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SOME PROPERTIES OF $\gamma$—AND $P$—SPACES

N. Kalamidas

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Abstract. A $\gamma$-space with a strictly positive measure is separable. An example of a non-separable $\gamma$—space with c.c.c. is given. A $P$—space with c.c.c. is countable and discrete.

In this paper by a space we mean a Tychonoff space. Our terminology is the standard one: any undefined term can be found in [1] or [2].

Is $X$ is a space, then $C_p(X)$ is the space of all real-valued continuous functions on $X$, with the topology of pointwise convergence.

It is well known that $C_p(X)$ is 1st—countable iff $X$ is countable [2]. On the other hand, there exist uncountable $X$’s with $C_p(X)$ satisfying a weaker property, the Frechet-Urysohn (F.U.) property (i.e. if $A \subseteq C_p(X)$, $f \in \overline{A}$ implies $\lim f_n = f$, for a suitable sequence $(f_n)$ with $f_n \in A$). Such spaces are the compact scattered, the Lindelöf $P$—spaces e.t.c. [3]. Spaces $X$ for which $C_p(X)$ has the F.U. property are exactly the spaces with the so-called $\gamma$—property that is the expression of the F.U. property on $C_p(X)$ in terms of covering axioms of $X$ [2].

It is easy to see that a compact scattered space $X$ with the countable chain condition (c.c.c.) is separable. Indeed the set $A = \{x \in X : \{x\} \text{ is clopen}\}$ is countable and dense. Below we prove the same result for a Lindelöf $P$—space (or simply a $P$—space). The question arises whether this is also true on the general class of $\gamma$—spaces or not.

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Theorem 1. There exists a non-separable $\gamma$–space with c.c.c.

Proof. Consider an uncountable set $\Gamma$ and the space

$$\Sigma_\omega = \{ x \in \{0,1\}^\Gamma : |\{ \gamma \in \Gamma : \pi_\gamma(x) \neq 0 \}| < \omega \}$$

($\pi_\gamma$ is the usual $\gamma$–projection).

Then $\Sigma_\omega$ has c.c.c. since it is dense in $\{0,1\}^\Gamma$, but it is not separable since the family $\{\pi_\gamma^{-1}\{1\} : \gamma \in \Gamma\}$ does not contain any infinite subfamily with non-empty intersection. Now we shall prove that $\Sigma_\omega$ is a $\gamma$–space.

For $k = 1, 2, \ldots$ set

$$\Sigma_k = \{ x \in \{0,1\}^\Gamma : |\{ \gamma \in \Gamma : x(\gamma) \neq 0 \}| \leq k \}.$$  

Every $\Sigma_k$ is compact and $\Sigma_\omega = \bigcup \Sigma_k$. It follows that $\Sigma_\omega$ is $\sigma$–compact, so it preserves the Lindelöf property on finite powers and consequently $t(C_p(\Sigma_\omega)) = \omega$ ($t =$ tightness). Besides the generated algebra of $\{\pi_\gamma : \gamma \in \Gamma\} \cup \{1\}$ is dense in $C_p(\Sigma_\omega)$. From these facts it follows that every continuous function on $\Sigma_\omega$ depends on a countable subset of $\Gamma$. So if $f \in \{f_n : n = 1, 2, \ldots\} \subset C_p(\Sigma_\omega)$ and $A$ is the countable subset of $\Gamma$ on which all $f, f_n$ depend, we may suppose that these functions are defined on $\Sigma_\omega \cap (\{0,1\}^A \times (0)_{\Gamma \setminus A})$ which is countable, and the result is immediate.

Remarks. (i) It is well known that $\{0,1\}^\Gamma$ does not have a countable dense subset, in case that $|\Gamma| > 2^\omega$. However, $\{0,1\}^\Gamma$ (hence every dyadic space too) contains a dense $\gamma$–subset.

(ii) Every $\Sigma_k$, is a $\gamma$–space as a closed subspace of a $\gamma$–space. It follows that $\Sigma_\omega$ is $\sigma$–compact, so it preserves the Lindelöf property on finite powers and consequently $t(C_p(\Sigma_\omega)) = \omega$ ($t =$ tightness). Besides the generated algebra of $\{\pi_\gamma : \gamma \in \Gamma\} \cup \{1\}$ is dense in $C_p(\Sigma_\omega)$. From these facts it follows that every continuous function on $\Sigma_\omega$ depends on a countable subset of $\Gamma$. So if $f \in \{f_n : n = 1, 2, \ldots\} \subset C_p(\Sigma_\omega)$ and $A$ is the countable subset of $\Gamma$ on which all $f, f_n$ depend, we may suppose that these functions are defined on $\Sigma_\omega \cap (\{0,1\}^A \times (0)_{\Gamma \setminus A})$ which is countable, and the result is immediate.

Theorem 2. Let $X$ be a $\gamma$–space.

(a) If $X$ has a s.p.m. then $X$ is separable.

(b) A Borel finite measure $\mu$ on $X$, with the property $\mu\{x\} = \inf\{\mu(U) : U \text{ is clopen, } x \in U\}$ is of the form $\sum \alpha_k \delta_{x_k}$.

Proof. (a) Let $\mu$ be a s.p.m. on $X$, that is defined on the $\sigma$–field generated by the pseudobase $B$. Notice that $X$ has a base of clopen subsets [2].

For $k = 1, 2, \ldots$, set

$$J_k = \left\{ U \subset X : U \text{ is clopen and } \exists V \in B \text{ such that } V \subset U \text{ and } \mu(V) \geq \frac{1}{k} \right\}.$$
Then every infinite family in $\mathcal{J}_k$ contains an infinite subfamily with non-empty intersection.

We claim that there exists a finite $F_k \subset X$ such that $F_k \cap U \neq \emptyset$, for every $U \in \mathcal{J}_k$. Suppose not. Then $0 \in \{\chi_U : \hat{U} \in \mathcal{J}_k\}$, so $\chi_{U_n} \to 0$, for a sequence $U_n$ in $\mathcal{J}_k$, contradictory to the property mentioned before for $\mathcal{J}_k$.

(b) Consider the countable set $A = \{x \in X : \mu\{x\} > 0\}$ (because of the finiteness of $\mu$). It is enough to prove that $\mu(X) = \mu(A)$.

Suppose that $\mu(A) < \mu(X)$ and let $0 < \delta < \mu(X) - \mu(A)$. For every finite $F \subset X$, choose a clopen $U_F$ such that $F \subset U_F$ and $\mu(U_F) < \mu(A) + \frac{\delta}{2}$. Then $1 \in \{\chi_{U_F} : F \subset X, \text{ finite}\}$, so $\chi_{U_{F_n}} \to 1$ for a sequence $(U_{F_n})$. It follows that $\mu(X) = \int 1d\mu \leq \sup_n \mu(U_{F_n}) \leq \mu(A) + \frac{\delta}{2} < \mu(X)$, which is absurd.

Remark. Theorem 2(b) extends a previous result of Rudin for the class of compact scattered spaces [3].

If $X$ has a Baire s.p.m $\mu$, a metric can be defined in a natural way on $C(X)$ by the type, $\rho(f, g) = \int \frac{|f - g|}{1 + |f - g|}d\mu$.

In case that $X$ is a $\gamma$–space the identity map $id : C_p(X) \to C_\rho(X)$ is continuous. What about the continuity of $id^{-1}$? Certainly for uncountable $\gamma$–spaces the answer is negative. But if $X$ is a $P$–space (where $G_\delta$–sets are open) then $id^{-1}$ is continuous.

[Suppose that $\int \frac{|f_n - f|}{1 + |f_n - f|}d\mu \to 0$. We claim that $f_n \to 0$. If not, then $\frac{|f_{n_k}(x_0) - f(x_0)|}{1 + |f_{n_k}(x_0) - f(x_0)|} > \delta$, for some $x_0 \in X$, $\delta > 0$ and a subsequence $(f_{n_k})$. For $k = 1, 2, \ldots$ set

$$U_k = \left\{x \in X : \frac{|f_{n_k}(x) - f(x)|}{1 + |f_{n_k}(x) - f(x)|} > \delta\right\}.$$ 

Then $\cap U_k$ is a non-empty open subset of $X$, so $\mu(\cap U_k) > 0$. On the other hand, $\mu(\cap U_k) \leq \frac{1}{\delta} \int \frac{|f_{n_k} - f|}{1 + |f_{n_k} - f|}d\mu \to 0$, contradiction.]

Consequently, a Lindelöf $P$–space with a Baire s.p.m. is countable and discrete. Certainly, this is also followed by Theorem 2(a). In fact, a more general result is valid:

**Theorem 3.** A $P$–space with c.c.c. is countable and discrete.

**Proof.** Let $X$ be a $P$–space with c.c.c. and $x \in X$. 

We claim that \( \{ x \} \) is clopen. Suppose not. Then we construct pairwise disjoint clopen sets \( V_\xi, \xi < \omega^+ \) such that \( x \notin V_\xi \) in the following way. If \( V_\xi, \xi < \ell \) have been defined, then \( \bigcap_{\xi < \ell} V_\xi^c \) is a clopen neighbourhood of \( x \). Since \( \bigcap_{\xi < \ell} V_\xi^c \neq \{ x \} \), we find a clopen (since \( X \) has a base of clopen subsets) subset \( V_\ell \subset \bigcap_{\xi < \ell} V_\xi^c \) with \( x \notin V_\ell \). The result follows from the fact that \( X \) has c.c.c.

**Remarks.** (i) From the proof of Theorem 3 it follows that a \( P_{k^+} \)-space \( X \) with \( k^+.c.c. \) has cardinality \( |X| \leq k \). We mention that this is not true if \( X \) is simply a \( P \)-space. For example, the space \( X = \left( \{ 0,1 \}^{(2^\omega)^+} \right)^{\omega^+} \) (the space \( \{ 0,1 \}^{(2^\omega)^+} \) with the \( \omega^+ \)-box topology) is a \( P \)-space with \( (2^\omega)^+.c.c. \). [2] and \( |X| \geq (2^\omega)^+ \).

(ii) If \( X \) is a \( P \)-space and \( A \subset X \) is countable then \( A \) is closed. We note that Shakhmatov constructed a non-separable, c.c.c. space, all countable subsets of which are closed [4].

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Department of Mathematics
Section of Mathematical Analysis and Applications
University of Athens
Panepistemiopolis
157 84 Athens
Greece

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