Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **24** (1998), 283-288

Serdica
Mathematical Journal

Institute of Mathematics
and Informatics, Bulgarian
Academy of Sciences

ON EXTENSION OF FUNCTORS ONTO THE KLEISLI CATEGORY OF THE INCLUSION HYPERSPACE MONAD

Andrii Teleiko

Communicated by S. P. Gul'ko

Abstract. It is proved that there exists no extension of any non-trivial weakly normal functor of finite degree onto the Kleisli category of the inclusion hyperspace monad.

0. In this note we consider the problem of extension of endofunctors in the category of compacta onto the Kleisli category of the inclusion hyperspace monad. The main result states that there is no such extension for weakly normal functors of finite degree. Note that the class of normal functors of finite degree is introduced by E.V. Shchepin [7] and contains, in particular, the finite power functors $(-)^n$ and G-symmetric power functors SP^n_G. Related results devoted to extensions of functors onto the Kleisli categories of some monads in the category of compacta can be found in [10, 11, 8], see also [12, 13].

1. A triple $\mathbb{T} = (T, \eta, \mu)$ is said to be a *monad* on a category $\mathcal{C}$ if T is an endofunctor acting in $\mathcal{C}$ and $\eta\colon 1_{\mathcal{C}} \to T$, $\mu\colon T^2 \to T$ are natural transformations satisfying $\mu \circ \eta T = \mu \circ T\eta = 1_T$, $\mu \circ \mu T = \mu \circ T\mu$ [2].

1991 *Mathematics Subject Classification:* 54B30, 18C20
Key words: Monad, Kleisli category, weakly normal functor, inclusion hyperspace

Recall the definition of *Kleisli category $\mathcal{C}_{\mathbb{T}}$ of the monad* $\mathbb{T}$ [4]. The objects of both categories $\mathcal{C}$ and $\mathcal{C}_{\mathbb{T}}$ are the same, and $\mathcal{C}_{\mathbb{T}}(X,Y) = \mathcal{C}(X,TY)$. The composition $g * f$ of morphisms $f \in \mathcal{C}_{\mathbb{T}}(X,Y)$, $g \in \mathcal{C}_{\mathbb{T}}(Y,Z)$ is defined by the formula: $g * f = \mu Z \circ Tg \circ f$.

Denote by $I\colon \mathcal{C} \to \mathcal{C}_{\mathbb{T}}$ the functor being the identity on objects and satisfying $If = \eta Y \circ f \in \mathcal{C}_{\mathbb{T}}(X,Y)$, $f \in \mathcal{C}(X,Y)$.

A functor $\overline{F}\colon \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}}$ is called an *extension of the functor $F\colon \mathcal{C} \to \mathcal{C}$ onto the Kleisli category $\mathcal{C}_{\mathbb{T}}$* of the monad $\mathbb{T}$ if $\overline{F}I = IF$.

In the sequel we shall consider only the category $\mathcal{C}omp$ of compacta (= compact Hausdorff spaces) and their continuous maps.

Recall the construction of the *inclusion hyperspace* monad $\mathbb{G}$. For a compactum X by $\exp X$ we denote the set of nonempty closed subsets of X endowed with the Vietoris topology. A base of this topology consists of the sets of the form

$$\langle U_1,\dots,U_n\rangle = \Big\{A \in \exp X \mid A \subset \bigcup_{i=1}^{n} U_i,\ A \cap U_i \neq \emptyset \text{ for each } i = 1,\dots,n\Big\},$$

where $U_1,\dots,U_n$ are open in X. A set $\mathcal{A} \in \exp^2 X$ is said to be an *inclusion hyperspace* if $[A \in \mathcal{A}, A \subset B \Rightarrow B \in \mathcal{A}]$ (see [1]). Denote by GX the set of inclusion hyperspaces with the inherited from $\exp^2 X$ topology.

For a map $f\colon X \to Y$ the map $Gf\colon GX \to GY$ is defined by the formula: $Gf(\mathcal{A}) = \{B \in \exp Y \mid B \supset f(A) \text{ for some } A \in \mathcal{A}\}$, $\mathcal{A} \in GX$. Obviously, we obtain a functor $G\colon \mathcal{C}omp \to \mathcal{C}omp$ (the inclusion hyperspace functor).

Setting $\eta X(x) = \{A \in \exp X \mid x \in A\}$, $\mu X = \bigcup\{\bigcap \mathcal{A} \mid \mathcal{A} \in \mathfrak{A}\}$, $x \in X$, $\mathfrak{A} \in G^2 X$, we obtain the monad $\mathbb{G} = (G,\eta,\mu)$ [6].

The following definition is due to E. Shchepin [7]. A functor $F\colon \mathcal{C}omp \to \mathcal{C}omp$ is called *weakly normal* if it is continuous, monomorphic, epimorphic, preserves weight, intersections, singletons and empty set. A monad is said to be weakly normal if so is its functorial part. See, e.g., [3] for a detailed exposition of properties of weakly normal monads.

Note that the monad $\mathbb{G}$ is weakly normal [5].

Recall also that for a weakly normal functor the *support* $\operatorname{supp}(a)$ of a point $a \in FX$ is defined [7]: $\operatorname{supp}(a) = \bigcap\{A \in \exp X \mid FA \ni a\}$ (here FA is identified with $Fi(A)$, where $i\colon A \to X$ is the natural imbedding). We say that the *degree* of functor F is $\leq n$ (briefly, ${}^{\circ}F \leq n$) if $|\operatorname{supp}(a)| \leq n$ for every

$a \in FX$, $X \in \mathcal{C}omp$. If, moreover, $^\circ F \not\leq n-1$, we say that the degree of F equals n, $^\circ F = n$.

2. Now we prove the following result.

Theorem. *No weakly normal functor of finite degree $n > 1$ extends onto the Kleisli category of the inclusion hyperspace monad.*

P r o o f. To the contrary, assume that a weakly normal functor $F: \mathcal{C}omp \to \mathcal{C}omp$, $^\circ F = n > 1$, extends onto $\mathcal{C}omp_{\mathbb{G}}$. From results of J. Vinárek [9] it follows that there exists a natural transformation $\xi: FG \to GF$ with

(1) $\xi \circ F\eta = \eta F$,

(2) $\xi \circ F\mu = \mu F \circ G\xi \circ \xi G$.

Let $a \in F\{1, \dots, n\}$ be a point with $\operatorname{supp} a = \{1, \dots, n\}$.

Let $r: \{1, \dots, n+1\} \to \{1, \dots, n\}$ be the retraction $r(n+1) = n$. Since $^\circ F = n$, the preimage $(Fr)^{-1}(a)$ of the point a consists of two points, namely, a and the copy of a in $F\{1, \dots, n-1, n+1\}$.

For every compactum K and distinct points $t_1, t_2, t_3 \in K$ denote by $t(t_1, t_2, t_3)$ the inclusion hyperspace $\{A \in \exp K \mid |A \cap \{t_1, t_2, t_3\}| \geq 2\}$ and by $a(t_1, t_2)$ the following point: $a(t_1, t_2) = Ff(a)$, where $f: \{1, \dots, n\} \to K \sqcup \{1, \dots, n-2\}$ is the embedding such that $f|\{1, \dots, n-2\} = \mathrm{id}$, $f(n-1) = t_1$, $f(n-2) = t_2$.

Set $K = \{x_1, x_2, x_3, y_1, y_2, y_3\}$, $Z = K \sqcup \{1, \dots, n-2\}$, and suppose all points x_i, y_j be distinct.

We shall write $\overline{x}$, $\overline{y}$, a_{ij} instead of $t(x_1, x_2, x_3)$, $t(y_1, y_2, y_3)$, $a(x_i, y_j)$ respectively. And we shall often identify the points i and $\eta Z(i)$, $i = 1, \dots, n-2$.

Claim 1. *The following equality holds: $\xi Z(a(\overline{x}, \eta Z(y_1))) = t(a_{11}, a_{21}, a_{31})$.*

Denote by $\mathcal{A}$ the left part of the previous equality. Consider the retraction $g: Z \to Z \setminus \{x_2\}$ such that $g(x_2) = x_1$. Then $Gg(\overline{x}) = \eta Z(x_1)$, $Gg(\eta Z(y_1)) = \eta Z(y_1)$. Therefore,

$$GFg(\mathcal{A}) = \xi Z \circ FGg(a(\overline{x}, \eta Z(y_1))) = \xi Z \circ F\eta Z(a_{11}) = \eta FZ(a_{11}).$$

Hence, the preimage $(Fg)^{-1}(a_{11}) = \{a_{11}, a_{21}\}$ belongs to $\mathcal{A}$, because $\mathcal{A}$ is an inclusion hyperspace. Since $\overline{x}$ is invariant with respect to all authomorphisms of $\{x_1, x_2, x_3\}$, we see that $\{a_{11}, a_{31}\}$ and $\{a_{21}, a_{31}\}$ also belong to $\mathcal{A}$.

To end the proof of this claim, it is sufficient to show that $\{a_{11}\} \notin \mathcal{A}$. It is easy. To the contrary, let $\{a_{11}\}$ belong to $\mathcal{A}$. Then $\{a_{31}\} \in \mathcal{A}$ and, consequently, $\eta FZ(a_{11}) = GFg(\mathcal{A}) \ni \{a_{31}\}$. Contradiction.

In similar manner one can prove the following fact.

Claim 2. *The inclusion hyperspace $\mathcal{B} = \xi Z(a(\overline{x}, \overline{y}))$ contains the set $\{a_{11}, a_{12}, a_{21}, a_{22}\}$.*

Claim 3. *The inclusion hyperspace $\mathcal{B}$ contains the set $\{a_{11}, a_{12}, a_{22}, a_{31}\}$.*

Consider the following points $\mathfrak{M}, \mathfrak{N} \in G^2 Z$:

$$\mathfrak{M} = t(\overline{x}, \eta Z(x_1), \eta Z(x_2)), \qquad \mathfrak{N} = t(\eta Z(y_1), \eta Z(y_2), \overline{y}).$$

Since $\mu Z(\mathfrak{M}) = \overline{x}$, $\mu Z(\mathfrak{N}) = \overline{y}$, we obtain $\xi Z \circ F\mu Z(a(\mathfrak{M}, \mathfrak{N})) = \xi Z(a(\overline{x}, \overline{y})) = \mathcal{B}$. Therefore

$$\mathcal{B} = \mu FZ \circ G\xi Z \circ \xi GZ(a(\mathfrak{M}, \mathfrak{N})).$$

It is easy to see that

$$L = \Big\{a(\overline{x}, \eta Z(y_1)), a(\overline{x}, \eta Z(y_2)), a(\eta Z(x_1), \eta Z(y_1)), a(\eta Z(x_1), \eta Z(y_2))\Big\} \in$$

$$\in \xi GZ(a(\mathfrak{M}, \mathfrak{N})).$$

Then $\bigcap \xi Z(L) \subset \mathcal{B}$. Now, using the equality $\xi Z(a(\eta Z(x_i), \eta Z(y_j))) = \xi Z \circ F\eta Z(a_{ij}) = \eta FZ(a_{ij})$ and Claim 1, we obtain that $\{a_{11}, a_{31}, a_{12}, a_{22}\} \in \mathcal{B}$.

Now consider the point $\mathcal{R} = a(t(\overline{x}, \eta Z(x_2), \eta Z(x_3)), \eta GZ(\overline{y})) \in FG^2 Z$. Since $\mu Z \circ \eta GZ = 1_{GZ}$, we have $\xi Z \circ F\mu Z(\mathcal{R}) = \mathcal{B}$, and thus

$$\mathcal{B} = \mu FZ \circ G\xi Z \circ \xi GZ(\mathcal{R}).$$

Then the equality

$$\xi GZ(\mathcal{R}) = t\Big(a(\overline{x}, \overline{y}), a(\eta Z(x_2), \overline{y}), a(\eta Z(x_3), \overline{y})\Big)$$

and Claim 3 imply the set $\{a_{11}, a_{12}, a_{22}, a_{31}\}$ must belong to at least two of the following inclusion hyperspaces $\xi Z(a(\overline{x}, \overline{y}))$, $\xi Z(a(\eta Z(x_2), \overline{y}))$, $\xi Z(a(\eta Z(x_3), \overline{y}))$. But it is easy to verify that this set belongs neither $\xi Z(a(\eta Z(x_2), \overline{y}))$ nor $\xi Z(a(\eta Z(x_3), \overline{y}))$. Contradiction. $\square$

3. Ending this note, we formulate the following questions

Problem 1. *Let $\mathbb{T}$ be a weakly normal monad and there exists a weakly normal functor F, $1 <^\circ F \leq n$, which extends onto the Kleisli category of monad $\mathbb{T}$. Is there an extension of the power functor $(-)^2$ onto $\mathcal{C}omp_{\mathbb{T}}$?*

Problem 2. *Let $\mathbb{T}$ be a weakly normal monad such that the power functor $(-)^2$ extends onto $\mathcal{C}omp_{\mathbb{T}}$. Is there an extension of the power functor $(-)^n$, $n \geq 3$, onto $\mathcal{C}omp_{\mathbb{T}}$?*

REFERENCES

[1] D. Curtis. Growth hyperspaces of Peano continua. *Trans. Amer. Math. Soc.* **238** (1978), 271-283.

[2] S. Eilenberg, J. S. Moore. Adjoint functors and triples. *Illinois J. Math.* **9**, *3* (1965), 381-398.

[3] V. V. Fedorchuk, V. V. Filippov. General Topology. Basic Constructions, Moscow, MGU, 1988 (in Russian).

[4] H. Kleisli. Every standard construction is induced by a pair of adjoint functions. *Proc. Amer. Math. Soc.* **16** (1965), 544-546.

[5] E. V. Moiseev. On spaces of closed growth and inclusion hyperspaces. *Vestn. MGU, ser. I. Matem., Mech. 3* (1988), 54-57 (in Russian).

[6] T. Radul. The inclusion hyperspace monad and its algebras. *Ukr. Mat. J.* **42**, *6* (1990), 806-811 (in Russian).

[7] E. V. Shchepin. Functors and uncountable powers of compacta. *Uspekhi Mat. Nauk* **36**, *3* (1981), 3-62 (in Russian).

[8] A. Teleiko. On extension of functors to the Kleisli category of some weakly normal monads. *Comment. Math. Univ. Carolinae* **37**, *2* (1996), 405-410.

[9] J. Vinárek. Projective monads and extensions of functors. *Math. Centr. Afd.* **195** (1983), 1-12.

[10] M. M. Zarichnyi. Characterization of G-symmetric power functors and extension of functors onto Kleisli categories. *Matem. zametki* **52**, *5* (1992), 42-48 (in Russian).

[11] M. M. Zarichnyi. On some categorical properties of probability measures. *Matematika: Zinatniskie raksti* **576** (1992), 81-88.

[12] M. M. Zarichnyi, T. M. Radul. Monads in the category of compacta. *Uspekhi mat. nauk* **50**, *3* (1995), 83-108 (in Russian).

[13] M. M. Zarichnyi, A. Teleiko. Semigroup and monads. In: Algebra and Topology (eds. M. Zarichnyi and M. Komarnytskyi), Lviv, 1996, 84-93 (in Ukrainian).

Lviv University
Department of Mathematics
University str., 1
290602 Lviv
Ukraina
e-mail: `topos@mf.franko.lviv.ua`

Received March 14, 1997