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ON THE BRILL–NOETHER THEORY OF SPANNED VECTOR BUNDLES ON SMOOTH CURVES

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ABSTRACT. Here we study the integers (d, g, r) such that on a smooth projective curve of genus g there exists a rank r stable vector bundle with degree d and spanned by its global sections.

Introduction. Let X be a smooth projective curve of genus $g \geq 2$. The first step of the usual Brill–Noether theory for rank r stable vector bundles on X is the description of the possible pairs $(\deg(E), h^0(X, E))$ with E rank r stable vector bundle. The first aim of this paper is to show that if $r \geq 2$ there is a huge difference (even for curves with general moduli) between this Brill–Noether theory and the Brill–Noether theory of stable and spanned vector bundles. The second aim of this paper is to show that a natural filtration introduced in [3] and used heavily in [6] to obtain Clifford's type theorems for rank r vector bundles, $2 \leq r \leq 5$, is quite relevant to our problem: the possible pairs $(\deg(E), h^0(X, E))$ depend on this filtration even for a curve with general moduli and the geometry of the corresponding map to the Grassmannian $G(r, v)$, $v = h^0(X, E)$, depends on this filtration (see Definition 1.2). A key point is a construction of spanned

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bundles which we make in Section 1. In Section 2 we consider the case in which X is a k-gonal curve. We stress the main difference between the results of this section and [4, Th. 4.2]. In [4] we proved the existence of a rank r generically spanned vector bundle E with many sections and with a certain degree. This result is obviously significant for the Brill–Noether theory of stable vector bundles. Furthermore, by the construction of such bundle given in the proof of Th. 4.2 [4], such bundle is generically spanned and completely weakly filtrable in the sense of [4] and [6] (see Definition 1.2). However, the subsheaf of such bundle E spanned by $H^0(X, E)$ is isomorphic to a direct sum of line bundles (usually $(L^{\otimes t})^{\oplus r}$ or $(L^{\otimes t})^{\oplus a} \oplus (L^{\otimes (t+1)})^{\oplus (r-a)})$) and in particular it is not stable. We will see that this restriction on the examples we found in [4, proof of Th. 4.2], is not due to the proof given in [4], but to the nature of the problem (see Proposition 2.4 and Remark 2.5). In Section 3 we consider stable weakly filtrable (in the sense of Definition 1.2) rank 2 spanned vector bundles, while in Section 4 we consider 2-generic (see Definition 1.3) rank 2 stable vector bundles on curves with general moduli (see Theorem 4.1) or on the general curve with gonality t, where t is any fixed integer at least 7 (see Proposition 4.5). We believe that the range covered by Theorem 4.1 (i.e. the case $\mathrm{rank}(E) = 2$, $h^0(X, E) = 4$) is the only case in which asymptotically there is no difference for the degrees of stable weakly filtrable and non weakly filtrable spanned vector bundles (see Remark 4.4), but we do not know how to prove this feeling.

1. A general construction. In this paper we will not distinguish between vector bundles and locally free sheaves. We work over an algebraically closed base field $\mathbf{K}$. In this section we make no restriction on $\mathrm{char}(\mathbf{K})$. In this section we will make the following construction of spanned vector bundles on a smooth projective curve X.

Construction 1.1. We fix a vector bundle A on X and a rank t vector bundle M on X which is spanned by its global sections. We fix an integer x with $t + 1 \leq x \leq h^0(X, M)$. We are looking for exact sequences on X:

$$0 \to A \to E \to M \to 0 \tag{1}$$

with $h^0(X, E) = h^0(X, A) + x$ and such that the image of $H^0(X, E)$ into $H^0(X, M)$ is a x-dimensional subspace of $H^0(X, M)$ spanning M. Notice that if A is spanned, then any such E is spanned. Here we will construct ALL such bundles E. Fix a linear subspace $W \subseteq H^0(X, M)$ with $\dim(W) = x$ and W spanning M; if one is interested in generically spanned bundles, one has just to assume A and M generically spanned, take W such that the subsheaf M' of M spanned by W has rank t and then consider M' instead of M in the construction below. We want to find ALL exact sequences (1) such that the image of the map $H^0(X,E) \to H^0(X,M)$ contains W. In particular we will obtain $h^0(X,E) = h^0(X,A) + x$. Consider the Euler's sequence on the Grassmannnian $G(t,x)$ seen as Grassmannian of t-dimensional quotient spaces of W:

$$0 \to S \to W \otimes \boldsymbol{O}_{G(t,x)} \to Q \to 0. \tag{2}$$

Since W spans M, the universal property of the Grassmannian $G(t,x)$ gives a morphism $f : X \to G(t,x)$ such that $M \cong f^*(Q)$. Set $U := f^*(S)$. Hence U is a vector bundle with $\mathrm{rank}(U) = x - t$, $\deg(U) = -\deg(M)$ and U^* is spanned by W^*. Thus if M has no trivial factor, we have $h^0(X,U) \geq x$. The case $x = t+1$ is very nice because in this case we have $U^* \cong \det(M)$. Fix $j \in H^0(X, Hom(U,A))$. The map j induces a map $u(j) : U \to A \oplus \boldsymbol{O}_X{}^{\oplus x}$. Recall that a subsheaf T' of a locally free sheaf T on X is said to be saturated in T/T' has no torsion, i.e. if either $T' = T$ or T/T' is locally free. Since W spans M, U is a saturated subsheaf of $W \otimes \boldsymbol{O}_X$ and $u(j)(U)$ is saturated in $A \oplus \boldsymbol{O}_X{}^{\oplus x}$. Hence $\mathrm{Coker}(u(j))$ is a vector bundle. Set $E := \mathrm{Coker}(u(j))$. We have $\mathrm{rank}(E) = \mathrm{rank}(A) + \mathrm{rank}(M)$. By construction E has A as a saturated subbundle. By construction E fits in an exact sequence (1). By construction the x chosen spanning sections of M are lifted to E. Now we check that this construction gives all such bundles. Take E fitting in (1) with $h^0(X,E) = h^0(X,A) + x$. Hence the image, W, of $H^0(X,E)$ into $H^0(X,M)$ has rank x; if E is spanned, then W spans M and E is a quotient of $A \oplus W$ with U as kernel, i.e. we have an exact sequence

$$0 \to U \to A \oplus W \to E \to 0 \tag{3}$$

in which the map $U \to W$ is induced by (2), while the map $U \to A$ obtained from (3) is our map j. Notice that (1) splits if and only if $j = 0$.

For reader's sake we repeat the following definitions introduced in [1] and [3] and used heavily in [6].

Definition 1.2. *Let E be a rank r generically spanned vector bundle on X. We will say that E is completely weakly filtrable if there is an increasing filtration $\{E_i\}_{0 \leq i \leq r}$ of E with $E_0 = \{0\}$, $E_r = E$, E_i saturated subbundle of E with* $\mathrm{rank}(E) = i$ *and $h^0(X, E_i/E_{i-1}) \geq 2$ for every $i \leq r$.*

Definition 1.3. *Let E be a rank r generically spanned vector bundle on X and $V \subseteq H^0(X,E)$. We will say that the pair (E,V) is r-generic if for every subsheaf F of E with* $\mathrm{rank}(F) < r$ *we have* $\dim(V \cap H^0(X,F)) \leq \mathrm{rank}(F)$. *We will say that E is r-generic if the pair $(E, H^0(X,E))$ is r-generic.*

Remark 1.4. Let X be a smooth projective curve, E a rank r vector bundle on X and $V \subseteq H^0(X,E)$ a linear subspace spanning E. Set $v := \dim(V)$.

See the Grassmannian $G(r, v)$ as embedded in $\mathbf{P}(\Lambda^r(V^*))$ by the Plücker embedding. Let $h_{V,E} : X \to G(r, v)$ be the morphism associated to the pair (E, V) by the universal property of the rank r quotient bundle of $G(r, v)$. The definition of r-genericity implies that the pair (E, V) is r-generic if and only if $h_{V,E}(X)$ is not contained in a certain Schubert cycle of $G(r, v)$. In particular if $h_{V,E}(X)$ spans $\mathbf{P}(\Lambda^r(V^*))$, then the pair (E, V) is r-generic. If $r = 2$ and $v = 4$ the spanned pair (E, V) is 2-generic if and only if $h_{V,E}(X)$ is not contained in a hyperplane, H, of $\mathbf{P}^5$ tangent to the quadric hypersurface $G(2, 4) \subset \mathbf{P}^5$.

Remark 1.5. Fix integers r, v with $v > r \geq 1$ and a smooth curve X of genus g. Then there are many rank r spanned vector bundles, E, on X with $h^0(X, E) \geq v$ and such that there is a linear space $V \subseteq H^0(X, E)$ with $\dim(V) = v$, V spanning E and such that the pair (E, V) is r-generic. Indeed it is very easy to construct spanned pairs (E, V) such that the associated morphism $h_{E,V} : X \to G(r, v) \subseteq \mathbf{P}(\Lambda^r(V^*))$ is an embedding and $h_{V,E}(X)$ spans $\mathbf{P}(\Lambda^r(V^*))$. By Remark 1.4 the pair (E, V) is r-generic. The non-trivial problem is to control the pair of integers $(\deg(E), h^0(X, E))$ and to show if (or when) E may have some stability property.

2. *k*-gonal curves. In this section we fix integers g, k, m with $k \geq 2$, $m \geq 2$, $g \geq 2km$. We fix a smooth k-gonal curve X such that there is $L \in \mathrm{Pic}^k(X)$ with $h^0(X, L) = 2$ such that $h^0(X, L^{\otimes t}) = t+1$ for every integer t with $1 \leq t \leq m$ and such that every base point free complete g^x_d with $d \leq km$ is associated to $L^{\otimes t}$ for some integer t (and hence $d = kt$ and $x = t$). If m is very small with respect to g/k, this is the case for a general k-gonal curve (see [7, Prop. 4.2], for the best possible result when $k = 4$ or 5). Call e the maximal integer $t > 0$ such that $h^0(X, L^{\otimes t}) = t+1$. We always have $e \leq [(g-1)/(k-1)]$ and $e = [(g-1)/(k-1)]$ if X is a general k-gonal curve (see [2] or, in characteristic 0, see [7, Prop. 1.1], for a much stronger statement). Call $h_L : X \to \mathbf{P}^1$ the degree k morphism induced by L. In this section we make no restriction on $\mathrm{char}(\mathbf{K})$.

Remark 2.1. Fix an integer $r \geq 2$ and a spanned weakly filtrable rank r vector bundle E on X. Let $\{E\}_{0 \leq i \leq r}$ be the corresponding filtration with $E_i/E_{i-1} \in \mathrm{Pic}(X)$ for $1 \leq i \leq r$ and $h^0(X, E_i/E_{i-1}) \geq 2$ for $1 \leq i \leq r$. It is obvious that $\deg(E) = \sum_{1 \leq i \leq r} \deg(E_i/E_{i-1}) \geq rk$. Now assume E stable. We have $\deg(E) \geq (r+1)k$ because E_i/E_{i-1} cannot have degree k for every integer i and hence there is an integer j with $\deg(E_j/E_{j-1}) \geq 2k$. Furthermore, if $\deg(E) = (r+1)k$, then $E_i/E_{i-1} \cong L$ except for one index, say j, and $E_j/E_{j-1} \cong L^{\otimes 2}$; by the stability of E we have $j > r/2$.

Remark 2.2. Fix an integer $r \geq 2$ and an integer $v < m$. Among the extensions of $L^{\otimes v}$ by $L^{\oplus(r-1)}$ there are the following ones which we call "coming trivially from the pencil". Consider all vector bundles, F, on $\mathbf{P}^1$ which are extensions of the degree v line bundle by the direct sum of $r-1$ line bundles of degree 1. Let $a_1 \geq \cdots \geq a_r$ be the splitting type of F. It is known that every sequence $(a_1, \ldots, a_r)$ with $a_1 \geq \cdots \geq a_r$ and $\sum_{1 \leq i \leq r} a_i = v + r - 1$ is the splitting type of at least one such extension. Every bundle $h_L^*(F)$ is the direct sum of r line bundles each of them isomorphic to suitable powers of L and we will say that the extension giving $h_L^*(F)$ comes from the pencil. No bundle coming from the pencil is stable and there is a semistable bundle coming from the pencil if and only if r divides $v + r - 1$, i.e. if and only if kr divides the degree of the bundles we are interested in. If this condition is satisfied, by the openness of stability we may at least say that the general extension of $L^{\otimes v}$ by $L^{\oplus(r-1)}$ is semistable. To prove the stability of the general extension, E, of $L^{\otimes v}$ by $L^{\oplus(r-1)}$ making numerical computations, this is an extremely useful information because allows one to work by contradiction assuming the existence of a proper subbundle, T, of E with $\mu(T) = \mu(E)$, i.e. with a prescribed slope. If this condition is not satisfied one can apply this remark for different degrees d divisible by k looking at extensions, say, of $L^{\otimes v}$ by $A := L^{\oplus a} \oplus (L^{\otimes 2})^{\oplus(r-1-a)}$ with $0 \leq a < r$ and $2r - 2 - a + v$ divisible by r. And many other alternatives are possible taking a different bundle A. All the extensions which come from the pencils are spanned. Hence by the openness of semistability we may apply the same remark to our search of extensions with spanned middle term, E. More precisely, we may fix an integer y with $2 \leq y \leq v + 1$ and use the Construction 1.1 using a spanning vector space $V \subseteq H^0(X, L^{\otimes v})$ with $\dim(V) = y$ and obtain E with $h^0(X, E) \geq h^0(X, L^{\oplus(r-1)}) + y = 2r - 2 + y$.

Remark 2.3. The method of Remark 2.2 may give non-existence results for stable spanned weakly filtrable bundles with certain numerical invariants for the following reason. Since on $\mathbf{P}^1$ there is no stable bundle of rank $r \geq 2$, it is sufficient to show that every such spanned weakly filtrable bundle is the pull-back by h_L of a bundle on $\mathbf{P}^1$. Suppose that we are looking only at rank r bundles given by extensions of $L^{\otimes v}$ by a rank $r-1$ bundle A coming from $\mathbf{P}^1$, say $A = h_L^*(B)$. The essential point is that the construction 1.1 gives ALL such bundles. Fix an integer y with $2 \leq y \leq v + 1$ and a vector space $V \subseteq H^0(X, L^{\otimes v})$ spanning $L^{\otimes v}$. Since $h^0(\mathbf{P}^1, \boldsymbol{O}_{\mathbf{P}^1}(v)) = v + 1$, there is a vector space $W \subseteq H^0(\mathbf{P}^1, \boldsymbol{O}_{\mathbf{P}^1}(v))$ with $\dim(W) = y$ and $h_L^*(W) = V$. We apply the Construction 1.1 on $\mathbf{P}^1$ to B and W and on X to A and V; call $T\Pi(-1)$ and $T\Pi'(-1)$ the corresponding universal quotient rank $y-1$ vector bundles on $\Pi := \mathbf{P}(W)$ and $\Pi' := \mathbf{P}(V)$. Since (A, V) comes from (B, W), we have $T\Pi'(-1) \cong h_L^*(T\Pi(-1))$. Let $a_1 \geq \cdots \geq a_{y-1}$ be the splitting type of $T\Pi(-1)$. Since $\Omega_\Pi(2)$ is spanned by its global sections, we have $a_1 \leq v$. Hence if $h^0(X, A \otimes L^{\otimes v}) = h^0(\mathbf{P}^1, B(v))$, every such spanned E

with $h^0(X,E) = h^0(X,A) + y$ comes from $\mathbf{P}^1$.

Applying Remarks 2.2 and 2.3 and the definition of e to the bundle $A := L^{\oplus a} \oplus (L^{\otimes 2})^{\oplus(r-1-a)}$ considered in Remark 2.2 we obtain the following result.

Proposition 2.4. *Fix integers r, a and v with $r \geq 2$, $v \geq 2$, $0 \leq a < r$ and $v + 1 + \min\{1, r-1-a\} \leq e$. Then every spanned extension of $L^{\otimes v}$ by $L^{\oplus a} \oplus (L^{\otimes 2})^{\oplus(r-1-a)}$ is the direct sum of line bundles isomorphic to tensor powers of L. In particular no such bundle is stable. There is a semistable bundle of that type if and only if $2r - 2 - a + v$ is divisible by r.*

Remark 2.5. In the range of ranks and degrees covered by Proposition 2.4 (and for many more degrees not divisible by k) there exist stable generically spanned weakly filtrable bundles [4, Th. 4.2]. In every example, E, constructed in the proof of [4], Th. 4.2, the subsheaf of E generated by $H^0(X,E)$ is the direct sum of r line bundles coming from $\mathbf{P}^1$. Proposition 2.4 shows that this is not due to the proof but to the nature of the problem. If $v + r \leq m$ we obtain a range in which there is no other way to obtain spanned weakly filtrable bundles. Hence in this range there is no such stable weakly filtrable spanned bundle. For non-weakly filtrable stable rank 2 bundles, see Proposition 4.5.

3. Rank 2 weakly filtrable bundles. Here we apply the Construction 1.1 to the case $\mathrm{rank}(A) = \mathrm{rank}(M) = 1$ and $x = 2$. Hence $U \cong M^*$. We will always assume A spanned. Hence any such E is spanned and we have $\mathrm{rank}(E) = 2$, $\deg(E) = \deg(A) + \deg(M)$ and $h^0(X,E) \geq h^0(X,A) + 2$. Furthermore, $h^0(X,E) = h^0(X,A) + 2$ if $h^0(X,M) = 2$. In this section we do not make any restriction on $\mathrm{char}(\mathbf{K})$. We fix an integer $g \geq 2$. Let $\rho(g,x,d) := g - (x+1)(g + x - d)$ be the Brill–Noether number. For every integer $s \geq 1$ let $d\{s\}$ be the minimal integer d with $\rho(g,s,d) \geq 0$, i.e. with $g - (s+1)(g+s-d) \geq 0$. In particular we have $d\{1\} := [(g+3)/2]$ and $d\{2\} = 2 + [(2g+2)/3]$.

(3.1) Here we discuss the stability of a rank 2 vector bundle E fitting in an extension (1) and given by the Construction 1.1. If $\deg(A) > \deg(M)$, then E is always unstable. If $\deg(A) = \deg(M)$, then E is always semistable but not stable. If $\deg(M) > \deg(A)$ we will see in 3.4 and 3.6 a few cases in which E is stable. Only the case $\deg(M) = \deg(A) + 1$ is trivial by the following obvious remark.

Remark 3.2. Assume $\mathrm{rank}(A) = \mathrm{rank}(M) = 1$ and $\deg(M) = \deg(A) + 1$. A bundle E fitting in an exact sequence (1) is stable if and only if the extension (1) is not the splitted extension.

Definition 3.3. *Let X be a smooth projective curve. For every integer $t \geq 2$ let $X[\{t\}]$ be the first integer x such that there exists $L \in \mathrm{Pic}^x(X)$ with*

$h^0(X,L) \geq t$.

If X has general moduli we have $X[\{t\}] = d\{t-1\}$ for every integer $t \geq 2$.

Proposition 3.4. *Let E be a rank 2 vector bundle on X fitting in a non-splitted exact sequence (1) with A and M spanned, $h^0(X,M) = 2$ and $h^0(X,E) = h^0(X,A) + 2 \geq 4$. Assume $\deg(A) < \deg(M) < X[\{h^0(X,A)+2\}]$ and $\deg(A) + \deg(M) < 2(X[\{h^0(X,A)+1\}])$. Then E is stable. Assume also $\deg(M) < X[\{h^0(X,A)+1\}]$; then A is the unique rank 1 subsheaf of E with maximal degree.*

Proof. By assumption we have $h^0(X,E) = h^0(X,A) + h^0(X,M)$ and E is spanned. Let D be a maximal degree rank 1 subsheaf of E. Since $\deg(D)$ is maximal, D is saturated in E and hence $E/D \in \mathrm{Pic}(X)$. Since $\deg(A) > 0$, we have $\deg(D) > 0$. Since $\deg\,(E/D) \leq \deg(M) < X[\{h^0(X,A)+2\}]$, we have $h^0(X,E/D) < h^0(X,E)$. Hence $h^0(X,D) \neq 0$. Take $s \in H^0(X,D) \subseteq H^0(X,E)$ with $s \neq 0$. D is the saturation of $s(\boldsymbol{O}_X)$ in E. If $D = A$, then E is stable because $\deg(A) < \deg(M)$. Assume $D \neq A$ as subsheaves of E; we allow the case in which A and D are isomorphic. Then there is an inclusion of $A \oplus D$ in E (as subsheaf!). First assume $h^0(X,D) \geq 2$. Since $h^0(A \oplus D) \geq h^0(X,A) + 2 = h^0(X,E)$ and E is spanned, we have $E = A \oplus D$, i.e. (1) splits, contradiction. Hence $h^0(X,D) = 1$. Thus $h^0(X,E/D) \geq h^0(X,A) + 1$. Since $(\deg(A) + \deg(M))/2 < X[\{h^0(X,A) + 1\}]$, we obtain $\deg(E/D) > \deg(E)/2$, i.e. we obtain the stability of E. If $\deg(M) < X[\{h^0(X,A)+1\}]$, the inequality $\deg(E/D) \leq \deg(M)$ gives that the case $D \neq A$ is impossible. Hence A is the unique rank 1 subsheaf of E with degree at least $\deg(A)$. $\square$

Definition 3.5. *Let X be a curve of genus $g \geq 3$. Fix integers d, r with $r > 0$ and $d > 0$. We will say that X is general from the point of view of $g^r_d{}'s$ if the set $G^r_d(X)''$ of all base point free $g^r_d{}'s$ on X has the same dimension as for a curve with general moduli, i.e. $G^r_d(X)'' \neq \emptyset$ if and only if $\rho(g,r,d) \geq 0$ and $\dim(G^r_d(X)'') = \rho(g,r,d)$ if $\rho(g,r,d) \geq 0$ and $d < g + r$. We will say that a smooth curve X of genus $g \geq 3$ is general from the point of view of pencils if the following conditions are satisfied:*

(*i*) *$G^1_d(X) \neq \emptyset$ if and only if $2d \geq g+1$;*

(*ii*) *$\dim(G^1_d(X)) = 2d - g - 1$ for every integer d with $g + 1 \leq 2d \leq 2g$.*

Notice that if, for some fixed r and d, the curve X is general from the point of view of $g^r_d{}'s$, then there are appropriate ranges of pairs (u,v) and (a,b) such that X has base point free complete $g^u_v{}'s$ and no g^a_b.

Using this definition we obtain at once the following particular case of Proposition 3.4.

Corollary 3.6. *Let X be a smooth curve of genus g which is general from the point of view of $g_y^x{}'s$ for every $x \le 3$. Let E be a rank 2 vector bundle on X fitting in a non-splitted exact sequence* (1) *with A and M spanned, $h^0(X,A) = h^0(X,M) = 2$ and $h^0(X,E) = 4$. Assume* $\deg(A) < \deg(M) < (3g+12)/4$ *and* $\deg(A)+\deg(M) < 4g/3+4$. *Then E is stable. If we have also* $\deg(M) < 2+2g/3$, *then A is the unique rank* 1 *subsheaf of E with maximal degree.*

Remark 3.7. Let X be a smooth genus g curve which is general from the point of view of pencils. Since for every integer $d \ge g/2+1$ we have $\rho(g,1,d+1) - \rho(g,1,d) = 2 > 1 = \dim(X)$, for every integer $d \ge g/2+1$ there is a base point free g_d^1 on X. Furthermore, if $d \le g+1$ there is a base point free complete g_d^1 on X.

In the first step of the proof of Theorem 4.1 we will use the following result.

(3.8) Let X be a general curve of genus g, $3 \le g \le 8$. We want to find a spanned and stable rank 2 vector bundle E fitting in (1) and with as degree any integer d with $g+5 \le d \le 2g+10$. Set $a := \deg(A)$ and $m := \deg(M)$. If d is odd, take $a = (d-1)/2$ and $m = (d+1)/2$; it works because $a \ge d\{1\}$ and we may apply Remark 3.2; if $(d+1)/2 \le g+1$ (resp. $(d-1)/2 \le g+1$), i.e. if $d \le 2g-1$ (resp. $d \le 2g+1$), we may take M (resp. A) with $h^0(X,M) = 2$ (resp. $h^0(X,A) = 2$); hence if $d \le 2g-1$ or $d = 2g+1$ we find a spanned stable E fitting in (1) and with $h^0(X,E) = 4$. If d is even, take $a = d/2-1$ and $m = d/2+1$; if $m \le g+1$, i.e. if $d \le 2g$ and $(g,d) \ne (6,12), (7,14)$ or $(8,16)$, it works because we have $a \ge d\{1\}$, while we may take M with $h^0(X,M) = 2$ and hence we may apply Corollary 3.6. Now assume $(g,d) = (6,12)$, $(7,14)$ or $(8,16)$; take $(a,m) = (g-1, g+1)$, $A \in \mathrm{Pic}^a(X)$ and $M \in \mathrm{Pic}^m(X)$ with $h^0(X,A) = h^0(X,M) = 2$ and both A and M spanned; since $h^0(X, A \otimes M) = d+1-g$, Construction 1.1 gives a spanned vector bundle E fitting in a non-trivial extension (1); take A such that $h^0(X,A(P)) = 2$ for every $P \in X$, i.e. such that for every $P \in X$ the line bundle $A(P)$ is not spanned; hence there is no P such that $A(P)$ is a quotient of E, i.e. such that E is an extension of $A(P)$ by $M(-P)$; it is easy to check that every such E is stable; alternatively, use that for every $P \in X$ we have $\dim(\mathrm{Ext}^1(X; A(P), M(-P))) + \dim(X) = \dim(\mathrm{Ext}^1(X;A,M)) - 2 + 1 < \dim(\mathrm{Ext}^1(X;A,M))$ since $A(P)$ and $M(-P)$ are not isomorphic because $h^0(X,A(P)) \ge 2 > 1 = h^0(X, M(-P))$. Now assume $2g+2 \le d \le 2g+10$. If d is odd we take $a = (d-1)/2$, $m = (d+1)/2$, A and M not special and take as E any bundle fitting in a non-splitted extension (1). E is stable by Remark 3.2. The vector bundle E is spanned because A and M are spanned and $h^1(X,A) = 0$. We have $h^1(X,E) = 0$ and hence $h^0(X,E) = d+2-2g$. Take a general linear subspace V of $H^0(X,E)$ such that $\dim(V) = 4$ and $\dim(V \cap H^0(X,A)) = 2$. We obtain a spanned pair (E, V) mapping X into $G(2,4)$. Since $\dim(V \cap H^0(X, A)) = 2$, the spanned pair (E, V) is not 2-generic. This part of the proof works verbatim even for every genus $g \geq 3$. Now assume d even. We make the same construction with respect to the integers $a := d/2 - 1$ and $m := d/2 + 1$. We need to prove that we may obtain in this way a stable bundle E. Assume the existence of a saturated rank 1 subbundle R of E with $\deg(R) \geq d/2$. Since the extension (1) does not split, we have $\deg(R) = d/2$ and hence $E/R \in \mathrm{Pic}^{d/2}(X)$. By Riemann–Roch and Serre duality we have $\dim(\mathrm{Ext}^1(M, A)) = g + 1$, $\dim(\mathrm{Ext}^1(E/R, R)) = g - 1$ if E/R and R are not isomorphic and $\dim(\mathrm{Ext}^1(E/R, R)) = g$ if $E/R \cong R$. We vary $A \in \mathrm{Pic}^{(d/2-1)}(X)$ and M in $\mathrm{Pic}^{(d/2+1)}(X)$. We obtain that a general extension (1) cannot be associated to any such R unless the general extension (1) contains also a two-dimensional family of non-isomorphic saturated rank 1 subbundles, A', of degree $d/2 - 1$. Since E is extension of E/R by R, each such A' induces a subsheaf of E/R with $\deg(A') = \deg(E/R) - 1$. Since any such sheaf is of the form $E/R(-P)$ for some $P \in X$ and $\dim(X) = 1$, this is impossible.

Remark 3.9. Let X be a general curve of genus $g \geq 3$ and E be one of the spanned bundles on X constructed in 3.8 with $d := \deg(E) \geq 2g + 4$ but no other restriction on d and g. We assume $\deg(M) > \deg(A) \geq g + 1$ and $h^1(X, A) = h^1(X, M) = 0$. In particular we have $h^1(X, E) = 0$ and $h^0(X, E) = d + 2 - 2g \geq 7$. If $\deg(M) = \deg(A) + 1$, the bundle E is stable by Remark 3.2. If $\deg(M) - \deg(A)$ is small, then usually the proof of 3.8 gives that a general such E is stable; this is true if $\deg(M) = \deg(A) + 2$. Let $V \subseteq H^0(X, E)$ be a general linear subspace with $\dim(V) = 4$. Hence V spans E. For general V we have $\dim(V \cap H^0(X, A)) \leq 1$; for general V we have $\dim(V \cap H^0(X, A_i)) \leq 1$ for any finite family $\{A_i\}_{i \in I}$ of rank 1 subbundles of E with $\deg(A_i) = (d-1)/2$ for every i. E has only finitely many rank 1 subbundles of degree $(d-1)/2$ ([12, Prop. 4.2]). Hence for d odd the general pair (E, V) is 2-generic. If d is even, we get the same result since there is at most a one-dimensional family, $\mathbf{T}$, of maximal degree line subbundles of E ([12, Cor. 4.6]) and for general V we have $\dim(V \cap H^0(X, A)) \leq 1$ for every $A \in \mathbf{T}$. For large d it would be possible to obtain that the image of $\Lambda^2(V)$ in $H^0(X, \det(E))$ has dimension 6, i.e. that $h_{V,E}(X)$ is not contained in a hyperplane of $\mathbf{P}^5$. But we send the reader to Theorem 4.1 proved in a different way for such type of assertions.

Remark 3.10. Let X be a curve of genus g which is general from the point of view of pencils. By Remark 3.2 for every odd integer $d \geq 2[(g+3)/2]$ there is a stable, spanned and weakly filtrable rank 2 vector bundle E on X with $h^0(X, E) \geq 2$: just take A and M spanned with $\deg(A) = (d-1)/2 \geq d\{1\}$ and $\deg(M) = (d+1)/2$.

4. Curves with general moduli. Here we prove the existence of 2-generic spanned pairs (E, V) with E rank 2 stable vector bundle of low degree and $\dim(V) = 4$ on a genus g curve with general moduli (see Theorem 4.1) or on a general curve with prescribed gonality, t, at least 7 (see Proposition 4.5). In this section we assume $\mathrm{char}(\mathbf{K}) = 0$. These results clarify completely the picture in the case of invariants $(\mathrm{rank}(E), \dim(V)) = (2, 4)$ for 2-generic spanned pairs. Hence they fit very well after the corresponding study for non 2-generic rank 2 vector bundles made in Section 3. The following result covers the corresponding case of [16, Th. 2], for the case g odd not considered there.

Theorem 4.1. *Fix an integer $g \geq 3$. Let X be a general curve of genus g. Then for every integer $d \geq g + 21$ there is a spanned pair (E, V) on X with* $\mathrm{rank}(E) = 2$, $\deg(E) = d$, *E stable,* $\dim(V) = 4$ *and such that the induced morphism $h_{V,E} : X \to G(2, 4) \subset \mathbf{P}^5$ is an embedding with $h_{V,E}(X)$ spanning $\mathbf{P}^5$.*

Proof. Call Q the universal rank 2 quotient bundle of $G(2, 4)$. We stress that the non-degenericity condition for $h_{V,E}(X) \subset \mathbf{P}^5$ is obvious when $h_{V,E}(X)$ is obtained as a general smoothing inside $G(2, 4)$ of a reducible curve $A = Z \cup D \subset G(2, 4)$ with A spanning $\mathbf{P}^5$. Notice that such curve may span $\mathbf{P}^5$ even if neither Z nor D span $\mathbf{P}^5$ and even if both $Q|Z$ and $Q|D$ are not induced by a 2-generic spanned pair. We divide the proof into 3 steps.

Step 1. Here we fix a pair of integers (d', g') with $3 \leq g' \leq 8$ and $2g' + 5 \leq d' \leq 2g' + 10$. We want to prove the existence of a non-degenerate smooth curve $X \subset G(2, 4)$ with $p_a(X) = g'$ and $\deg(X) = d'$. First assume d' odd. We take a smooth rational normal curve $Y \subset G(2, 4) \subset \mathbf{P}^5$. Hence $p_a(Y) = 0$ and $\deg(Y) = 5$. Let $T \subset G(2, 4)$ be the union of Y, g' general smooth conics intersecting quasi-transversally Y at two points and $(d' - 5 - 2g')/2$ general smooth conics intersecting Y at one point. If d' is even, instead of T we take the union of Y, g' general smooth conics intersecting quasi-transversally Y at two points, $(d' - 4 - 2g')/2$ general conics intersecting Y' at one point and a line intersecting Y at one point. Then we ask the reader to read the smoothing technique described in Steps 2 and 3 to obtain a smooth $X \subset G(2, 4)$; each time we add a conic intersecting Y at two points (resp. one point) instead of $P_1, \dots, P_6$ as in Step 3 we take P_1, P_2 (resp. P_1). We claim that for this set of integers the general such curve X has general moduli. As in the following steps our curve is built inductively in the following way. We have two smooth curves Z, $D \subset G(2, 4)$ intersecting quasi-transversally at $t \geq 1$ points. We assume D rational, $H^1(Z, TG(2, 4)|Z) = 0$ and that $TG(2, 4)|D$ is the direct sum of line bundles of degree at least $t - 1$ (see the structure of $TG(2, 4)|D$ for D the curves appearing in the claim in Steps 3). Hence

$H^1(D, (TG(2,4)|D)(-Z \cap D)) = 0$. From the Mayer–Vietoris exact sequence

$$(4)\quad 0 \to TG(2,4)|(Z \cup D) \to TG(2,4)|Z \oplus TG(2,4)|D \to TG(2,4)|(Z \cap D) \to 0$$

we obtain $H^1(Z \cup D, TG(2,4)|(Z \cup D)) = 0$. If X is a general smoothing of $Z \cup D$ we obtain $H^1(X, TG(2,4)|X) = 0$ by semicontinuity. Hence the coboundary map $H^0(X, N_{X,G(2,4)}) \to H^1(X, TX)$ is surjective. If X is smooth the deformation space of X is smooth and it has $H^1(X, TX)$ as tangent space. The Hilbert scheme $\mathrm{Hilb}(G(2,4))$ of $G(2,4)$ is smooth at the point parametrizing X because we have also $H^1(X, N_{X,G(2,4)}) = 0$. Since $H^0(X, N_{X,G(2,4)})$ is the tangent space at X to $\mathrm{Hilb}(G(2,4))$, the surjectivity of the coboundary map gives that near X $\mathrm{Hilb}(G(2,4)$ contains curves with general moduli. Thus we checked the claim. Now we claim that to obtain that for general X the associated rank 2 spanned vector bundle $Q|X$ is stable we may use 3.8 and 3.9. Indeed, we are considering bundles, E, on a smooth curve X with $h^1(X, E) = 0$. For such bundles $h^0(X, E)$ is constant for small deformations and hence the spannedness condition is open. Such bundles form an open subset of the deformation space of any bundle on X with the same property. Such bundles form a non-empty Zariski open subset of the corresponding moduli space; notice that this open set is irreducible and that every bundle on a smooth curve is the flat limit of a flat family of stable bundles ([13, Prop. 2.6], or [11, Cor. 2.2]). Since stability is an open condition, we obtain the last claim.

Step 2. Here we cover the cases $3 \le g \le 8$ and $d \ge 2g + 11$. Furthermore, for these pairs of integers (d, g) we will find an example $Z \subset G(2,4)$ with $h^1(Z, TG(2,4)|Z) = h^1(Z, N_{Z,G(2,4)}) = 0$, where $N_{Z,G(2,4)}$ is the normal bundle of Z in $G(2,4)$. We use induction on d assuming that we know the existence of a genus g smooth curve $Z \subset G(2,4)$ with $\deg(Z) = d - 6$ and with $Q|Z$ nice. The starting cases of the induction are the cases $2g + 5 \le d - 6 \le 2g + 10$ given in Step 1. Now we take a general $P \in Z$ and a general smooth rational curve $D \subset G(2,4)$ with $P \in D$ and $\deg(D) = 6$; we could take $\deg(D) = 5$ without any further modification. Now we ask the reader to copy the proof of the next step taking P instead of $P_1, \dots, P_7$; here the cohomological vanishing needed for the Mayer–Vietoris type proofs given in [10] and [15] and used in this step are much easier than the ones needed for the next step; see Step 1 for an example of Mayer–Vietoris exact sequence and an explanation of its use. We showed in Step 1 how to use the vanishing of $H^1(D, (TG(2,4)|D)(-Z \cap D))$ and semicontinuity to obtain the existence of curves, X, with general moduli and such that $Q|X$ is stable and 2-generic.

Step 3. Now we assume $g \ge 9$. Fix an integer $d \ge g + 9$. We assume the existence of a smooth curve $Z \subset G(2,4)$ with $\deg(Z) = d - 6$, $p_a(Z) = g - 6$, Z

with general moduli and such that $Q|Z$ is stable. We assume $h^1(Z, N_{Z,G(2,4)}) = 0$; we obtained this condition in the examples given in Step 2 and the proof below will show how to obtain inductively this condition. We assume that for any 7 general points $P_1, \dots, P_7$ of $G(2,4)$ there is a curve Z as above with $P_i \in Z$ for every i; the proof below will show how to obtain inductively this condition and the same inductive proof shows why we obtained this condition in the examples given in Step 2. Fix any 7 general points $P_1, \dots, P_7$ of Z; by the choice of Z we may assume that the points $P_1, \dots, P_7$ are general in $G(2,4)$.

Claim. *There is a smooth rational curve D of $\mathbf{P}^5$ with $P_i \in D$ for every i, $\deg(D) = 6$, $D \subset G(2,4)$ and such that $Q|D$ is the direct sum of 2 line bundles of degree 3.*

P r o o f o f t h e c l a i m. A general linear subspace $V \subset H^0(\mathbf{P}^1, \boldsymbol{O}_{\mathbf{P}^1}(3) \oplus \boldsymbol{O}_{\mathbf{P}^1}(3))$ with $\dim(V) = 4$ embeds $\mathbf{P}^1$ into $G(2,4)$ as a degree 6 curve Y. Hence $TG(2,4)|Y$ is the direct sum of 4 line bundles of degree 6 and $N_{Y,G(2,4)}$ is the direct sum of three line bundles of degree at least six. In particular for any 7 points $A_1, \dots, A_7$ of Y we have $H^1(Y, (TG(2,4)|Y)(-A_1 - \cdots - A_7)) = 0$ and $H^1(Y, N_{Y,G(2,4)}(-A_1 - \cdots - A_7)) = 0$. By a theorem of Kleppe (see [14, Th. 1.5]), for general $P_i \in G(2,4)$, $1 \le i \le 7$, there is a deformation, D, of Y, i.e. a degree 6 smooth rational curve D with $P_i \in D$ for every i. By semicontinuity we have $H^1(D, (TG(2,4)|D)(-P_1 - \cdots - P_7)) = 0$ and $H^1(D, N_{D,G(2,4)}(-P_1 - \cdots - P_7)) = 0$, $TG(2,4)|D$ is the direct sum of 4 line bundles of degree 6 and $N_{D,G(2,4)}$ is the direct sum of three line bundles of degree at least six. Taking sufficiently general the points P_i and Z we may assume that Z and D intersects quasi-transversally and exactly at the points P_i. Hence $A := Z \cup D$ is a degree d curve of $G(2,4)$ spanning $\mathbf{P}^5$ and with $p_a(A) = g$. By the general theory proved in [15, Lemma 5.1 and Th. 5.2], and independently in [10] we have $h^1(A, N_{A,G(2,4)}) = 0$ and the curve A is smoothable inside $G(2,4)$, i.e. it is a flat limit of smooth curves of $G(2,4)$. Call X the general member of this flat family degenerating to A. Hence X is a smooth curve of genus g and degree d. By semicontinuity we have $h^1(X, N_{X,G(2,4)}) = 0$; here we obtain again curves with non-special normal bundle to continue the inductive construction. Assume $h^1(Z, (TG(2,4)|Z)) = 0$. This condition was obtained in the examples given in Steps 1 and 2. As in Step 1, i.e. copying [15, Cor. 5.3] and proofs of 6.2 and 6.3, by a Mayer–Vietoris exact sequence we obtain $h^1(A, TG(2,4)|A) = 0$. Hence by semicontinuity we may assume $h^1(X, TG(2,4)|X) = 0$. Since $Q|Z$ is stable and $Q|D$ is semistable, $Q|(Z \cup D)$ is stable ([5, Lemma 1.1]). By [9, Th. 2.4], for general X the bundle $Q|X$ is stable. Consider the normal bundle exact sequence

$$0 \to TX \to TG(2,4)|X \to N_{X,G(2,4)} \to 0 \tag{5}$$

Since $h^1(X, TG(2,4)|X) = 0$, the coboundary map $H^0(X, N_{X,G(2,4)}) \to H^1(X, TX)$ is surjective. As in [15], 3.3 and 6.3, this means that the map, Ψ, from a neighborhood of the point representing X in the Hilbert scheme $\mathrm{Hilb}(G(2,4))$ into the moduli scheme M_g of smooth genus g curves is smooth and in particular it is open and dominant. Hence we may assume that X has general moduli and we conclude by the universal property of the Grassmannian $G(2,4)$. To conclude the proof we need the existence of the nice curve Z with invariants $(g-6, d-6)$. More precisely, it is sufficient to prove the existence of the nice curve Z with invariants $(g'-6, d'-6)$ with $9 \leq g' \leq 14$ and low d' and then use the proof of Step 2 to increase d without increasing g. If $g' = 9$ Step 1 covers the pairs $(g'-6, d'-6)$ with $11 \leq d' \leq 16$ and Step 2 all cases with $d' \geq 17$. Similarly, for $10 \leq g' \leq 14$; for more details, see the last part of the next remark.

Remark 4.2. The last few lines of the proof of 4.1 show that cover in that way a few cases not claimed by the statement of 4.1. If we drop the assertion "general moduli" we will show here how to modify Step 3 of the proof of 4.1 to obtain the existence of curves $X \subset G(2,4)$ with all the other results for all pairs (d,g) with $g \geq 3$ and $d \geq (4/5)g + 13$. We may assume $g \geq 9$. Fix an integer $d \geq (4/5)g + 13$ and assume the existence of a smooth curve $Z \subset G(2,4)$ with $\deg(Z) = d-4$, $p_a(Z) = g-5$, and such that $Q|Z$ is stable. We assume $h^1(Z, N_{Z,G(2,4)}) = 0$; we obtained this condition in the examples given in Step 2 of the proof of 4.1 and the proof below will show how to obtain inductively this condition. Fix any 6 general points $P_1, \dots, P_6$ of a general hyperplane section $H \cap Z$ of Z. Take a general linear subspace W of dimension 4 of $H^0(\mathbf{P}^1, \boldsymbol{O}_{\mathbf{P}^1}(2) \oplus \boldsymbol{O}_{\mathbf{P}^1}(2))$. Hence W spans $\boldsymbol{O}_{\mathbf{P}^1}(2) \oplus \boldsymbol{O}_{P^1}(2)$. We claim that we obtain in this way the existence of a smooth rational curve $D \subset H$ with $P_i \in D$ for every i, $\deg(D) = 4$ and $D \subset G(2,4) \cap H$. Indeed any two ordered sets, say $(P_1, \dots, P_6)$ and $(P'_1, \dots, P'_6)$, of $\mathbf{P}^4$ in linearly general position are projectively equivalent. Take a rational normal curve, D', of $\mathbf{P}^4$ and any 6 points of D' in linearly general position. Since D' is contained in a smooth quadric hypersurface, Q', of $\mathbf{P}^4$ and Q' is a hyperplane section of a smooth quadric hypersurface of $\mathbf{P}^5$, the claim follows. Alternatively, the claim may be proved as the claim in Step 3 of the proof of 4.1. We have $Q|D \cong \boldsymbol{O}_{\mathbf{P}^1}(2) \oplus \boldsymbol{O}_{\mathbf{P}^1}(2)$ because $G(2,4) \cap H$ is a smooth quadric. In particular $Q|D$ is semistable. Since $D \subset G(2,4) \cap H$, $N_{D,G(2,4)}$ has $\boldsymbol{O}_D(1)$ as a direct factor. $N_{D,G(2,4)}$ fits in an exact sequence

$$0 \to \boldsymbol{O}_{\mathbf{P}^1}(2) \to \boldsymbol{O}_{\mathbf{P}^1}(4)^{\oplus 4} \to N_{D,G(2,4)} \to 0. \tag{6}$$

We obtain easily that $N_{D,G(2,4)}$ is the direct sum of two line bundles of degree 7 and one line bundle of degree 4. Taking sufficiently general the points P_i and Z we may assume that Z and D intersect quasi-transversally and exactly at the points P_i. Hence $A := Z \cup D$ is a degree d curve of $G(2,4)$ spanning $\mathbf{P}^5$ and with $p_a(A) = g$. $N_{A,G(2,4)}|D$ is obtained from $N_{D,G(2,4)}$ making 6 positive elementary transformations supported by the points $P_1, \dots, P_6$ (see [10]). For general H and P_i we obtain that every direct factor of $N_{A,G(2,4)}|D$ has degree at least 5. By the general theory proved in [15, Lemma 5.1 and Th. 5.2], and independently in [10] we have $h^1(A, N_{A,G(2,4)}) = 0$ and the curve A is smoothable inside $G(2,4)$, i.e. it is a flat limit of smooth curves of $G(2,4)$. A general member, X, of a smoothing family of A gives a solution of our problem for the following reason. Since $Q|Z$ is stable and $Q|D$ is semistable, for general C the bundle $Q|C$ is stable ([5, Lemma 1.1] and [9, Th. 2.4]). To start the induction we need to prove the existence of the nice curve Z with invariants $(g-5, d-4)$. More precisely, it is sufficient to have the nice curve Z with invariants $(g-5, d'-4)$ for every integer d' with $[(4g+4)/5] + 13 \le d' \le [(4g+4)/4] + 16$ and then add smooth rational curves, say of degree 4 linked for a point repeating the construction with one point, P_1, instead of 6 points $P_1, \dots, P_6$. If $g = 9$ we need the existence of the curve Z with invariants $(4, d')$ with $16 \le d' \le 19$; this is proved in Step 1 and Step 2 which would cover also the cases $d' = 13, 14$ and 15 not claimed in the statement of 4.1. If $g = 10$ we need to cover the pairs $(5, d')$ with $17 \le d' \le 20$; this is done in Step 1. If $g = 11$ we need to cover the pairs $(6, d')$ with $18 \le d' \le 21$; Step 1 of the proof of 4.1 covers the cases $17 \le d' \le 21$; just to warn the reader that the statements of 4.1 and 4.2 are not sharp we remark that if $(g', d') = (6, 16)$ we may use 3.9. If $g = 12$ we need to cover the pairs $(7, d')$ with $19 \le d' \le 22$; these cases are covered in Step 1 of the proof 4.1. If $g = 13$ we need to cover the pairs $(8, d')$ with $20 \le d' \le 23$; these cases are covered in Step 1 of the proof of 4.1. If $g \ge 14$ we have $g - 5 \ge 9$ and, as explained before the starting cases for $g = 9$, at this point we have proved the existence of a nice curve Z with invariants $(g-5, d-4)$.

Remark 4.3. Consider the Euler's sequence on $G(2,4)$:

$$0 \to S \to \boldsymbol{O}_{G(2,4)}{}^{\oplus 4} \to Q \to 0 \tag{7}$$

with S rank 2 universal subbundle. We claim that the proof of Theorem 4.1 show the existence of smooth curves $X \subset G(2,4)$ of genus g with general moduli, with $\deg(Q|X) = d$ and such that both $Q|X$ and $S|X$ are stable, where S is the tautological rank 2 subbundle of $G(2,4)$, i.e. such that there is an exact sequence

$$0 \to F \to \boldsymbol{O}_X{}^{\oplus 4} \to E \to 0 \tag{8}$$

with $\deg(E) = d$, $\deg(F) = -d$ and both E and F stable. Let $D \subset G(2,4)$ be the smooth rational curve of degree 6 considered in Step 3 of the proof of Theorem 4.1. Consider the restriction to D of the Euler's exact sequence (7) on $G(2,4)$. Since $S|D$ is the direct sum of two line bundles of degree -3, it is semistable.

Now we follow the proof of 4.1. At each step in which we obtain a reducible curve, T, such that $Q|T$ is stable, we obtain the same curve T with both $Q|T$ and $S^*|T$ stable. Then we smooth the reducible curve T inside $G(2,4)$. By the openness of stability and semistability ([9, Th. 2.4]) we obtain a nearby smooth curve Y such that both $Q|Y$ and $S^*|T$ are stable, proving the claim. We may apply verbatim this observation in the set-up of Remark 4.2.

Remark 4.4. Let $X \subset G(2,4)$ be a smooth curve of genus g and degree d. We have $\chi(N_{X,G(2,4)}) = 4d + 1 - g$. Assume $h^1(X, N_{X,G(2,4)}) = 0$. Then the Hilbert scheme $\mathrm{Hilb}(G(2,4))$ of $G(2,4)$ is smooth of dimension $4d + 1 - g$ at X. If the unique irreducible component of $\mathrm{Hilb}(G(2,4))$ containing X contains curves with general moduli, we have $4d + 1 - g - \dim(\mathrm{Aut}(G(2,4)) \geq 3g - 3$. Hence the statement of Theorem 4.1 is quite good for embeddings with $h^1(X, N_{X,G(2,4)}) = 0$: d must be at least of order g. With the notations of 4.1, for large d step 3 of its proof give curves C and X and bundles F and E with $h^0(C, \boldsymbol{O}_C(1)) = h^0(X, \boldsymbol{O}_X(1)) = 6$, i.e with $h^0(C, \det(F)) = h^0(X, \det(E)) = 6$ and with $h^0(C, F) = h^0(X, E) = 4$.

Fix an integer t with $7 \leq t \leq g/2 + 1$. Look at the last few lines of the proof of 4.1. We obtain an open map Ψ. Hence if for the reducible curve A considered in the proof of 4.1 is in the closure in M_g^- of the irreducible variety of all t-gonal smooth genus g curves, we may take as X not a curve with general moduli, but a general t-gonal curve. Assume that the curve Z considered in the proof of 4.1 is t-gonal. Then the general theory of admissible coverings given in [8, §4], shows that A is in the closure in M_g^- of the irreducible variety of all t-gonal smooth genus g curves. Since the curves constructed in Steps 1 and 2 of the proof of 4.1 are t-gonal, we obtain the following result; to copy the proof of 4.1 we use the convention that if $t \geq [(g+3)/2]$ "general t-gonal curve of genus g" means "general curve of genus g".

Proposition 4.5. *Fix integers g, t, d with $g \geq 2$, $14 \leq 2t \leq g+3$ and $d \geq g+21$. Let X be a general t-gonal curve of genus g. There is a rank 2 stable vector bundle E on X with $\deg(E) = d$ and a subspace $V \subseteq H^0(X, E)$ with $\dim(V) = 4$ such that V spans E, the pair (E, V) is 2-generic, the associated morphism $h_{V,E} : X \to G(2,4) \subset \mathbf{P}^5$ is an embedding and $h_{V,E}(X)$ is not contained in a hyperplane.*

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