SOME PARA-HERMITIAN RELATED COMPLEX
STRUCTURES AND NON-EXISTENCE OF
SEMI-RIEMANNIAN METRIC ON SOME SPHERES

Sadettin Erdem

Communicated by V. Kanev

Abstract. It is shown that the spheres $S^{2n}$ (resp: $S^k$ with $k \equiv 1 \mod 4$) can be given neither an indefinite metric of any signature (resp: of signature $(r, k-r)$ with $2 \leq r \leq k-2$) nor an almost paracomplex structure. Further for every given Riemannian metric on an almost para-Hermitian manifold with the associated 2-form $\phi$ one can construct an almost Hermitian structure (under certain conditions, two different almost Hermitian structures) whose associated 2-form(s) is $\phi$.

1. Introduction. Semi-Riemannian metric is quite important in differential geometry as well as in physics in which it plays a central role in the theory of relativity, especially as a Lorentz metric. Almost paracomplex structure is one of the basic ingredient in the geometry resting on the ring of paracomplex numbers. It also provides a link between paraholomorphicity and harmonicty of maps (as solutions of hyperbolic systems) among certain almost para-Hermitian manifolds. These are the few things that one may site among many others. As

1991 Mathematics Subject Classification: 53C15, 53C50, 53C55

Key words: Hermitian, para-Hermitian and indefinite metric.
Sadettin Erdem

for this reason, we state here some non-existence results and introduce some para-Hermitian related complex structures.

Acknowledgement. I would like to thank to the referee for bringing out a work of Matsushita, [2] to my attention and making valuable suggestions.

2. Definition and results. Let \((V, h) \to M\) denote throughout a vector bundle of rank \(k\) endowed with a semi-Riemannian metric \(h\) of signature \((r, s)\) over a paracompact manifold \(M\). A semi-Riemannian metric \(h\) of signature \((r, s)\) on \(V\) with \(r + s = k\) is a globally defined continuous symmetric section of the bundle \((V \otimes V)^*\) such that for every \(p \in M\) if \(h_p(X, Y) = 0, \ \forall Y \in V_p\) then \(X = 0\). (This is the nondegeneracy condition for \(h_p\) on \(V_p\)). It is the standard fact that if \(e_1, \ldots, e_k\) form an \(h_p\)-orthogonal basis for \(V_p\) then there are exactly \(r\) many of them with \(h_p(e_i, e_i) < 0\) and the rest \(s\) many are with \(h_p(e_i, e_i) > 0\). When \(r = 0\) or \(r = k\) then \(h\) is called a Riemannian metric. When \(r = s\) then \(h\) is called a neutral metric.

Let \(F\) be a globally defined continuous section of the bundle \(V^* \otimes V\) over \(M\). Then

- \(a^\circ\) \(F\) is called a product structure on \(V\) if \(F^2 = I\) with \(F \neq \pm I\), where \(I\) is the identity.
- \(b^\circ\) \(F\) is called a complex structure on \(V\) if \(F^2 = -I\).

A product structure \(F\) gives rise two complementary subbundles \(F^+\) and \(F^-\) over \(M\) which are eigensubbundles of \(F\) corresponding to eigenvalues 1 and -1 respectively. That is

\[ F^+ = \{v \in V : F(v) = v\} \quad \text{and} \quad F^- = \{v \in V : F(v) = -v\} \]

Clearly \(F^+ \oplus F^- = V\) and for \(\text{rank}(F^+) = \ell, \ \text{rank}(F^-) = t\) we have \(\ell + t = k = \text{rank}(V)\), and that \(F\) is said to have signature \((\ell, t)\). If \(F\) is of signature \((\ell, \ell)\) then it is called a paracomplex structure on \(V\). We designate the letters \(P\) and \(J\) for paracomplex and complex structures respectively.

Note that if \(V\) can be endowed with a structure \(P\) or \(J\) then the \(\text{rank}(V)\) is necessarily even. In the cases where \(V = TM\) (the tangent bundle of \(M\)) the structures \(F, P\) and \(J\) are called almost product, almost paracomplex and almost complex structure of \(M\) respectively. Also

- \(c^\circ\) A pair \((P, h)\) (resp: \((J, g)\)) is called para-Hermitian (resp: Hermitian) structure on \(V\) if the semi-Riemannian metric \(h\) (resp: Riemannian metric \(g\)) satisfies

\[ h(P(X), P(Y)) = -h(X, Y), \quad (\text{resp:} \ g(J(X), J(Y)) = g(X, Y)) \]

for every sections \(X, Y\) of \(V\).
A skew-symmetric, nondegenerate continuous section $\phi$ of $(V \otimes V)^*$ is called a symplectic structure on $V$. The bundle $V$ with $\phi$ is then called a symplectic vector bundle.

When $V = TM$, $\phi$ is then called almost symplectic structure on $M$. Note that the structures $(P, h)$ and $(J, g)$ define an associated symplectic structures on $V$ via

$$\phi(X, Y) = h(X, P(Y)) \quad \text{and} \quad \Omega(X, Y) = g(X, J(Y))$$

respectively.

Consider now a vector bundle $V$ of rank $k$ endowed with a semi-Riemannian metric $h$ of signature $(r, s)$. Let $G$ be a Riemannian metric on $V$, (which always exists). Define a global section $L$ of the bundle $V^* \otimes V \to M$ by $G_p(L_p(X), Y) = h_p(X, Y)$ for every $p \in M$ and $X, Y \in V_p$.

Now we set $V^+_p$ (resp: $V^-_p$) to be the sum of the eigenspaces corresponding to the positive (resp: negative) eigenvalues of the symmetric endomorphism $L$ of $V$. Then it is easy to prove the following:

**Lemma 2.1.** A pair $(h, G)$ of a semi-Riemannian and a Riemannian metrics gives rise to subbundles $V^+$ and $V^-$ satifying

(a°) $V = V^+ \oplus V^-$,
(b°) rank $(V^+) = s$ and rank $(V^-) = r$,
(c°) $h$ is positive definite on $V^+$ and negative definite on $V^-$,
(d°) $V^-$ is $h$-orthogonal (and also $G$-orthogonal) complement of $V^+$.

We call $V^+$ and $V^-$ the $(h, G)$-induced subbundles.

Our first result is the following:

**Theorem 2.2.** The $k$-sphere $S^k$ does not admit any semi-Riemannian metric of signature $(r, s)$ if $k$ is even and $1 \leq r \leq k - 1$ or $k \equiv 1 \pmod{4}$ and $2 \leq r \leq k - 2$.

**Proof.** Suppose $S^k$ admits a semi-Riemannian metric of signature $(r, s)$. Then by Lemma 2.1, the tangent bundle $TS^k$ splits into two subbundles of ranks $r$ and $s$. But it is well known that (see e.g [3, Theorem 27.18]) $TS^k$ does not admit a subbundle of rank $r$ if $k$ satisfies the hypothesis of the theorem. So the result follows immediately. □

**Corollary 2.3.** A sphere of even dimension does not admit a semi-Riemannian metric of any signature.

It is now straightforward to see that

**Theorem 2.4.** The $k$-sphere $S^k$ does not admit any almost product structure of signature $(\ell, t)$ if $k$ is even and $1 \leq \ell \leq k - 1$ or $k \equiv 1 \pmod{4}$ and
\[ 2 \leq \ell \leq k - 2. \]

**Corollary 2.5.** A sphere of even dimension does not admit an almost product structure of any signature. Therefore it does not admit an almost para-complex structure.

**Remark.** Recall also that \( S^2 \) and \( S^6 \) are the only spheres that admit almost complex structures.

**Theorem 2.6.** Let \( V \) be a symplectic vector bundle with a symplectic structure \( \phi \) and let \( G \) be a Riemannian metric on \( V \). Then the pair \( (\phi, G) \) induces a complex structure \( J \) which is also compatible with \( \phi \), that is, \( \phi(J(X), J(Y)) = \phi(X, Y) \). If further \( \phi \) is the associated one with a para-Hermitian structure \( (P, h) \) on \( V \) then the pair \( (\phi, G) \) gives rise to two more complex structures \( J_+ \) and \( J_- \) which are not, in general, \( \phi \)-compatible. However, the following are equivalent:

- \( (a^\circ) \) \( J_+ \) is \( \phi \)-compatible,
- \( (b^\circ) \) \( J_- \) is \( \phi \)-compatible,
- \( (c^\circ) \) \( J_- = -J_+ \),
- \( (d^\circ) \) \( V^+ = P(V^-) \),
- \( (e^\circ) \) \( V^- = P(V^+) \)

where \( G \) is an arbitrary Riemannian metric and \( V^+ \), \( V^- \) are \((G, h)\)-induced sub-bundles.

**Proof.** (For a detailed proof see [4, Theorem 2.3.3.]). Define a tensor field \( W \), a section of \( V^* \otimes V \), by \( G(W(X), Y) = \phi(X, Y) \) for every sections \( X, Y \). Choose a local frame field \( B = \{v_1, \cdots, v_{2n}\} \) such that the matrix representation \([W]_B \) of \( W \) in \( B \) is

\[
\begin{bmatrix}
0 & \cdots & N \\
\cdots & \cdots & \cdots \\
-N^t & \cdots & 0
\end{bmatrix}
\]

where

\[
N = \begin{bmatrix}
0 & \cdots & 0 & \lambda_1 \\
0 & \cdots & \lambda_2 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
\lambda_n & 0 & \cdots & 0
\end{bmatrix}
\]

with \( \lambda_i \neq 0 \ \forall i = 1, \cdots, n \). (One can show the existence of such a basis). For a section \( K \) of \( V^* \otimes V \) with \([K]_B = \text{diag} \ (\lambda_1, \cdots, \lambda_n, \lambda_n, \cdots, \lambda_1) \), set

\[
J = WK^{-1} \quad \text{and} \quad g(X, Y) = \phi(X, J(Y)).
\]

Then \( (J, g) \) is the required Hermitian structure on \( V \).

Further, since \( V^\pm \cap P(V^\pm) = 0 \), the vector bundle \( V \) has two splittings

\[
V = V^+ \oplus P(V^+) \quad \text{and} \quad V = V^- \oplus P(V^-),
\]
where $V^+$ and $V^-$ are the $(h, G)$-induced subbundles. Thus we have two complex structures $J_+$ and $J_-$ on $V$ given by

$$J_±(X + P(Y)) = -Y + P(X) \quad \text{for} \quad X, Y \in C(V^±).$$

Moreover, for $u, v, w, z \in C(V^+)$ set $X = u + P(v)$ and $Y = w + P(z)$ and observe that

$$\phi(J_+X, J_+Y) = \phi(v, z) - \phi(u, w) + \phi(u, P(z)) + \phi(P(v), w),$$

$$\phi(X, Y) = \phi(u, w) - \phi(v, z) + \phi(u, P(z)) + \phi(P(v), w).$$

Thus we get $\phi(J_+X, J_+Y) = \phi(X, Y)$ if and only if $\phi \equiv 0$ on $V^+$, that is, $\phi(u, w) = h(u, P(w)) = 0 \forall u, w \in C(V^+)$. Hence the equivalence of $(a_0^°), (b^°), (c^°), (d^°)$ and $(e^°)$ follows easily. \(\square\)

We call a complex structure on a vector bundle induced by the pair $(\phi, G)$ with $(P, h)$-associated $\phi$ the $(P, h, G)$-related complex structure.

**Remarks.**

1°) The statement of the above theorem is no longer true for the case of paracomplex structures, i.e. that symplectic vector bundle may not admit any paracomplex structure at all, e.g. $S^2$ and $S^6$ admit almost symplectic structures and yet do not admit any almost paracomplex structure.

2°) The $\phi$-compatible complex structure $J$ in the above theorem gives rise to a Hermitian structure $(J, g)$ via $g(X, Y) = \phi(X, J(Y))$ whose associated symplectic structure coincides with $\phi$.

3°) If the $(P, h, G)$-related complex structures $J_+$ and $J_-$ on $V$ satisfy that $J_+ = -J_-$ then, by the above theorem, they are $\phi$-compatible. (For the case where $J_+ \neq -J_-$, see Example 3.1 at the end). Thus $J_±$ together with the metric $g_±(X, Y) = \phi(X, J_±(Y))$ defines a Hermitian structure $(J_±, g_±)$ whose associated symplectic structure coincides with $\phi$. Also the Hermitian metric has the following properties:

$a^°)$ $g_+ = h$ on $V^+$, $g_- = -h$ on $V^-$ and $V^+, V^-$ are $g_+$-orthogonal.

$b^°)$ For a $G$ and $h$ orthogonal local frame field $\{u_1, \cdots, u_{2n}\}$ of $V$ we have $g_+(u_i, u_j) = \lambda_i G(u_i, u_j)$ for some nonzero $\lambda_i(p) \in \mathbb{R}$; $\forall i, j = 1, 2, \cdots, (2n)$; $p \in M$.

$c^°)$ $g_+ = -g_-$, where $g_-(X, Y) = \phi(X, J_-(Y))$

**Theorem 2.7.** For a vector bundle $V$ the following are equivalent:

i) $V$ admits a product structure $F$ of signature $(\ell, t)$

ii) $V$ admits a semi-Riemannian metric $h$ of signature $(\ell, t)$.

**Proof.** Assume (i), then we have a splitting $V = F^+ \oplus F^-$. For a
Riemannian metric $g$ on $V$ set $h(X,Y) = g(X,Y)$ if $X,Y \in F^+$; $h(X,Y) = -g(X,Y)$ if $X,Y \in F^-$ and $h(X,Y) = 0$ otherwise. Then $h$ is the required semi-Riemannian metric on $V$.

Conversely assume (ii), then $V$ admits a splitting $V = V^+ \oplus V^-$ with $\dim V^+ = \ell$ and $\dim V^- = t$ by Lemma (1.1). So set $F(X) = X$ if $X \in V^+$ and $F(X) = -X$ if $X \in V^-$. Then $F$ is the required product structure. □

Remark. In the case where $\ell = t$ the above theorem states that:

$N$ admits an almost paracomplex structure $Q$ (of signature $(\ell, \ell)$) if and only if $N$ admits a neutral metric $H$ (of signature $(\ell, \ell)$).

3. A special case and an example. Let $N$ be an oriented 4-manifold and set $V = TN$. Then the following two conditions are equivalent, [2]:

i) $N$ admits two distinct mutually commuting almost complex structures

ii) $N$ admits non-degenerate field of oriented 2-planes.

On the other hand, from the last Remark, we have the following two statements that are equivalent to each other

iii) $N$ admits an almost paracomplex structure $Q$ (of signature $(2,2)$)

iv) $N$ admits a neutral metric $H$ (of signature $(2,2)$)

Also note that $(Q,H)$ does not form an almost para-Hermitian structure on $N$. It is easy to see that (i) (and therefore (ii)) implies (iii) (and therefore (iv)). But the converse is not true as the field of 2-planes induced by either $Q$ or $H$ is not necessarily orientable. However, the fact that “(i) implies (iii)” enables us to consider some complex surfaces such as some certain classes of minimal rational surfaces, Hopf surfaces and Inoue surfaces, ruled surfaces of genus $\mu \geq 1$, Enriques surfaces, hyperelliptic surfaces, Kodaira surfaces, $K3$ surfaces, Tori and minimal property elliptic surfaces as examples of paracomplex (and also semi-Riemannian) compact 4-manifolds which are not product of real surfaces, (for detail see [2]). Note that the tangent bundle of a product manifold $M$ has a canonical splitting and therefore $M$ has a product structure and a semi-Riemannian metric.

Example 3.1. For $X = (x_i), Y = (y_i) \in \mathbb{R}^7$ set

$$H(X,Y) = -\sum_{i=1}^{3} x_i y_i + \sum_{i=4}^{7} x_i y_i \quad \text{and} \quad S_3^6 = \{X \in \mathbb{R}^7 : H(X,X) = 1\}$$

and denote by $h_0$ the neutral metric (which is naturally of signature $(3,3)$) on $S_3^6$ induced by $H$.

Also for $X = (x_i) \in S_3^6 \subseteq \mathbb{R}^7$ and $Y = (y_i) \in T_X S_3^6 \subseteq \mathbb{R}^7$ Let $Q \in C(T^* S_3^6 \otimes TS_3^6)$
be a tensor field such that at \( X \in S^6_3 \), \( Q_X : T_X S^6_3 \to T_X S^6_3 \) is given by

\[
Q_X(Y) = (a_1(X, Y), \cdots, a_7(X, Y))
\]

where

\[
\begin{align*}
a_1 &= x_2y_3 - x_3y_2 + x_4y_5 - x_5y_4 - x_6y_7 + x_7y_6 \\
a_2 &= -x_1y_3 + x_3y_1 + x_4y_7 - x_7y_4 - x_5y_6 + x_6y_5 \\
a_3 &= x_1y_2 - x_2y_1 + x_4y_6 - x_6y_4 + x_5y_7 - x_7y_5 \\
a_4 &= x_1y_5 - x_5y_1 + x_2y_7 - x_7y_2 + x_3y_6 - x_6y_3 \\
a_5 &= -x_1y_4 + x_4y_1 - x_2y_6 + x_6y_2 + x_3y_7 - x_7y_3 \\
a_6 &= -x_1y_7 + x_7y_1 + x_2y_5 - x_5y_2 - x_3y_4 + x_4y_3 \\
a_7 &= x_1y_6 - x_6y_1 - x_2y_4 + x_4y_2 - x_3y_5 + x_5y_3
\end{align*}
\]

This tensor field \( Q \), together with the metric \( h_\circ \) defines an almost para-Hermitian structure \((Q, h_\circ)\) on \( S^6_3 \) which is first described by Libermann [1] by using the Cayley’s split octaves. Denote the \((Q, h_\circ)\)-associated almost symplectic structure on \( S^6_3 \) by \( \phi_\circ \).

Now let \( \alpha = (0, 0, 1; 1, 1, 0, 0) \in S^6_3 \), then the set \( U = \{ E_1, \cdots, E_6 \} \) forms a basis for \( T_\alpha = T_\alpha S^6_3 \) where

\[
E_1 = (1, 0, \cdots, 0), \quad E_2 = (0, 1, 0, \cdots, 0), \quad E_3 = (0, 0, 1; 1, 0, 0, 0) \\
E_4 = (0, 0, 1; 0, 1, 0, 0), \quad E_5 = (0, \cdots, 0, 1, 0), \quad E_6 = (0, \cdots, 0, 1)
\]

The matrix representation \( A = [W_\alpha]_U \) in \( U \) of a linear transformation \( W_\alpha : T_\alpha \to T_\alpha \) defined by the equation

\[
G_\circ(W(Z), Y) = h_\circ(Z, Y)
\]

takes the form

\[
A = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/3 & -2/3 & 0 & 0 \\
0 & 0 & -2/3 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

where \( G_\circ \) is the Riemannian metric on \( S^6_3 \) obtained by restricting the standard inner product on \( \mathbb{R}^7 \) to \( T_X S^6_3 \). Hence the \((h_\circ, G_\circ)\)-induced subbundles \( V^+, V^- \) of \( TS^6_3 \) have the fibres at \( \alpha \) given by:

\[
V^+_\alpha = \{ Z \in T_\alpha S^6_3 : W(Z) = Z \} = \text{span} \ \{ E_3 - E_4, E_5, E_6 \}
\]
\[ V^- = \{ Z \in T_\alpha S^6_3 : W(Z) = -Z \} = \text{span} \{ E_1, E_2, E_3 + E_4 \} \]

We see that \( Q_\alpha(E_1) = (0, 1, 0; 1, 1, 0) \not\in V^+ \) while \( E_1 \in V^- \). So \( Q(V^-) \neq V^+ \), thus by Theorem 2.6, the pair \((\phi_o, G_o)\) induces three different almost complex structures \( J_o, J_o^+ \) and \( J_o^- \) on \( S^6_3 \) and that \((J_o, g_o)\) defines an almost Hermitian structure whose associated almost symplectic structure coincides with \( \phi_o \), where \( g_o(X, Y) = h_o(X, QJ_o(Y)) = \phi(X, J_o(Y)) \) which is positive definite. On the other hand \((J_o^\pm, g_o^\pm)\) associated symplectic structure \( \phi_o^\pm \) does not coincide with \( \phi_o \).

REFERENCES


Department of Mathematics
Middle East Technical University
06531 Ankara
Turkey

e-mail: serdem@arf.math.metu.edu.tr

Received September 7, 1998

Revised January 28, 1999