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THE PERTURBED GENERALIZED TIKHONOV'S ALGORITHM

P. Alexandre

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Abstract. We work on the research of a zero of a maximal monotone operator on a real Hilbert space. Following the recent progress made in the context of the proximal point algorithm devoted to this problem, we introduce simultaneously a variable metric and a kind of relaxation in the perturbed Tikhonov's algorithm studied by P. Tossings. So, we are led to work in the context of the variational convergence theory.

1. Introduction. Let $\mathcal{H}$ be a real Hilbert space and T be a maximal monotone operator on $\mathcal{H}$.

We consider the problem

$$\text{(P)} \qquad \text{“To find } \overline{x} \in \mathcal{H} \text{ such that } 0 \in T\overline{x}.\text{”}$$

The practical importance of this problem is well known, thanks to its applications (nondifferentiable convex optimization, minimax problems, variational inequalities, ...).

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Two algorithms for solving problem (P) are well known: *the proximal point algorithm*, on the one hand, and *the Tikhonov's algorithm*, on the other hand.

Many authors worked on the first one, less on the second one. Nevertheless, they have complementary advantages and desadvantages and are, therefore, both interesting.

Let us recall some great steps of the history of these algorithms.

– Using the variational convergence theory, B. Lemaire [10] gave, in the eighties, a perturbed version of the proximal point algorithm related to convex optimization.

– In 1990, P. Tossings [14] extended the notion of variational metric between proper closed convex functions to the context of maximal monotone operators and studied the perturbed version of both general algorithms.

– In the same time, more and more authors began to modify the metric appearing in the proximal regularization. Two kinds of modifications arose. Some authors replaced the classical metric by a nonlinear fixed metric, based on an entropic method (G. Chen and M. Teboulle [5], J. Eckstein [8], S. Kabbadj [9], . . .). Others let the metric change at each iteration (G. Cohen [6] and [7], M. Qian [11] and [12], J. F. Bonnans, J. C. Gilbert, C. Lemaréchal and C. Sagastizabal [4], A. Renaud [13], . . .)

– In [11] and [12], M. Qian introduced also a kind of relaxation.

– To conclude, we combined recently, in the proximal point algorithm, all the notions of perturbation, variable metric and relaxation (see P. Alexandre, V. H. Nguyen and P. Tossings [3]).

In the present paper, we make the same work for the Tikhonov's algorithm.

In section 2, we recall some definitions and results of the *generalized* variational convergence theory. In section 3, we study the Tikhonov's algorithm associated with a linear, continuous, self-adjoint positive definite transformation H, with linear continuous inverse H^{-1}. Section 4 is devoted to the *perturbed variable metric Tikhonov's algorithm* or, more simply, *perturbed generalized Tikhonov's algorithm* and its properties of convergence.

The applications of our algorithm to convex optimization and variational inequalities will be studied in following papers.

Notations and conventions. *In the following text, $\mathcal{H}$ will always denote a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. T or T_n ($n \in \mathbb{N}^*$) will denote a maximal monotone operator on $\mathcal{H}$ and H or H_n ($n \in \mathbb{N}^*$) a linear, continuous, self-adjoint positive definite transformation on $\mathcal{H}$, with linear* *continuous inverse* H^{-1} *or* H_n^{-1}.

2. The generalized variational convergence theory. In this section are stated some extensions of the variational convergence theory introduced in P. Alexandre [1] and summarized in P. Alexandre and P. Tossings [2].

Let us first fix some notations.

⋆ We denote by $\langle\cdot,\cdot\rangle_H$ the inner product associated with H:

$$\langle x,y\rangle_H \underset{def.}{=} \langle x,Hy\rangle, \quad \forall\, x,y\in\mathcal{H},$$

and by $\|\cdot\|_H$ the associated norm:

$$\|x\|_H \underset{def.}{=} \sqrt{\langle x,x\rangle_H}, \quad \forall\, x\in\mathcal{H}.$$

This norm is connected with the initial one by the relations

$$\|x\|_H \le \sqrt{\|H\|}\,\|x\|, \quad \forall\, x\in\mathcal{H}, \tag{2.1}$$

and

$$\|x\| \le \sqrt{\|H^{-1}\|}\,\|x\|_H, \quad \forall\, x\in\mathcal{H}. \tag{2.2}$$

From a functional point of view, let us recall a property which is fundamental in view of the proof of Theorem 3.1.

The set of solutions of (P),

$$S=\{x\in\mathcal{H}: 0\in Tx\},$$

being, thanks to the properties of T, a closed convex subset of $\mathcal{H}$, *there is a unique point* $\overline{x}\in S$ *such that*

$$\|\overline{x}\|_H=\min_{x\in S}\|x\|_H.$$

In other words, we may write

$$(0\in Tx \ \text{ and } \ \|x\|_H\le\|\overline{x}\|_H) \quad\Leftrightarrow\quad x=\overline{x}. \tag{2.3}$$

⋆ We denote by $J_\lambda^{H^{-1}T}$ the *generalized resolvent operator associated with* T*, with parameter* λ (*i.e.* the resolvent operator associated with $H^{-1}T$, with parameter λ),

$$J_\lambda^{H^{-1}T} \underset{def.}{=} \left(I+\lambda H^{-1}T\right)^{-1}, \tag{2.4}$$

which has sense because, under the assumptions on H, T is maximal monotone for the initial inner product on $\mathcal{H}$ if and only if $H^{-1}T$ is maximal monotone for the inner product associated with H.

We denote by $A_\lambda^{H^{-1}T}$ the *generalized Yosida approximate of* T, *with parameter* λ,

$$A_\lambda^{H^{-1}T} \underset{def.}{=} \frac{I - J_\lambda^{H^{-1}T}}{\lambda}. \tag{2.5}$$

Note that, when H is the identity, these operators are nothing else but the classical resolvent operator associated with T, with parameter λ, and the corresponding Yosida approximate.

Definitions (2.4) and (2.5) imply

$$0 \in T\overline{x} \Leftrightarrow J_\lambda^{H^{-1}T}\overline{x} = \overline{x},\ \forall\, \lambda > 0 \Leftrightarrow A_\lambda^{H^{-1}T}\overline{x} = 0,\ \forall\, \lambda > 0 \tag{2.6}$$

and

$$A_\lambda^{H^{-1}T}x \in H^{-1}T\left(J_\lambda^{H^{-1}T}x\right),\ \forall\, \lambda > 0,\ \forall\, x \in \mathcal{H}. \tag{2.7}$$

The generalized resolvent operator is useful to define a *generalized variational metric* between operators on $\mathcal{H}$.

Definition 2.1. *Let* $\lambda > 0$ *and* $\rho \geq 0$ *be given. The* generalized variational metric between T_1 and T_2, with parameters λ and ρ, *is the metric*

$$\delta_{\lambda,\rho}\left(H_1^{-1}T_1, H_2^{-1}T_2\right) \underset{def.}{=} \sup_{\|x\| \leq \rho} \left\| J_\lambda^{H_1^{-1}T_1}x - J_\lambda^{H_2^{-1}T_2}x \right\|.$$

Note that, once more, for $H_1 = H_2 = I$, I denoting the identity on $\mathcal{H}$, this metric is nothing else but the classical variational metric between T_1 and T_2, with parameters λ and ρ, introduced in P. Tossings [14].

It is possible to compare $\delta_{\lambda,\rho}\left(H_1^{-1}T_1, H_2^{-1}T_2\right)$ and $\delta_{\lambda,\rho}\left(T_1, T_2\right)$ ($\lambda > 0$, $\rho \geq 0$). We give here below a practical version of this comparison.

Proposition 2.2. *Assume that* T *has at least one zero* $\overline{x}$ *and*

(i) $0 < \underline{\lambda} \leq \lambda_n,\ \forall\, n \in \mathbb{N}^*$,

(ii) $H_n \overset{u}{\to} H$,

(iii) $\|I - H\| < 1$.

Then, for every $\rho > 0$, there are a range $N \in \mathbb{N}^*$ and strictly positive real numbers ρ^*, C and ε such that

$$\delta_{\lambda_n,\rho}(H_n^{-1}T_n, H^{-1}T) \leq \frac{1}{\varepsilon}\left[C\|H_n - H\| + \frac{\lambda_n}{\underline{\lambda}}\delta_{\underline{\lambda},\rho^*}(T_n, T)\right], \ \forall\, n \geq N.$$

P r o o f. See P. Alexandre [1], Proposition I.3.16, or P. Alexandre and P. Tossings [2], Proposition 6.2. □

3. The Tikhonov's algorithm associated with H.

Let us work like in the context of the proximal point algorithm (see P. Alexandre, V. H. Nguyen and P. Tossings [3]).

The classical Tikhonov's algorithm generates a sequence $\{y_n\} \subset \mathcal{H}$ by the rule

$$y_n = J_{\lambda_n}^{T} 0, \ \forall\, n \in \mathbb{N}^*.$$

It needs two transformations to become the Thikhonov's algorithm associated with H.

First, using the norm associated with H and the related notions, we replace, in iteration $n \in \mathbb{N}^*$ of the classical algorithm, the resolvent operator $J_{\lambda_n}^{T}$ by the generalized resolvent operator $J_{\lambda_n}^{H^{-1}T}$, H denoting, with the previous conventions, a linear, continuous, symetric positive definite transformation on $\mathcal{H}$, with linear continuous inverse H^{-1}.

Then, to introduce a relaxation, we replace the generalized resolvent operator $J_{\lambda_n}^{H^{-1}T}$ by the combination

$$I + \vartheta_n\left(J_{\lambda_n}^{H^{-1}T} - I\right),$$

ϑ_n being a strictly positive real number.

So, we are led to consider the algorithm that generates a sequence $\{x_n\} \subset \mathcal{H}$ defined by the rule (called *basic generalized Tikhonov's rule*)

$$\text{(BGTR)} \qquad x_n = S_{\lambda_n,\vartheta_n}^{H^{-1}T} 0 = \vartheta_n J_{\lambda_n}^{H^{-1}T} 0, \quad \forall\, n \in \mathbb{N}^*.$$

Before giving the fundamental result of convergence for the sequence generated by (BGTR), let us recall two properties of the operator

$$S_{\lambda,\vartheta}^{H^{-1}T} \underset{def.}{=} I + \vartheta\left(J_{\lambda}^{H^{-1}T} - I\right) \ (\lambda, \vartheta > 0).$$

⋆ On the one hand, the previous definition and the properties of the generalized resolvent operator imply

$$0 \in T\overline{x} \;\Leftrightarrow\; S^{H^{-1}T}_{\lambda_n,\vartheta_n}\overline{x} = \overline{x}. \tag{3.1}$$

⋆ On the other hand, it is possible to prove that, *when $\vartheta \leq 2$, the operator $S^{H^{-1}T}_{\lambda,\vartheta}$ is a contraction for the norm generated by H* (see P. Alexandre [1], Proposition II.2.1).

Theorem 3.1. *Assume that problem* (P) *has at least one solution and*

(i) $0 < \lambda_n,\ \forall\, n \in \mathbb{N}^*,$ *with* $\lim_{n\to+\infty} \lambda_n = +\infty,$

(ii) $0 < \vartheta_n,\ \forall\, n \in \mathbb{N}^*,$ *with* $\lim_{n\to+\infty} \vartheta_n = \vartheta,\ \vartheta \in]0,2[.$

Then, the sequence $\{x_n\}$ generated by (BGTR) *strongly converges to $\vartheta\overline{x}$ where $\overline{x}$ is the solution of* (P) *with minimum H-norm.*

Proof. We will establish this result in four steps.

$\boxed{1}$ *The sequence $\{x_n\}$ is bounded.*

Since hypothesis (ii) ensures the existence of a range $N \in \mathbb{N}^*$ from which $0 < \vartheta_n < 2$, we may write successively, by using the properties of $\|\cdot\|_H$ and $S^{H^{-1}T}_{\lambda_n,\vartheta_n}$ $(n \in \mathbb{N}^*)$,

$$\begin{aligned}
\|x_n - \overline{x}\| &= \left\| S^{H^{-1}T}_{\lambda_n,\vartheta_n} 0 - S^{H^{-1}T}_{\lambda_n,\vartheta_n}\overline{x} \right\| \\
&\leq \sqrt{\|H^{-1}\|}\, \left\| S^{H^{-1}T}_{\lambda_n,\vartheta_n} 0 - S^{H^{-1}T}_{\lambda_n,\vartheta_n}\overline{x} \right\|_H \\
&\leq \sqrt{\|H^{-1}\|}\, \|\overline{x}\|_H \\
&\leq \sqrt{\|H\|\,\|H^{-1}\|}\, \|\overline{x}\|, \quad \forall\, n \geq N.
\end{aligned}$$

It follows that

$$\|x_n\| \leq \left(1 + \sqrt{\|H\|\,\|H^{-1}\|}\right) \|\overline{x}\|, \quad \forall\, n \geq N.$$

$\boxed{2}$ *Every weak cluster point x^* of $\{x_n\}$* (and, from $\boxed{1}$, there is at least one) *is such that $\dfrac{x^*}{\vartheta}$ is a solution of* (P).

Let $x^* \in \mathcal{H}$ be a weak cluster point of $\{x_n\}$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$x_{n_k} \xrightarrow{w} x^*.$$

Since the sequence $\{\vartheta_n\}$ goes to $\vartheta > 0$, we have

$$\frac{x_{n_k}}{\vartheta_{n_k}} \xrightarrow{w} \frac{x^*}{\vartheta}.$$

The continuity of H and the conditions imposed on the sequence $\{\lambda_n\}$ imply therefore

$$\frac{Hx_{n_k}}{\lambda_{n_k}\vartheta_{n_k}} \xrightarrow{s} 0.$$

The rule (BGTR) being equivalent to

$$-\frac{Hx_n}{\lambda_n\vartheta_n} \in T\frac{x_n}{\vartheta_n}, \ \forall\, n \in \mathbb{N}^*, \tag{3.2}$$

and the graph of T being closed in $H_w \times H_s$, we deduce thence

$$0 \in T\frac{x^*}{\vartheta}.$$

$\boxed{3}$ *The sequence $\{x_n\}$ weakly converges to $\vartheta\overline{x}$.*

Let $x^* \in \mathcal{H}$ be a weak cluster point of the sequence $\{x_n\}$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ weakly convergent to x^*.

Relation (3.2) and the definition of $\overline{x}$, joined to the monotonicity of T and the positivity of λ_{n_k} and ϑ_{n_k} $(k \geq N)$, allow us to write

$$\langle Hx_{n_k}, \overline{x} - \frac{x_{n_k}}{\vartheta_{n_k}}\rangle \geq 0, \quad \forall\, k \geq N.$$

Taking into account the definitions of the inner product associated with H and the corresponding norm, we get

$$\langle x_{n_k}, \overline{x}\rangle_H \geq \frac{1}{\vartheta_{n_k}}\|x_{n_k}\|_H^2, \quad \forall\, k \geq N.$$

We deduce thence

$$\|x_{n_k}\|_H \leq \vartheta_{n_k}\|\overline{x}\|_H, \quad \forall\, k \geq N,$$

and, consequently,

$$\limsup_{k\to+\infty} \|x_{n_k}\|_H \leq \vartheta\|\overline{x}\|_H.$$

But, the norm associated with H being a proper closed convex function, we also have

$$\|x^*\|_H \leq \liminf_{k\to+\infty} \|x_{n_k}\|_H.$$

The two last inequalities lead to

$$\|x^*\|_H \leq \vartheta\|\overline{x}\|_H$$

and, from step $\boxed{2}$ and implication (2.3), to the announced result:

$$x^* = \vartheta\overline{x}.$$

The sequence $\{x_n\}$ is therefore a bounded sequence that owns a unique weak cluster point $\vartheta\overline{x}$: it weakly converges to $\vartheta\overline{x}$.

$\boxed{4}$ *The sequence $\{x_n\}$ strongly converges to $\vartheta\overline{x}$.*

By applying the development made in $\boxed{3}$ to the whole sequence $\{x_n\}$, we obtain

$$\vartheta\|\overline{x}\|_H \leq \liminf_{n\to+\infty} \|x_n\|_H \leq \limsup_{n\to+\infty} \|x_n\|_H \leq \vartheta\|\overline{x}\|_H,$$

and, consequently,

$$\lim_{n\to+\infty} \|x_n\|_H = \vartheta\|\overline{x}\|_H.$$

Since the sequence $\{x_n\}$ weakly converges to $\vartheta\overline{x}$, we deduce thence

$$\lim_{n\to+\infty} \|x_n - \vartheta\overline{x}\|_H = 0.$$

Overestimate (2.2) leads to the conclusion. $\square$

Remark 3.2. The two first steps of the proof here above bring out that, under hypothesis (i) and (ii) of Theorem 3.1, problem (P) admits at least one solution if and only if the sequence $\{x_n\}$ generated by the rule (BGTR) is bounded.

4. The perturbed generalized Tikhonov's algorithm. Let us introduce, in (BGTR), the notions of *variable metric*, *perturbation* and *numerical error*. We obtain the rule (PGTR), called *perturbed variable metric Tikhonov's rule* or, more simply, *perturbed generalized Tikhonov's rule*:

(PGTR) $$z_n = S_{\lambda_n,\vartheta_n}^{H_n^{-1}T_n} 0 + e_n = \vartheta_n J_{\lambda_n}^{H_n^{-1}T_n} 0 + e_n, \quad \forall\, n \in \mathbb{N}^*,$$

$\{H_n\}$ and $\{T_n\}$ being sequences having to go respectively to H and T in an appropriate way, $\{e_n\}$ being a sequence of $\mathcal{H}$ taking into account the errors due to numerical computation.

Theorem 4.1. *Assume that*

(i) $0 < \lambda_n,\ \forall\, n \in \mathbb{N}^*,$ *with* $\lim\limits_{n\to+\infty} \lambda_n = +\infty,$

(ii) $0 < \vartheta_n,\ \forall\, n \in \mathbb{N}^*,$ *with* $\lim\limits_{n\to+\infty} \vartheta_n = \vartheta,\ \vartheta \in]0, 2[,$

(iii) $T_n \overset{G}{\to} T,$

(iv) *the sequence* $\{z_n\}$ *generated by the rule* (PGTR) *is bounded,*

(v) $\lim\limits_{n\to+\infty} \|e_n\| = 0,$

(vi) $H_n \overset{u}{\to} H.$

Then, every weak cluster point z^* *of the sequence* $\{z_n\}$ *is such that* $\dfrac{z^*}{\vartheta}$ *is a solution of* (P)*.*

Proof. By setting

$$u_n = S_{\lambda_n,\vartheta_n}^{H_n^{-1}T_n} 0,\ \forall\, n \in \mathbb{N}^*,$$

we may write

$$z_n = u_n + e_n,\ \forall\, n \in \mathbb{N}^*.$$

From hypothesis (v), each weak cluster point of $\{z_n\}$ is a weak cluster point of $\{u_n\}$ and conversely.

Moreover, thanks to hypothesis (iv), $\{z_n\}$ has at least one weak cluster point.

Let thus $z^* \in \mathcal{H}$, be such a point and $\{u_{n_k}\}$ be a subsequence of $\{u_n\}$ weakly convergent to z^*.

The conditions imposed on $\{\vartheta_n\}$ and $\{\lambda_n\}$ imply, on the one hand,

$$\frac{u_{n_k}}{\vartheta_{n_k}} \overset{w}{\to} \frac{z^*}{\vartheta} \tag{4.1}$$

and, on the other hand,

$$\frac{u_{n_k}}{\lambda_{n_k}\vartheta_{n_k}} \overset{s}{\to} 0.$$

Therefore, we get, by using hypothesis (vi),

$$H_{n_k}\frac{u_{n_k}}{\lambda_{n_k}\vartheta_{n_k}} \overset{s}{\to} 0. \tag{4.2}$$

The definition of the sequence $\{u_n\}$ implying

$$-H_{n_k}\frac{u_{n_k}}{\lambda_{n_k}\vartheta_{n_k}} \in T_{n_k}\frac{u_{n_k}}{\vartheta_{n_k}}, \quad \forall\, k \in \mathbb{N}^*,$$

we deduce, from relations (4.1) and (4.2), hypothesis (iii) and the closedness of the graph of T in $\mathcal{H}_w \times \mathcal{H}_s$, that

$$0 \in T\frac{z^*}{\vartheta}. \qquad \square$$

In view of establishing a result of strong convergence for the sequence generated by the perturbed generalized Tikhonov's algorithm we will now first study the distance between the corresponding iterates of rules (BGTR) and (PGTR).

Proposition 4.2. *Assume that* $\{x_n\}$ *is generated by* (BGTR) *and* $\{z_n\}$ *is generated by* (PGTR). *If*

(i) $0 < \underline{\lambda} \le \lambda_n$, $\forall\, n \in \mathbb{N}^*$,

(ii) $0 < \vartheta_n$, $\forall\, n \in \mathbb{N}^*$,

(iii) $H_n \xrightarrow{u} H$,

(iv) $\|I - H\| < 1$,

then, there are strictly positive constants ε, C *et* ρ^* *such that*

$$\|x_n - z_n\| \le \frac{\vartheta_n}{\varepsilon}\left[C\|H_n - H\| + \frac{\lambda_n}{\underline{\lambda}}\delta_{\underline{\lambda},\rho^*}(T_n, T)\right] + \|e_n\|, \; \forall\, n \in \mathbb{N}^*.$$

Proof. The definitions of the two sequences $\{x_n\}$ et $\{z_n\}$ imply

$$\begin{aligned}\|x_n - z_n\| &\le \vartheta_n \left\| J_{\lambda_n}^{H^{-1}T}0 - J_{\lambda_n}^{H_n^{-1}T_n}0 \right\| + \|e_n\| \\ &\le \vartheta_n \delta_{\lambda_n,0}\left(H_n^{-1}T_n, H^{-1}T\right) + \|e_n\|, \; \forall\, n \in \mathbb{N}^*,\end{aligned}$$

and Proposition 2.2 leads to the conclusion. $\square$

Proposition 4.2, Theorem 3.1 and Remark 3.2 imply the next results.

Theorem 4.3. *Assume that problem* (P) *has at least one solution and*

(i) $0 < \lambda_n$, $\forall\, n \in \mathbb{N}^*$, *with* $\lim\limits_{n\to+\infty} \lambda_n = +\infty$,

(ii) $0 < \vartheta_n$, $\forall\, n \in \mathbb{N}^*$, *with* $\lim\limits_{n\to+\infty} \vartheta_n = \vartheta$, $\vartheta \in]0, 2[$,

(iii) *there is* $\underline{\lambda} > 0$ *such that* $\lim_{n\to+\infty} \lambda_n \delta_{\underline{\lambda},\rho}(T_n, T) = 0,\ \forall\, \rho \geq 0,$

(iv) $\lim_{n\to+\infty} \|e_n\| = 0,$

(v) $H_n \xrightarrow{u} H,$

(vi) $\|I - H\| < 1.$

Then, the sequence $\{z_n\}$ *generated by* (PGTR) *strongly converges to* $\vartheta\overline{x}$ *where* $\overline{x}$ *is the solution of* (P) *with minimum* H*-norm.*

Remark 4.4 Hypothesis (i) to (vi) of Theorem 4.3 imply that problem (P) has at least one solution if and only if the sequence $\{z_n\}$ generated by (PGTR) is bounded.

Remark 4.5. The adaptation of the previous results to the *nonperturbed* context is immediate : it suffices to replace everywhere T^n $(n \in \mathbb{N}^*)$ by T and to note that

$$\delta_{\lambda,\rho}(T, T) = 0,\ \forall\, \lambda > 0,\ \forall\, \rho \geq 0,$$

what implies, in particular,

$$T \xrightarrow{G} T.$$

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P. Alexandre
Université de Liége
F.E.G.S.S.
Boulevard du Rectorat, 7
4000 Liège
Belgique
`e-mail: P.Alexandre@ulg.ac.be`

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