero.*

b) The linear in λ_k ($k \neq i$) polynomial $\tilde{S}_N^{(i)} = \sum_{r=1}^{N} \frac{\prod_{s\neq i}(\lambda_s - Q_r)}{(\lambda_i - Q_r)\prod_{s\neq r}(Q_s - Q_r)}$ *where $N \geq 1$ coincides with the expression* $\frac{\prod_{s\neq i}(\lambda_i - \lambda_s)}{\prod_{s=1}^{N}(\lambda_i - Q_s)}$.

P r o o f. a) If $N = 2$ then $S_2^{(1,2)} = \frac{1}{Q_2 - Q_1} + \frac{1}{Q_1 - Q_2} = 0$. Suppose $S_{N-1}^{(i,j)} \equiv 0$ for some $N \geq 3$ and every pair $i \neq j$. Then, for each k ($k \neq i,j$) and w, we have

$$S_N^{(i,j)}\Big|_{\lambda_k = Q_w} = \sum_{r\neq w} \frac{\prod_{s\neq i,j,k}(\lambda_s - Q_r)}{\prod_{s\neq r,w}(Q_s - Q_r)}.$$

However, this is zero according to the assumption for $S_{N-1}^{(i,j)}$. In other words, $S_N^{(i,j)}$ is a linear polynomial in λ_k which has N different zeros, $\lambda_k = Q_w$ ($1 \leq w \leq N$). Now $N \geq 2$ implies $S_N^{(i,j)} \equiv 0$.

b) The statement follows directly from a) since the expression

$$\sum_{r=1}^{N} \frac{\prod_{s\neq i}(\lambda_s - Q_r)}{(\lambda_i - Q_r)\prod_{s\neq r}(Q_s - Q_r)} - \frac{\prod_{s\neq i}(\lambda_i - \lambda_s)}{\prod_{s=1}^{N}(\lambda_i - Q_s)}$$

is nothing but $S_{N+1}^{(i,N+1)}$ where Q_{N+1} has been replaced by λ_i. $\square$

Now we are in a position to prove the above theorem.

P r o o f o f T h e o r e m 2.1. (ii) Let us calculate the (i,j)-th element of $\mathcal{A}^{-1}\tilde{V}\mathcal{A}$:

$$\sum_{r=1}^{N} \overline{\alpha}_{ir} Q_r \alpha_{rj} = \frac{\prod_{s=1}^{N}(\lambda_i - Q_s)}{\prod_{s\neq i}(\lambda_i - \lambda_s)} \sum_{r=1}^{N} \frac{(Q_r - \lambda_j + \lambda_j)\prod_{s\neq j}(\lambda_s - Q_r)}{(\lambda_i - Q_r)\prod_{s\neq r}(Q_s - Q_r)} =$$

$$= \frac{\prod\limits_{s=1}^{N}(\lambda_i - Q_s)}{\prod\limits_{s\neq i}(\lambda_i - \lambda_s)}\left(-\sum_{r=1}^{N}\frac{\prod\limits_{s\neq i}(\lambda_s - Q_r)}{\prod\limits_{s\neq r}(Q_s - Q_r)}\right) + \lambda_j\delta_{ij} = \lambda_j\delta_{ij} - \frac{1}{2}F_i.$$

Here we used Eq. (4.7) and Lemma 4.3 (below).

Lemma 4.3. *The expression* $\overline{S}_N^{(i)} = \sum\limits_{r=1}^{N}\frac{\prod_{s\neq i}(\lambda_s - Q_r)}{\prod_{s\neq r}(Q_s - Q_r)}$ *where* $N \geq 1$ *is identically equal to 1.*

The proof is similar to that of Lemma 4.2. Note that Lemma 4.3 ensures compliance with (4.1) for each row $\alpha_{s1}, \alpha_{s2}, \ldots, \alpha_{sN}$ of the matrix $\mathcal{A}$ due to Eq. (4.5).

The statement (i) is also a direct result of (4.5) and Lemma 4.3 (with the interchange $\lambda_i \leftrightarrow Q_i$ for all $i = 1, \ldots, N$).

As for proving (iii), we start with calculating the (i, j)-th element of $\mathcal{A}^{-1}\tilde{W}\mathcal{A}$:

$$(4.8)\quad \sum_{r=1}^{N}\overline{\alpha}_{ir}\frac{-Q_{r,x}}{2(\lambda_r - Q_r)}\left(\prod_{s\neq r}\frac{Q_s - Q_r}{\lambda_s - Q_r}\right)\alpha_{rj} = \frac{\prod\limits_{s=1}^{N}(\lambda_i - Q_s)}{2\prod\limits_{s\neq i}(\lambda_i - \lambda_s)}\sum_{r=1}^{N}\frac{-Q_{r,x}}{(\lambda_i - Q_r)(\lambda_j - Q_r)}.$$

For $i \neq j$ that is equal to $\dfrac{F_j}{4(\lambda_j - \lambda_i)}\left(\dfrac{F_i}{F_j}\right)_x = w_{ij}$ due to Eqs. (4.5) and (2.22). As for $i = j$, by using Lemma 4.3 with the interchange $\lambda_i \leftrightarrow Q_i$ we find that the expression (4.8) is equal to

$$\frac{1}{2}\sum_{s=1}^{N}\frac{-Q_{s,x}}{\lambda_i - Q_s}\left[1 + \frac{\prod\limits_{k\neq s}(\lambda_i - Q_k)}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)} - \sum_{r=1}^{N}\frac{\prod\limits_{k\neq s}(\lambda_r - Q_k)}{\prod\limits_{k\neq r}(\lambda_r - \lambda_k)}\right]$$

which, in turn, coincides with $\dfrac{F_{i,x}}{2F_i} + \sum\limits_{r\neq i}\dfrac{F_i}{4(\lambda_r - \lambda_i)}\left(\dfrac{F_r}{F_i}\right)_x = w_{ii}$.

Finally, Eq. (4.6) is an immediate corollary of (i), (ii), (iii) and Eq. (2.24). $\square$

Eq. (4.6) is nothing but the spatial part of the Dubrovin equations for the N-gap solution of KdV (cf. [5]). Indeed, it can be written as

$$Q_{k,x} = 2\sqrt{T(Q_k)}\frac{\prod\limits_{i=1}^{N}(\lambda_i - Q_k)}{\prod\limits_{i\neq k}(Q_i - Q_k)}, \qquad k = 1,\dots,N \tag{4.9}$$

where

$$T(Q) = -Q - \sum_{i=1}^{N}\frac{d_i}{2(\lambda_i - Q)} + \sum_{i=1}^{N}\frac{c_i}{4(\lambda_i - Q)^2}. \tag{4.10}$$

In order to get the standard form for that equation, one just needs to bring the product $\prod\limits_{i=1}^{N}(\lambda_i - Q_k)$ in (4.9) under the square root sign.

For the N-soliton solution we have $c_i{=}d_i{=}0$ and, therefore, $T(Q){=}-Q$.

We also need to know how the time evolution equation (2.15) transforms under the change of variables (4.5). For that purpose we calculate the sum

$$\sum_{r=1}^{N}\alpha_{ir}F_{r,x} = \sum_{r=1}^{N}\frac{\prod\limits_{s=1}^{N}(\lambda_r - Q_s)}{\prod\limits_{s\neq r}(\lambda_r - \lambda_s)}\sum_{j=1}^{N}\frac{-Q_{j,x}}{(\lambda_r - Q_i)(\lambda_r - Q_j)} = Q_{i,x}\frac{\prod\limits_{s\neq i}(Q_i - Q_s)}{\prod\limits_{s=1}^{N}(Q_i - \lambda_s)}$$

(here we used Eq. (4.7) after interchanging λ_m and Q_m for all m in it). Then we obtain a similar expression for $\sum\limits_{r=1}^{N}\alpha_{ir}F_{r,t}$ and, by comparing the two and using the relation $\mathcal{A}\hat{F}_t = \left(\mathcal{A}R\mathcal{A}^{-1}\right)\mathcal{A}\hat{F}_x = \left(4\tilde{V} + 2\sum\limits_{i=1}^{N}F_i\right)\mathcal{A}\hat{F}_x$, we come to $Q_{k,t} = \left(4Q_k + 2\sum\limits_{i=1}^{N}F_i\right)Q_{k,x}$. The final form of the time evolution equation,

$$Q_{k,t} = 4\left(\sum_{i=1}^{N}\lambda_i - \sum_{i\neq k}Q_i\right)Q_{k,x}, \qquad k = 1,\dots,N, \tag{4.11}$$

is a result of yet another lemma.

Lemma 4.4. *The sums of the old and the new variables in Eq.* (4.5) *are connected via the relation* $\sum\limits_{i=1}^{N}F_i = 2\sum\limits_{i=1}^{N}(\lambda_i - Q_i)$, *i.e., the following identity* *takes place:*

$$\check{S}_N = \sum_{i=1}^{N} \frac{\prod_{s=1}^{N}(\lambda_i - Q_s)}{\prod_{s\neq i}(\lambda_i - \lambda_s)} - \sum_{i=1}^{N}(\lambda_i - Q_i) \equiv 0.$$

The proof is similar to that of Lemmas 4.2 and 4.3.

Note that Eq. (4.11) is the temporal part of the Dubrovin equations as one can see after substituting for $Q_{k,x}$ from (4.9). The trace formula

$$u = \sum_{k=1}^{N} 2(\lambda_k - Q_k)$$

results from Eq. (2.7) after using Lemma 4.4.

The matrix form of Eq. (4.11) is

$$\hat{Q}_t = \tilde{R}\hat{Q}_x \qquad \text{where} \qquad \tilde{R} = \mathcal{A}R\mathcal{A}^{-1} = 4\left[\tilde{V} + \sum_{i=1}^{N}(\lambda_i - Q_i)\right]. \tag{4.12}$$

Also, one can show that (4.11) leads to the commutation relations $\left[\partial_t - \tilde{R}\partial_x, \tilde{V}\right] = \left[\partial_t - \tilde{R}\partial_x, \tilde{W}\right] = 0$ for the generating operators $\tilde{V}$ and $\tilde{W}$ of Eq. (4.6) thus ensuring the compatibility of the Dubrovin equations (4.9) and (4.11). This time, however, $\tilde{V}$ and $\tilde{W}$ are only two of a much wider set of operators commuting with $\partial_t - \tilde{R}\partial_x$. Such are, for example, all polynomials of $\tilde{U}_1$ and $\tilde{U}_2\partial_x$ where

$$\tilde{U}_1 = \begin{pmatrix} Y_1(Q_1) & & 0 \\ & \ddots & \\ 0 & & Y_N(Q_N) \end{pmatrix}, \quad \tilde{U}_2 = \begin{pmatrix} \prod_{i\neq 1}(Q_1 - Q_i) & & 0 \\ & \ddots & \\ 0 & & \prod_{i\neq N}(Q_N - Q_i) \end{pmatrix}$$

with $Y_1, \ldots, Y_N$ being arbitrary functions.

Now we turn our attention to integrating the Dubrovin equations (4.9) and (4.11). A way of doing that is suggested by the N-soliton solution. Indeed, according to Eq. (3.18), each function $(\log \varphi_k) - p_k x$, $k = 1, \ldots, N$, contains an additive constant representing a new parameter not present in Eq. (4.9). So we can think of it as an integration constant and the respective function as a first integral of (4.9). However, the problem is to express them in terms of $Q_1, \ldots, Q_N$.

For instance, at $N = 1$ and $T(Q) = -Q$ Eq. (4.9) becomes $Q_{1,x} = 2(\lambda_1 - Q_1)\sqrt{-Q_1}$ which, after integration, yields

$$\frac{1}{2p_1}\log\left(\frac{\sqrt{-Q_1}+p_1}{\sqrt{-Q_1}-p_1}\right) - x = \text{const}\,. \tag{4.13}$$

That can be obtained from Eq. (3.11′) as well. For $N = 1$ we get $f_1 = \varphi_1 + \frac{\left(\partial_x^{-1}\varphi_1^2\right)}{4p_1} f_1 = \varphi_1 + \frac{\varphi_1^2}{8p_1^2} f_1$ which leads to

$$\varphi_1 = \frac{-4p_1^2 - \sqrt{16p_1^4 + 8p_1^2 f_1^2}}{f_1} = \frac{-4p_1\left(p_1 + \sqrt{-Q_1}\right)}{\sqrt{2\left(-p_1^2 - Q_1\right)}} = -2\sqrt{2}p_1\sqrt{\frac{\sqrt{-Q_1}+p_1}{\sqrt{-Q_1}-p_1}}$$

and then we take a logarithm on both sides to come to (4.13).

For $N = 2$ Eq. (3.11′) yields

$$f_k = \frac{\varphi_k\left[1 - \dfrac{\varphi_i^2(p_k - p_i)}{8p_i^2(p_k + p_i)}\right]}{1 - \dfrac{\varphi_1^2}{8p_1^2} - \dfrac{\varphi_2^2}{8p_2^2} + \dfrac{\varphi_1^2\varphi_2^2(p_1-p_2)^2}{64p_1^2p_2^2(p_1+p_2)^2}} \qquad (i \neq k;\quad k, i = 1, 2).$$

From here we express φ_1 and φ_2 via f_1 and f_2 by using auxiliary variables $\psi_{1,2}$:

$$f_k = \frac{(p_i - p_k)\psi_i}{\psi_{k,x}\psi_i - \psi_{i,x}\psi_k} \qquad \text{with} \qquad \psi_k = \frac{1}{\varphi_k}\left[1 - \frac{\varphi_k^2(p_i - p_k)}{8p_k^2(p_i + p_k)}\right] \qquad (i \neq k;\quad k, i = 1, 2)$$

resulting in $\psi_k = \dfrac{(p_k - p_i)f_i}{f_{k,x}f_i - f_{i,x}f_k}$ and, therefore,

$$\varphi_k = -2p_k\sqrt{\frac{2(p_i+p_k)}{(p_i-p_k)}}\sqrt{\left(\frac{\sqrt{-Q_1}+p_k}{\sqrt{-Q_1}-p_k}\right)\left(\frac{\sqrt{-Q_2}+p_k}{\sqrt{-Q_2}-p_k}\right)} \quad (i \neq k;\quad k, i = 1, 2).$$

Obviously, the functions $\overline{I}_k = \left[\sum_{s=1}^{N} \frac{1}{2p_k}\log\left(\frac{\sqrt{-Q_s}+p_k}{\sqrt{-Q_s}-p_k}\right)\right] - x$, $k = 1, \ldots, N$, are candidates for being first integrals of (4.9). The next theorem states that result in the general case when $T(Q)$ has the form in (4.10).

Theorem 4.2. *The functions* $\tilde{I}_k = -(x + 4\lambda_k t) + \sum_{s=1}^{N} \eta_k(Q_s)$, $k = 1, \ldots, N$ *where* $\eta_k(Q)$ *are determined by*

$$\eta_k'(Q) = \frac{1}{2(\lambda_k - Q)\sqrt{T(Q)}} \tag{4.14}$$

constitute N first integrals for the Dubrovin equations (4.9) *and* (4.11). (*Here the sign of* $\sqrt{T(Q_k)}$ *for any fixed* $k = 1, \dots, N$ *is the same as in Eq.* (4.9).)

Proof. By differentiating $\tilde{I}_k$ in x and using (4.9) we obtain

$$\frac{d\tilde{I}_k}{dx} = -1 + \sum_{s=1}^{N} \frac{Q_{s,x}}{2(\lambda_k - Q_s)\sqrt{T(Q_s)}} = -1 + \sum_{s=1}^{N} \frac{\prod\limits_{i\neq k}(\lambda_i - Q_s)}{\prod\limits_{i\neq s}(Q_i - Q_s)} = -1 + \overline{S}_N^{(k)} = 0$$

(see Lemma 4.3). In a similar way we find that $\frac{d\tilde{I}_k}{dt} = 0$ due to Lemmas 4.3 and 4.4. $\square$

The implicit function theorem guarantees the existence of at least a local solution for the system of functional equations $\tilde{I}_k = a_k = \text{const}$ $(k = 1, \dots, N)$ since we have

$$\det\left(\left\{\frac{\partial \tilde{I}_k}{\partial Q_i}\right\}_{k,i=1}^{N}\right) = \det\left(\left\{\frac{1}{2(\lambda_k - Q_i)\sqrt{T(Q_i)}}\right\}_{k,i=1}^{N}\right) = \frac{\det\mathcal{A}}{\prod\limits_{i=1}^{N}\sqrt{T(Q_i)}} \neq 0.$$

A global solution exists [7] as well, in terms of θ-functions (see also [20]).

For $N = 1$ we obtain the equation

$$x + 4\lambda t + a = \int \frac{dQ}{\sqrt{-4Q(\lambda - Q)^2 - 2d(\lambda - Q) + c}}$$

which yields the 1-gap solution of KdV,

$$u = F = 2(\lambda - Q) = 2\left(\frac{\lambda}{3} + \mathcal{P}(x + 4\lambda t + a)\right) = -2\frac{d^2}{dx^2}\theta(x + 4\lambda t + a) + \text{const}$$

(cf. (1.4)). The connection between the constants c, d and the coefficients in the ODE $\mathcal{P}'^2 = 4\mathcal{P}^3 - g_2\mathcal{P} - g_3$ satisfied by the Weierstrass elliptic function $\mathcal{P}(z)$ is $g_2 = \dfrac{4}{3}\lambda^2 + 2d$, $g_3 = \dfrac{8}{27}\lambda^3 + \dfrac{2}{3}\lambda d - c$.

5. Dubrovin equations for PKdV. In this section we will repeat the entire procedure of deriving the Dubrovin equations and solving them for the case of the polynomial KdV equation (1.8).

According to [15], both u_x and the eigenfunctions of the recursion operator Λ in (1.7) satisfy the linearized (perturbed) PKdV as a result of the hereditary symmetry property [16] possessed by Λ. On the other hand, it is easy to see that the eigenfunctions of Λ have the form $J\mathcal{F}$ where

$$J = \begin{pmatrix} -j(u_1) & \cdots & -j(u_{M-1}) & \partial_x \\ -j(u_2) & \cdot^{\cdot^{\cdot}} & \partial_x & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ \partial_x & 0 & \cdots & 0 \end{pmatrix}, \qquad \mathcal{F} = \begin{pmatrix} F \\ \lambda F \\ \vdots \\ \lambda^{M-1} F \end{pmatrix} = \sigma(\lambda) F$$

with $\sigma(\lambda) = (1, \lambda, \ldots, \lambda^{M-1})^\top$, and F is a solution of the linear spectral problem

$$\text{(5.1)} \quad \frac{1}{4} F_{xxx}(x,\lambda) + j(U(x,\lambda))F(x,\lambda) = 0, \qquad U(x,\lambda) = \lambda^M - \sum_{r=0}^{M-1} \lambda^r u_r(x)$$

(see also [8]). We remind that $j(u) = u\partial_x + \frac{1}{2}u_x$.

Therefore, the analogs of Eq. (2.6) and (the derivative of) (2.7) are, respectively, Eq. (5.1) for $\lambda = \lambda_k,\ k = 1, \ldots, N$, and

$$\text{(5.2)} \quad u_x = J\mathcal{F}_1 + J\mathcal{F}_2 + \ldots + J\mathcal{F}_N, \qquad \mathcal{F}_k = \sigma(\lambda_k) F_k \equiv \sigma(\lambda_k) F(x, \lambda_k).$$

The time evolution of F_k is provided by the linearized PKdV and has the form

$$\text{(5.3)} \qquad F_{k,t} = 4\lambda_k F_{k,x} + 2\left(u_{M-1} F_{k,x} - u_{M-1,x} F_k\right)$$

(see also [12]). Eq. (5.2) can be integrated in x to yield expressions for $u_0, \ldots, u_{M-1}$ in terms of $F_1, \ldots, F_N$,

$$\text{(5.4)} \quad u_{M-1} = -b_1 + \sum_{k=1}^N F_k, \quad u_{M-2} = -b_2 + b_1 \sum_{k=1}^N F_k + \sum_{k=1}^N \lambda_k F_k - \frac{3}{4}\left(\sum_{k=1}^N F_k\right)^2, \ \ldots$$

where $b_1, b_2, \ldots, b_M$ are the integration "constants", i.e., $b_k = b_k(t)$.

A simple way of producing the formulas in (5.4) is provided by

Lemma 5.1. *The equations in* (5.4) *are coefficients in the series equation*

$$\text{(5.5)} \qquad LZ^2 = 4B + O\left(\varepsilon^{M+1}\right)$$

where

$$L = 1 - \varepsilon u_{M-1} - \varepsilon^2 u_{M-2} - \varepsilon^3 u_{M-3} - \ldots \qquad (u_{-1} = u_{-2} = \ldots = 0)$$

$$Z = 2 + \varepsilon\left(\sum_{k=1}^{N} F_k\right) + \varepsilon^2\left(\sum_{k=1}^{N}\lambda_k F_k\right) + \varepsilon^3\left(\sum_{k=1}^{N}\lambda_k^2 F_k\right) + \dots$$
$$B = 1 + \varepsilon b_1 + \varepsilon^2 b_2 + \varepsilon^3 b_3 + \dots$$

P r o o f. Eq. (5.2) yields the following relation between L and Z:

$$2j(L)Z \equiv 2LZ_x + L_x Z = O\left(\varepsilon^{M+1}\right) \tag{5.6}$$

which is then multiplied by Z and integrated to Eq. (5.5). $\square$

Just as in the case of the KdV equation, the constants b_k $(k = 1,\dots,N)$ are constants in time as well. Indeed, Eqs. (1.8), (5.3), (5.5) and (5.6) imply that

$$4B_t = L_t Z^2 + 2LZZ_t + O\left(\varepsilon^{M+1}\right) =$$
$$= \left[\frac{4}{\varepsilon}(L_x + \varepsilon u_{M-1,x}) + 4j(L-1)u_{M-1} + \varepsilon^M u_{M-1,xxx} + O\left(\varepsilon^{M+1}\right)\right] Z^2 +$$
$$+2LZ\left[\frac{4}{\varepsilon}\left(Z_x - \varepsilon\sum_{k=1}^{N} F_{k,x}\right) + 2u_{M-1}Z_x - 2u_{M-1,x}(Z-2) + O\left(\varepsilon^{M+1}\right)\right] =$$
$$= \varepsilon^M u_{M-1,xxx}Z^2 + \left(\frac{4}{\varepsilon} + 2u_{M-1}\right) Z(L_xZ + 2LZ_x) + 8LZ\left(u_{M-1,x} - \sum_{k=1}^{N} F_{k,x}\right) +$$
$$+O\left(\varepsilon^{M+1}\right) = \varepsilon^M\left(\sum_{k=1}^{N} F_{k,xxx}\right) Z^2 + \frac{4}{\varepsilon}Z(L_xZ + 2LZ_x) + O\left(\varepsilon^{M+1}\right).$$

Now Eqs. (5.1) and (5.6) lead to $4B_t = O\left(\varepsilon^{M+1}\right)$, i.e., $b_1'(t) = \ldots = b_M'(t) = 0$.

The analogs of Eqs. (2.11) and (2.12) for the case of PKdV are obtained by replacing $\lambda_k - u$ with U_k in them where $U_k = U(x, \lambda_k)$ (see (5.1)). Here again c_k is a constant in both x and t.

As for the analog of (2.16), it is

$$f_{k,x}^2 + \sum_{i\neq k}\frac{(f_{k,x}f_i - f_k f_{i,x})^2}{2(\lambda_k - \lambda_i)} - \frac{c_k}{f_k^2} - \sum_{i\neq k}\frac{c_k f_k^{-2} f_i^2 + c_i f_i^{-2} f_k^2}{2(\lambda_k - \lambda_i)} + d_k +$$

$$+f_k^2\left[U_k + \frac{1}{2}\sum_{s=0}^{M-1}\left(\lambda_k^{M-1-s} - \sum_{r=0}^{M-2-s}\lambda_k^r u_{r+1+s}\right)\sum_{i=1}^{N}\lambda_i^s f_i^2\right] = 0, \qquad k = 1,\dots,N \tag{5.7}$$

where $d_k = d_k(t)$.

Lemma 5.2. *Eq.* (5.7) *has the matrix representation* (*cf.* (2.24))

$$W^2\hat{F}+V^M\hat{F}+\sum_{s=1}^{M} b_s V^{M-s}\hat{F}+\sum_{k=1}^{N}\frac{d_k}{2}(\lambda_k-V)^{-1}\hat{F}-\sum_{k=1}^{N}\frac{c_k}{4}(\lambda_k-V)^{-2}\hat{F}=0. \tag{5.8}$$

Proof. Obviously, we just need to show that the last term in (5.7) is the k-th element of the vector $V^M\hat{F}+\sum_{s=1}^{M} b_s V^{M-s}\hat{F}$ since that term is the only difference between Eqs. (2.16) and (5.7). We start with the fact that $V^n\hat{F}$ has the form

$$V^n\hat{F}=\sum_{k=0}^{n} C_{n-k}V_0^k\hat{F} \tag{5.9}$$

where $V_0 = \operatorname{diag}\{\lambda_1,\ldots,\lambda_N\}$ is the constant part of the operator V and the functions C_k, $k = 0, 1, \ldots$ are defined by the recursion formula

$$C_0 = 1, \qquad C_k = -\frac{1}{2}\sum_{s=0}^{k-1} C_{k-1-s}\sum_{i=1}^{N}\lambda_i^s f_i^2 \qquad (k = 1, 2, \ldots). \tag{5.10}$$

Just like in Lemma 5.1, we can write down (5.10) by using series,

$$CZ = 2, \qquad C = C_0 + \varepsilon C_1 + \varepsilon^2 C_2 + \ldots \tag{5.11}$$

From Eq. (5.11) we obtain $(2BC)Z = 4B$ which, after a comparison with (5.5), shows that $2BC - LZ = O\left(\varepsilon^{M+1}\right)$ and, therefore,

$$\sum_{s=0}^{j} b_s C_{j-s}=\frac{1}{2}\left(-2u_{M-j}-\sum_{s=1}^{j-1}u_{M-j+s}\sum_{i=1}^{N}\lambda_i^{s-1}f_i^2+\sum_{i=1}^{N}\lambda_i^{j-1}f_i^2\right),\ j=1,\ldots,M \tag{5.12}$$

(here we have $b_0 = 1$). Now we can calculate the last term in Eq. (5.7):

$$f_k^2\left[U_k+\frac{1}{2}\sum_{s=0}^{M-1}\lambda_k^{M-1-s}\sum_{i=1}^{N}\lambda_i^s f_i^2-\frac{1}{2}\sum_{s=0}^{M-2}\sum_{r=0}^{M-2-s}\lambda_k^r u_{r+1+s}\sum_{i=1}^{N}\lambda_i^s f_i^2\right]=$$

$$=f_k^2\left[\lambda_k^M+\sum_{r=0}^{M-1}\lambda_k^r\left(-u_r+\frac{1}{2}\sum_{i=1}^{N}\lambda_i^{M-1-r}f_i^2-\frac{1}{2}\sum_{s=0}^{M-2-r}u_{r+1+s}\sum_{i=1}^{N}\lambda_i^s f_i^2\right)\right]=$$

$$\overset{(5.12)}{=} f_k^2 \left[\sum_{r=0}^{M} \lambda_k^r \sum_{s=0}^{M-r} b_s C_{M-r-s} \right] = \sum_{s=0}^{M} b_s \sum_{r=0}^{M-s} C_{M-r-s} \lambda_k^r f_k^2.$$

That is, however, the k-th component of the vector

$$\sum_{s=0}^{M} b_s \sum_{r=0}^{M-s} C_{M-r-s} V_0^r \hat{F} = \sum_{s=0}^{M} b_s V^{M-s} \hat{F}$$

(see (5.9)). With that, the proof of Lemma 5.2 is completed. □

The time evolution of $F_1, \dots, F_N$ for PKdV is essentially the same as that for KdV. From Eqs. (5.3) and (5.4) we obtain

$$(5.13) \qquad (\partial_t - R_P \partial_x)\, \hat{F} = 0, \qquad R_P = R - 2b_1$$

(cf. (2.18) for $\hat{Q} = \hat{F}$). In fact, the matrix R corresponding to KdV would also contain a constant if b was not excluded from Eq. (2.7).

The substitution $\lambda_k \to \lambda_k - \dfrac{b_1}{2}$ in (2.19) shows that $\partial_t - R_P \partial_x$ also commutes with V and W so that Eqs. (5.8) and (5.13) are compatible when $d_k'(t) = 0$, $k = 1, \dots, N$.

Now apparent becomes the duality between the pairs (KdV, PKdV) and (KdV, GKdV) mentioned in the introduction. Namely, KdV and PKdV exhibit the same time evolution for the functions F_k but the space evolution is different. For the pair (KdV, GKdV), the role of space and time is reversed.

It is also interesting to know that Eq. (3.4) has an analog for PKdV as well,

$$f_{k,x} - \frac{1}{2} \sum_{i=1}^{N} f_i \left[\partial_x^{-1} f_k f_i \left(\sum_{s=0}^{M-1} \lambda_k^{M-1-s} \lambda_i^s - \sum_{r=1}^{M-1} u_r \sum_{s=0}^{r-1} \lambda_k^{r-1-s} \lambda_i^s \right) \right] = p_k f_k.$$

However, that equation cannot be easily solved order by order (as in the KdV case) even for $M = 2$ when an N-soliton solution is known to exist.

The results obtained for PKdV are summarized in the next theorem.

Theorem 5.1. *The system of ODEs* (5.1) (*with* $\lambda = \lambda_1, \dots, \lambda_N$) *and* (5.2) *for the N-gap solution of PKdV is integrated to the first-order system* (5.7) *which has the matrix representation* (5.8). *The time behavior* (5.13) *of the functions F_k is essentially the same as that for KdV. The constants of integration*

b_k, c_k, d_k are constants in time as well. Under the change of variables (4.5), *the Dubrovin equations* (4.9) *with*

$$T(Q) = -Q^M - \sum_{s=1}^{M} b_s Q^{M-s} - \sum_{i=1}^{N} \frac{d_i}{2(\lambda_i - Q)} + \sum_{i=1}^{N} \frac{c_i}{4(\lambda_i - Q)^2}, \tag{5.14}$$

and

$$Q_{k,t} = 4\left(-\frac{b_1}{2} + \sum_{i=1}^{N} \lambda_i - \sum_{i \neq k} Q_i\right) Q_{k,x} \tag{5.15}$$

are obtained. Then they are integrated to the system of functional equations $\tilde{I}_k = a_k,\ k = 1, \ldots, N$ *(*$a_k = \text{const}$ *) where* $\tilde{I}_k = -\left[x + (4\lambda_k - 2b_1)\, t\right] + \sum_{s=1}^{N} \eta_k(Q_s)$ *with* η_k *defined by Eqs.* (4.14) *and* (5.14). *Trace formulas are provided by Eq.* (5.4) (*or* (5.5)) *after replacing* $F_1, \ldots, F_N$ *with* $Q_1, \ldots, Q_N$ *from Eq.* (4.5).

Proof. The change of variables (4.5) transforms Eq. (5.8) into (4.6) with a second term $\tilde{V}$ replaced by $\tilde{V}^M + \sum_{s=1}^{M} b_s \tilde{V}^{M-s}$ which results in the expression (5.14) for $T(Q)$ in (4.9).

As for Eq. (5.15), we use the fact that (2.15) is transformed into (4.11) under the change (4.5). Then the substitution $\lambda_j \to \lambda_j - \frac{b_1}{2}$, $Q_j \to Q_j - \frac{b_1}{2}$ ($j = 1, \ldots, N$) leaves (4.5) unchanged and shows that Eq. (5.13) (in its scalar form) is transformed into (5.15).

That same substitution yields the above expression for the functions $\tilde{I}_k$, $k = 1, \ldots, N$. $\Box$

The 1-soliton case corresponds to $N = 1,\ b_1 = \ldots = b_M = c_1 = d_1 = 0$. Then we have $T(Q) = -Q^M$ and Eq. (4.9) becomes $Q_x = 2(\lambda - Q)\sqrt{-Q^M}$ which is equivalent to Eq. (A3) from the Appendix for $Q = q^2$.

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Appendix. 1-soliton solution for PKdV. The substitution $u_t = 4\lambda u_x$ in Eq. (1.8) leads to

$$4\lambda u_{r,x} = 4u_{r-1,x} + 4u_r u_{M-1,x} + 2u_{r,x} u_{M-1} - \delta_{r0} u_{M-1,xxx} \tag{A1}$$

for $r = 0, 1, \ldots, M-1$ (we have $u_{-1} = 0$). From here we obtain another linear recursion relation for the finite sequence $u_{M-1}, u_{M-2}, \ldots, u_1, u_0$:

$$(A2) \quad \left[4u_{r-1} + 4u_r(u_{M-1} - 2\lambda) + u_{r+1}(u_{M-1} - 2\lambda)^2\right]_x = 0, \quad r = 1, \ldots, M-1$$

($u_M = -1$) which, after an integration in x, allows one to express the general term u_{M-s} $(s = 2, 3, \ldots, M)$ through the first one $w = u_{M-1}$,

$$u_{M-s} = -\left(\lambda - \frac{w}{2}\right)^s + s\left(\lambda + \frac{w}{2}\right)\left(\lambda - \frac{w}{2}\right)^{s-1} + \sum_{k=1}^{s-1} \gamma_k (s-k)\left(\lambda - \frac{w}{2}\right)^{s-k-1}$$

($\gamma_k = \text{const}$). By substituting that result for $s = M$ into Eq. (A1) for $r = 0$ we find an equation for w,

$$w_{xxx} = 2w_x \left[-2(M+1)\left(\lambda - \frac{w}{2}\right)^M + (M+1)M\left(\lambda + \frac{w}{2}\right)\left(\lambda - \frac{w}{2}\right)^{M-1} + \right.$$
$$\left. + \sum_{k=1}^{M-1} \gamma_k (M-k+1)(M-k)\left(\lambda - \frac{w}{2}\right)^{M-k-1}\right].$$

Now we integrate in x and, after multiplying by $2w_x$, integrate one more time:

$$w_x^2 + 2\beta_1 w + \beta_2 = 16\left(\lambda + \frac{w}{2}\right)\left(\lambda - \frac{w}{2}\right)^{M+1} + \sum_{k=1}^{M-1} 16\gamma_k \left(\lambda - \frac{w}{2}\right)^{M-k+1}$$

($\beta_{1,2} = \text{const}$). If we now make the substitution $q^2 = \lambda - \dfrac{w}{2}$ and choose $\gamma_1 = -\lambda^2$, $\gamma_2 = \gamma_3 = \ldots = \gamma_{M-1} = \beta_1 = \beta_2 = 0$ then the equation

$$(A3) \qquad q_x^2 = -\left(\lambda - q^2\right)^2 q^{2M-2}$$

is obtained. It, too, can be integrated yielding, for odd M,

$$G_M = \frac{1}{2\lambda^{M/2}} \log\left(\frac{\sqrt{\lambda} + q}{\sqrt{\lambda} - q}\right) - \sum_{k=1}^{\frac{M-1}{2}} \frac{1}{\lambda^k (M-2k) q^{M-2k}} = \pm ix + \beta_3, \qquad \beta_3 = \text{const}$$

and, for even M,

$$G_M = \frac{1}{2\lambda^{M/2}} \log\left(\frac{q^2}{\lambda - q^2}\right) - \sum_{k=1}^{\frac{M}{2}-1} \frac{1}{\lambda^k (M-2k) q^{M-2k}} = \pm ix + \beta_4, \qquad \beta_4 = \text{const}.$$

Obviously, for $M > 2$ these equations cannot be solved explicitly with respect to the unknown function q. However, a local solution does exist by the implicit function theorem since we have $\dfrac{\partial G_M}{\partial q} = \dfrac{1}{(\lambda - q^2)\, q^{M-1}} \neq 0$.

REFERENCES

[1] C. Gardner, J. Greene, M. Kruskal, R. Miura. The Korteweg-de Vries equation and generalizations, VI. Methods for exact solution. *Comm. Pure Appl. Math.* **27** (1974), 97-133.

[2] M. J. Ablowitz, P. A. Clarkson. Solitons, Nonlinear Evolution Equations and Inverse Scattering. Cambridge Univ. Press, Cambridge 1991.

[3] A. C. Newell. Solitons in Mathematics and Physics. SIAM, 1985.

[4] V. E. Zakharov, S. V. Manakov, S. P. Novikov, L. P. Pitaevskii. Theory of Solitons: Inverse Scattering Method. Nauka, Moscow, 1980 (in Russian).

[5] B. A. Dubrovin. Periodic problem for the Korteweg-de Vries equation in the class of finite-gap potentials. *Func. Anal. and Appl.* **9** (1975), 41-51 (in Russian).

[6] V. A. Marchenko. The periodic Korteweg-de Vries problem. *Mat. Sb.* **95** (1974), 331-356 (in Russian).

[7] A. Its, V. Matveev. Hill's operators with a finite number of gaps and multiple-soliton solutions of the KdV equation. *Teor. Mat. Fiz.* **23**, *1* (1975), 51-67.

[8] L. M. Alonso. Schrödinger spectral problems with energy-dependent potentials as sources of nonlinear Hamiltonian evolution equations. *J. Math. Phys.* **21**, *9* (1980), 2342-2349.

[9] M. Boiti, P. Caudrey, F. Pempinelly. A Hierarchy of Hamiltonian Evolution Equations Associated with a Generalized Schrödinger Spectral Problem. *Nuovo Cimento* **83B**, *1*, (1984), 71-87.

[10] M. Jaulent, C. Jean. The inverse s-wave scattering problem for a class of potentials depending on energy. *Commun. Math. Phys.* **28** (1972), 177-220.

[11] R. Yordanov. About some spectral properties of the Schrödinger equations with energy-dependent potentials generating fully integrable Hamiltonian systems. *Annuaire Univ. Sofia Fac. Math. Mec.* book 2 **78** (1984), 229-252 (in Russian).

[12] R. Yordanov. Cauchy problem for the linearized version of the Generalized Polynomial KdV equation. *J. Math. Phys.* **33**, *6* (1992), 2013-2022.

[13] R. Yordanov. On the Cauchy problem for the linearized GPKdV and gauge transformations for a quadratic pencil and AKNS system. Ph.D. dissertation, Virginia Polytechnic Institute, Blacksburg, VA, 1992.

[14] Ma Wen-xiu. Lax representations and Lax operator algebras of isospectral and nonisospectral hierarchies of evolution equations. *J. Math. Phys.* **33** (1992), 2464-2476.

[15] R. Yordanov. Why do soliton equations come in hierarchies? *J. Math. Phys.* **34**, *9* (1993), 4045-4049.

[16] B. Fuchssteiner. Application of hereditary symmetries to nonlinear evolution equations. *Nonlinear Analysis, Theory, Methods and Applications* **3**, *6* (1979), 849-862.

[17] M. Jaulent, I. Miodek. Nonlinear evolution equations associated with "energy-dependent Schrödinger potentials". *Lett. Math. Phys.* **1**, *3* (1975/77), 243-250.

[18] D. Mumford. Tata Lectures on Theta I, II. Birkäuser, Boston 1983, 1984.

[19] G. L. Lamb. Elements of Soliton Theory. Wiley, New York 1980.

[20] B. Dubrovin. Theta-functions and nonlinear equations. *Uspehi Matem. Nauk* **36**, *2* (1981), 12-80.

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