Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **25** (1999), 141-176

Serdica
Mathematical Journal

Institute of Mathematics
and Informatics, Bulgarian
Academy of Sciences

ON THE MAXIMUM OF A BRANCHING PROCESS CONDITIONED ON THE TOTAL PROGENY

Tzvetozar B. Kerbashev

Communicated by J.-P. Dion

Abstract. The maximum M of a critical Bienaymé-Galton-Watson process conditioned on the total progeny N is studied. Imbedding of the process in a random walk is used. A limit theorem for the distribution of M as $N \to \infty$ is proved. The result is trasferred to the non-critical processes. A corollary for the maximal strata of a random rooted labeled tree is obtained.

1. Introduction. Consider a Bienaymé-Galton-Watson (BGW) process $\{Z_t\}$ defined by the reccurrence

$$Z_t = \sum_{i=1}^{Z_{t-1}} X_i(t), \quad t = 1, 2, \dots; \quad Z_0 \equiv 1 \text{ a.s.},$$

where $X = \{X_i(t)\}$, $i, t = 1, 2, \dots$ are independent and identically distributed (i.i.d.) random variables, taking non-negative integer values.

As usual, we may think of Z_t as the number of particles existing in the moment $t = 0, 1, 2, \dots$ Following the terms of standard interpretation we will

1991 *Mathematics Subject Classification:* Primary 60J80. Secondary 60J15, 05C05

Key words: Bienaymé-Galton-Watson branching process; maximum; total progeny; left-continuous random walk; random rooted labeled trees

consider the distribution of X as offspring distribution of one particle, the moment when Z_t first becomes zero-moment of extinction and the sum N of all Z_t – total progeny of the process.

In general three cases are considered, according to the mean of the offspring distribution – the subcritical, the critical and the supercritical one. The obtained results appear to differ significantly from each other.

Our main object of study is the random variable M, the maximal number of particles existing at the same time. The probability that M does not exceed a given level has been obtained by Bishir [4] and Adke [1]. A different approach to the same problem has been used by Lindvall [15]. In fact Lindvall has proved a limit theorem for the maximum of a suitably conditioned random walk and exploiting the connection between the random walks and the BGW processes has estimated the limit distribution of M.

Some interesting results about the partial maximum and its expectation are due to Weiner [22], Kämmerle and Schuh [10], Pakes [16] and Athreya [2]. We also mention the recent papers by Borovkov and Vatutin [3] and Vatutin and Topchii [21] where previous results are shown to be still valid under weaker restrictions.

Nevertheless, the usual way of getting interesting results about the various characteristics of the BGW process is to condition on some set of non-degenerating trajectories. Most often conditioning on non-extinction has been used. Unfortunately a limit theorem in this case has not yet been proved, though Spătaru [18] has made some steps in that direction.

Sometimes, similar results to those in the critical case are obtained via conditioning on the total progeny. It has been shown that when such conditioning is used the criticality of the process has little influence on the final formulas. Moreover the results for the critical process are easily transferred to the non-critical ones with minimum extra requirements. The conditioning on the total progeny is even more interesting in the light of the connection between the BGW processes and the random rooted labelled trees (see e.g. [13, 6.2] and [19]).

Our main goal is to estimate the limit behaviour of M when the process is conditioned on the total progeny. At first we will consider only the critical case. Again it is natural to use imbedding of the process into a random walk. The problem for the asymptotic behaviour of the maximum of a random walk conditioned on the first return to zero has been an object of study in several papers. Kaigh [9] and Smith and Diaconis [17] have found the limit distribution in the case of a simple random walk. The invariance principle of Kaigh [8] shows that these limits are valid for essentially any random walk on the integers. A different approach to the problem has been used by Chung [5] and Durret and Iglehart [6], who have explored the limiting process, called Brownian excursion.

In Section 4 we are going to prove directly a limit theorem for the maximum of a left-continuous random walk under second moment assumption. The interim results are used essentially in the proof of Theorem 2.2, which describes the asymptotic behaviour of M when the branching process is conditioned on the total progeny. Applying the results of Kennedy [11] we will transfer the result to the non-critical cases.

It is well known that for many physical, biological and other processes the maximum appears to be one of the most important and most easily measurable characteristics. For example, imagine that in a computer system one of the users starts a process, which lasts a definite period of time and dies out with probability p_0, or with probability p_i starts i similar processes, $i = 1, 2, 3, \ldots$. Thus the total progeny, i.e. the total number of these transactions is proportional to the running time of the program. The obtained in this paper correlation will allow us to estimate whether the total running time of our program could exceed a given level if the number of the processes existing at the same time exceeds another given level. This is interesting, because in many simulation programs the time limit is crucial. Of course one might use modeling with random walk but that would require many more checks as to whether the maximum exceeds the given level, and consequently – loss of time.

A tree can be defined as a connected non-ordered graph without cycles. When we choose one of its nodes for a root and number the rest by $1, 2, 3, \ldots, n$ it becomes rooted and labelled. If we define uniform distribution on the set of all labelled rooted trees with n nodes they become random, $n = 1, 2, 3, \ldots$. Each node is connected to the root by an unique path, which length is called height of the node. The set of all nodes with height $t = 1, 2, 3, \ldots$ is called t-th strata of the tree. The height of the tree can be defined as the maximal height of its nodes.

The asymptotic results for the Brownian excursion have already been a basis (see e.g. [20]) for obtaining limit results for the height of some classes of random trees, like the plain ones. In Section 6 we will draw a corollary for the asymptotic behaviour of the maximal strata of a random rooted labelled tree.

One more fact attracts out attention. It is that the asymptotic distribution of the normalised maximum is the same as that of the normalised height (with different constants of normalisation). It should be no surprise that in one case the variance appears in the numerator while in the other – in the denominator, since it is intuitively clear that when the total number of particles (or nodes) is fixed and we increase the width, the height decrease and the opposite.

2. Main results. Let $f(s) = \sum_{i=0}^{\infty} p_i s^i$, $|s| \leq 1$ denote the offspring probability generating function of the process, $p_0 + p_1 < 1$ and $p_0 > 0$.

Further on we will suppose that

$$A) \qquad \begin{cases} f^{'}(1) = 1, \\ 0 < f^{''}(1) = \sigma^2 < \infty, \\ g.c.d.\{k : p_k > 0\} = 1. \end{cases}$$

Let $S = \{S_n\},\ n = 0, 1, 2, \dots$ be a random walk defined by

$$S_n = 1 + \sum_{i=1}^{n} \xi_i, \quad S_0 \equiv 1 \qquad \text{a.s.},$$

where $\xi = \{\xi_i\}_1^\infty$ are i.i.d. random variables taking values in $\{-1, 0, 1, \dots\}$ with $P(\xi = j) = P(X = j + 1)$ for $j = -1, 0, 1, \dots$.

Denote

$$\begin{aligned} M &= \max_{t>0} Z_t, \\ \tau &= \min\{i : S_i = 0\}, \\ M' &= \max_{i \leq \tau} S_i. \end{aligned}$$

First we will prove

Theorem 2.1. *Suppose that A) holds. If $n \to \infty$ then uniformly for all x, $0 < x < \infty$*

$$P\left(\frac{2}{\sigma\sqrt{n}} M^{'} > x | \tau = n\right) \to 2 \sum_{i=1}^{\infty} ((ix)^2 - 1) e^{-(ix)^2/2}.$$

The main result of this paper is

Theorem 2.2. *Suppose that A) holds. If $n \to \infty$ then uniformly for all x, $0 < x < \infty$*

$$P\left(\frac{2}{\sigma\sqrt{n}} M > x | N = n\right) \to 2 \sum_{i=1}^{\infty} ((ix)^2 - 1) e^{-(ix)^2/2}.$$

Now using [11, Lemma 1] it is not difficult to extend the assertion of Theorem 2.2 to the subcritical and the supercritical cases.

We need the extra condition.

$$B)\begin{cases} \text{there exists } \alpha > 0 \text{ with } f(\alpha) = \alpha f'(\alpha) < \infty, \\ f''(\alpha) < \infty. \end{cases}$$

Theorem 2.3. *Suppose that A) holds with $f'(1) = a < \infty$. Under the condition B) if $n \to \infty$ then uniformly for all x, $0 < x < \infty$*

$$P\left(\frac{2}{\sqrt{\beta n}} M > x | N = n\right) \to 2\sum_{i=1}^{\infty}((ix)^2 - 1)e^{-(ix)^2/2},$$

where

$$\beta = \alpha^2 f''(\alpha)/f(\alpha).$$

Finally, we will establish a corollary, exploiting the connection between the random trees and the BGW processes.

Let $Z_t(T_n)$ be the number of the nodes in the t-th strata of the random rooted labelled tree T_n. Denote $\max_{1\le t\le n} Z_t(T_n)$ by $M(T_n)$. In the particular case when the process $\{Z_t\}$ has a Poisson offspring distribution of one particle with parameter 1 we will obtain

Corollary 2.1. *If $n \to \infty$ then uniformly for all x, $0 < x < \infty$*

$$P\left(\frac{2}{\sqrt{n}} M(T_n) > x\right) \to 2\sum_{i=1}^{\infty}((ix)^2 - 1)e^{-(ix)^2/2}.$$

3. Preliminaries. Consider two sequences $\nu = \{\nu_i\}_0^\infty$ and $\overline{Z} = \{\overline{Z}_i\}_0^\infty$ of random variables defined as follows.

$$\begin{cases} \nu_i = \nu_{i-1} + \overline{Z}_{i-1} \quad \text{for} \quad i = 1, 2, 3, \ldots; \nu_0 \equiv 0 \ a.s., \\ \overline{Z}_i = S_{\nu_i} \quad \text{for} \quad i = 1, 2, 3, \ldots; \overline{Z}_0 \equiv 1 \ a.s. \end{cases}$$

Evidently for $k = 0, 1, 2, \ldots$

$$\overline{Z}_{k+1} \quad = \quad S_{\nu_k + \overline{Z}_k} - S_{\nu_k} + S_{\nu_k}$$

$$= \sum_{i=\nu_k+1}^{\nu_k+\overline{Z}_k} \xi_i + \overline{Z}_k$$

$$= \sum_{i=\nu_k+1}^{\nu_k+\overline{Z}_k} (\xi_i + 1),$$

hence $\overline{Z}$ is a BGW process, distributed just like Z.

Note that τ corresponds to the total progeny of the process.

Denote $\overline{M} = \max_{t>0}\{\overline{Z}_t : \nu_t < \tau\}$.

Denote by $\{Z_t(k)\}$ a BGW process with the same offspring distribution as $\{Z_t\}$ and $Z_0 \equiv k$ a.s., $k = 1, 2, 3, \ldots$. Let $N(k) = \sum_{t=0}^{\infty} Z_t(k)$ denote its total progeny.

Denote $\zeta_i = \xi_1 + \cdots + \xi_i$ for $i = 1, 2, 3, \ldots$.

Dwass [7] has shown that for $n \geq m \geq 0$, $n \geq 1$

$$P(N(m) = n) = \frac{m}{n} P(\zeta_n = -m). \tag{3.1}$$

It is well known that if $n \to \infty$ then uniformly for all integer k

$$\sqrt{n} P(\zeta_n = k) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{k^2}{2\sigma^2 n}\right\} + o(1). \tag{3.2}$$

Lindvall [15] has shown that as $k \to \infty$

$$kP(M' > k) = 1 + o(1). \tag{3.3}$$

Let us for $s, n, k = 1, 2, 3, \ldots$ denote the events

$$\{\min_{1<j<n} S_j > 1 - k - s;\ S_n = 1 - s\} \quad \text{by} \quad A(s, k, n),$$

$$\{\min_{1<j<n} S_j \leq 1 - k - s;\ S_n = 1 - s\} \quad \text{by} \quad B(s, k, n),$$

$$\{1 > S_j > 1 - k, j = \overline{1, n-1};\ S_n = 1 - k\} \quad \text{by} \quad C(k, n),$$

$$\{1 > S_j > 1 - k - s, j = \overline{1, n-1};\ S_n = 1 - s\} \quad \text{by} \quad D(s, k, n).$$

Denote also $\varphi(\theta) = e^{-i\theta} f(e^{i\theta})$.

It was proved in [12] that if $k \to \infty$ in a way that *A)* holds then

$$P(N(k) = n) = O\left(\frac{1}{k^2}\right) \tag{3.4}$$

uniformly for all $n \geq k$. From (3.1) it is equivalent to

(3.5) $$P(\zeta_n = -k) = O\left(\frac{n}{k^3}\right).$$

We are going to prove

Lemma 3.1. *If $k \to \infty$ in a way that* A) *holds then*

$$P(\zeta_n = k) = O\left(\frac{\sqrt{n}}{k^2}\right).$$

uniformly for all $n \geq k$.

Proof. Clearly for $n, k = 1, 2, 3, \ldots$

$$P(\zeta_n = k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\theta k}\varphi^n(\theta)d\theta.$$

By puting $z = \dfrac{k}{\sigma\sqrt{n}}$ and $x = \theta\sigma\sqrt{n}$ we get

$$P(\zeta_n = k) = \frac{1}{2\pi\sigma\sqrt{n}}\int_{-\pi\sigma\sqrt{n}}^{\pi\sigma\sqrt{n}} e^{-ixz}\varphi^n\left(\frac{x}{\sigma\sqrt{n}}\right)dx.$$

Consider a sequence of functions $G_1(z), G_2(z), G_3(z), \ldots$, defined as follows

(3.6) $$G_r(z) = z^2\int_{-\pi\sigma\sqrt{r}}^{\pi\sigma\sqrt{r}} e^{-ixz}\varphi^r\left(\frac{x}{\sigma\sqrt{r}}\right)dx, \qquad r = 1, 2, 3, \ldots.$$

It is easy to see that for each choice of k and n, $1 \leq k \leq n$

$$P(\zeta_n = k) = \frac{\sqrt{n}}{k^2}\frac{\sigma}{2\pi}G_n\left(\frac{k}{\sigma\sqrt{n}}\right).$$

Hence the lemma will be proved if we show that uniformly for all z, $0 < \dfrac{\sigma z}{\sqrt{r}} \leq 1$ as $r \to \infty$

(3.7) $$G_r(z) = O(1).$$

When $\theta \to 0$

(3.8) $$\varphi(\theta) = 1 - \frac{\sigma^2\theta^2}{2}(1 + o(1)),$$

hence there exists $\varepsilon_1 > 0$, such that for $|\theta| < \varepsilon_1$

$$|\varphi(\theta)| \leq 1 - \frac{\sigma^2\theta^2}{4} \leq e^{-\sigma^2\theta^2/4}. \tag{3.9}$$

From the basic properties of $\varphi(\theta)$ there exists such ε_2 that for $|\theta| < \varepsilon_2$

$$|(\varphi(\theta))'| \leq 2\sigma^2|\theta| \qquad |(\varphi(\theta))''| \leq \sigma^2. \tag{3.10}$$

Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ and without loss of generality assume $r \geq 4$.
Since the process is aperiodic there exists such $q = q(\varepsilon) < 1$ that

$$\sup_{\varepsilon\leq|\theta|<\pi} |\varphi(\theta)| < q, \qquad 0 < \varepsilon < \pi.$$

Therefore we have

$$\begin{aligned}
\left| z^2 \int_{\varepsilon\leq\frac{|x|}{\sigma\sqrt{r}}\leq\pi} e^{-ixz}\varphi^r\left(\frac{x}{\sigma\sqrt{r}}\right)dx\right| &\leq \frac{r}{\sigma^2}\int_{\varepsilon\leq\frac{|x|}{\sigma\sqrt{r}}\leq\pi}\left|\varphi\left(\frac{x}{\sigma\sqrt{r}}\right)\right|^r dx \\
&= \frac{r}{\sigma^2}\sigma\sqrt{r}\int_{\varepsilon\leq|\theta|\leq\pi}|\varphi(\theta)|^r d\theta \\
&\leq \frac{2\pi}{\sigma}r^{3/2}q^r \quad \to 0, \qquad r\to\infty.
\end{aligned} \tag{3.11}$$

Now from (3.6), (3.7) and (3.11) it suffices to show that if $0 < \dfrac{\sigma z}{\sqrt{r}} \leq 1$ and $r \to \infty$ then

$$z^2\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} e^{-ixz}\varphi^r\left(\frac{x}{\sigma\sqrt{r}}\right)dx = O(1). \tag{3.12}$$

Using (3.9) and (3.10) one can obtain for $\left|\dfrac{x}{\sigma\sqrt{r}}\right| < \varepsilon$

$$\begin{aligned}
\left|\frac{d\varphi^r\left(\frac{x}{\sigma\sqrt{r}}\right)}{dx}\right| &= r\left|\varphi^r - 1\left(\frac{x}{\sigma\sqrt{r}}\right)\frac{d\varphi\left(\frac{x}{\sigma\sqrt{r}}\right)}{dx}\right| \\
&\leq 2|x|e^{-\frac{3}{16}x^2}
\end{aligned} \tag{3.13}$$

and

$$(3.14)\quad \left|\frac{d^2\varphi^r(\frac{x}{\sigma\sqrt{r}})}{dx^2}\right| = \left|r(r-1)\varphi^{r-2}\left(\frac{x}{\sigma\sqrt{r}}\right)\left(\frac{d\left(\varphi\left(\frac{x}{\sigma\sqrt{r}}\right)\right)}{dx}\right)^2 + \right.$$

$$\left. + r\varphi^{r-1}\left(\frac{x}{\sigma\sqrt{r}}\right)\frac{d^2\varphi\left(\frac{x}{\sigma\sqrt{r}}\right)}{dx^2}\right|$$

$$\leq 4x^2 e^{-\frac{1}{8}x^2} + e^{-\frac{3}{16}x^2}.$$

Next integrating by parts the left-hand side of (3.12) we get

$$z^2\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} e^{-ixz}\varphi^r\left(\frac{x}{\sigma\sqrt{r}}\right)dx = iz\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}}\varphi^r\left(\frac{x}{\sigma\sqrt{r}}\right)de^{-ixz}$$

$$= ize^{-ixz}\varphi^r\left(\frac{x}{\sigma\sqrt{r}}\right)\Bigg|_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} - iz\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} e^{-ixz}\frac{d\varphi^r\left(\frac{x}{\sigma\sqrt{r}}\right)}{dx}dx.$$

Finally, using (3.8), (3.9) and more integrating by parts we get uniformly for all z, $0 < \dfrac{z\sigma}{\sqrt{r}} \leq 1$ as $r \to \infty$

$$\begin{aligned} z^2\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} e^{-ixz}\varphi^r\left(\frac{x}{\sigma\sqrt{r}}\right)dx &= o(1) + \int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}}\frac{d\varphi^r\left(\frac{x}{\sigma\sqrt{r}}\right)}{dx}de^{-ixz} \\ &= o(1) + e^{-ixz}\frac{d\varphi^r\left(\frac{x}{\sigma\sqrt{r}}\right)}{dx}\Bigg|_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} \\ &\quad - \int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} e^{-ixz}\frac{d^2\varphi^r\left(\frac{x}{\sigma\sqrt{r}}\right)}{dx^2}dx \end{aligned}$$

and from (3.13) and (3.14) follows that (3.12) holds, which completes the proof of the lemma.

In the case when $0 < n \leq k$ applying the Chebyshev inequality one gets

$$(3.15)\qquad P(\zeta_n = k) \leq P(\zeta_n \geq k) < \frac{n\sigma^2}{k^2} = O\left(\frac{1}{k}\right) \qquad \text{as} \quad k \to \infty.$$

From Lemma 3.1 follows that (3.15) holds uniformly for all $n = 1, 2, 3, \ldots$ as $k \to \infty$.

Lemma 3.2. *Suppose that A) holds. If $n \to \infty$ then uniformly for all integer k*

$$n(P(\zeta_n = k) - P(\zeta_n = k+1)) = \frac{k}{\sqrt{2\pi}\sigma^3\sqrt{n}} e^{-k^2/2\sigma^2 n} + o(1).$$

P r o o f. Since for $n = 1, 2, 3, \ldots$ and every integer k

$$P(\zeta_n = k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\theta k}(\varphi(\theta))^n d\theta$$

after the substitutions $z = \dfrac{-k}{\sigma\sqrt{n}}$ and $x = \sigma\theta\sqrt{n}$ we get

$$P(\zeta_n = k) = \frac{1}{2\pi\sigma\sqrt{n}}\int_{-\pi\sigma\sqrt{n}}^{\pi\sigma\sqrt{n}} e^{ixz}\left(\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right)^n dx. \tag{3.16}$$

Several additional notes should to be made here.

Consider some $\varepsilon > 0$.

It is easy to see that there exists such $A_1 = A_1(\varepsilon)$ that uniformly for all z

$$\frac{1}{2\pi\sigma^2}\left|\int_{A_1<|x|} e^{ixz} x e^{-x^2/2} dx\right| < \varepsilon. \tag{3.17}$$

Obviously there exists also $A_2 = A_2(\varepsilon)$ that

$$\frac{1}{2\pi\sigma^2}\int_{A_2<|x|} |x| e^{-x^2/4} dx < \varepsilon. \tag{3.18}$$

Denote $A = \max(A_1, A_2)$.

When $\dfrac{x}{\sqrt{n}} \to 0$ we have

$$1 - e^{-ix/\sigma\sqrt{n}} = \frac{ix}{\sigma\sqrt{n}}(1 + o(1)). \tag{3.19}$$

Hence there exists such $\delta_1 > 0$ that if $\left|\dfrac{x}{\sigma\sqrt{n}}\right| < \delta_1$ then

$$|1 - e^{-ix/\sigma\sqrt{n}}| \le \frac{2|x|}{\sigma\sqrt{n}} \tag{3.20}$$

Moreover if $\frac{x}{\sqrt{n}} \to 0$ then

$$\varphi\left(\frac{x}{\sigma\sqrt{n}}\right) = 1 - \frac{x^2}{2n} + o\left(\frac{1}{n}\right) \tag{3.21}$$

and

$$\log\left(\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right)^n = n\log\left(1 - \frac{x^2}{2n} + o\left(\frac{1}{n}\right)\right) = -\frac{x^2}{2} + o(1). \tag{3.22}$$

From (3.21) there exists such $\delta_2 > 0$ that if $|\theta| < \delta_2$

$$|\varphi(\theta)| \leq 1 - \frac{\sigma^2\theta^2}{4} \leq e^{-\sigma^2\theta^2/4}. \tag{3.23}$$

Denote $\delta = \min(\delta_1, \delta_2)$.

Our proof proceeds by an estimation of the expression

$$R(n,k) = n(P(\zeta_n = k) - P(\zeta_n = k+1)) - \frac{k}{\sqrt{2\pi}\sigma^3\sqrt{n}} e^{-k^2/2\sigma^2 n}.$$

From (3.16) for $n = 1, 2, 3, \ldots$ and every integer k

$$\begin{aligned} P(\zeta_n = k) &- P(\zeta_n = k+1) \\ &= \frac{1}{2\pi\sigma\sqrt{n}} \int_{-\pi\sigma\sqrt{n}}^{\pi\sigma\sqrt{n}} (e^{-ix\frac{k}{\sigma\sqrt{n}}} - e^{-ix\frac{k+1}{\sigma\sqrt{n}}}) \left(\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right)^n dx \\ &= \frac{1}{2\pi\sigma\sqrt{n}} \int_{-\pi\sigma\sqrt{n}}^{\pi\sigma\sqrt{n}} (1 - e^{-ix/\sigma\sqrt{n}}) e^{ixz} \left(\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right)^n dx. \end{aligned}$$

Since for every $z > 0$

$$ze^{-z^2/2} = -\frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixz} x e^{-x^2/2} dx$$

we may write

$$R(n,k) = I_1 + I_2 + I_3 + I_4, \text{ say,}$$

where

$$I_j = I_j(n, k, A, \delta) \quad for\ j = 1, 2, 3, 4,$$

$$\begin{aligned} I_1 &= \frac{\sqrt{n}}{2\pi\sigma}\int_{-A}^{A}\left(1-e^{\frac{-ix}{\sigma\sqrt{n}}}\right)e^{ixz}\left(\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right)^n dx \;-\frac{i}{2\pi\sigma^2}\int_{-A}^{A}e^{ixz}xe^{-x^2/2}dx,\\ I_2 &= -\frac{i}{2\pi\sigma^2}\int_{A<|x|}e^{ixz}xe^{-x^2/2}dx,\\ I_3 &= \frac{\sqrt{n}}{2\pi\sigma}\int_{A<|x|\le\delta\sigma\sqrt{n}}\left(1-e^{\frac{-ix}{\sigma\sqrt{n}}}\right)e^{ixz}\left(\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right)^n dx\\ I_4 &= \frac{\sqrt{n}}{2\pi\sigma}\int_{\delta\sigma\sqrt{n}<|x|\le\pi\sigma\sqrt{n}}\left(1-e^{\frac{-ix}{\sigma\sqrt{n}}}\right)e^{ixz}\left(\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right)^n dx. \end{aligned}$$

Here we used the fact that $z=\dfrac{-k}{\sigma\sqrt{n}}$ and therefore $\dfrac{k}{\sqrt{2\pi\sigma^3}\sqrt{n}}=-\dfrac{z}{\sqrt{2\pi}\sigma^2}$.

Let $n\to\infty$.

With the help of (3.19) and (3.22) we get

$$\begin{aligned} I_1 &= \frac{\sqrt{n}}{2\pi\sigma}\int_{-A}^{A}\frac{ix}{\sigma\sqrt{n}}(1+o(1))e^{ixz}\left(\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right)^n dx\\ &\quad -\frac{i}{2\pi\sigma^2}\int_{-A}^{A}e^{ixz}xe^{-x^2/2}dx\\ &= \frac{i}{2\pi\sigma^2}\int_{-A}^{A}e^{ixz}xe^{-x^2/2}(1+o(1))dx\\ &\quad -\frac{i}{2\pi\sigma^2}\int_{-A}^{A}e^{ixz}xe^{-x^2/2}dx\\ &= o(1). \end{aligned} \tag{3.24}$$

From (3.17) follows that

$$|I_2|<\varepsilon. \tag{3.25}$$

Using (3.18), (3.20) and (3.23) we get

$$\begin{aligned} |I_3| &\le \frac{1}{\pi\sigma^2}\int_{A<|x|\le\delta\sigma\sqrt{n}}|x|\,\left|\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right|^n dx\\ &\le \frac{1}{\pi\sigma^2}\int_{A<|x|\le\delta\sigma\sqrt{n}}|x|e^{-x^2/4}dx\\ &\le \frac{1}{\pi\sigma^2}\int_{A\le|x|}|x|e^{-x^2/4}dx<\varepsilon. \end{aligned} \tag{3.26}$$

Since A) holds there exists such q, $0<q<1$ that when $\delta<|\theta|<\pi$

$$|\varphi(\theta)|<q.$$

Hence

$$|I_4| \leq \frac{\sqrt{n}}{\pi\sigma}\int_{\delta\sigma\sqrt{n}<|x|\leq\pi\sigma\sqrt{n}}\left|\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right|^n dx \tag{3.27}$$

$$= \frac{n}{\pi}\int_{\delta<|\theta|\leq\pi}|\varphi(\theta)|^n d\theta$$

$$\leq 2nq^n = o(1).$$

Now from (3.24)–(3.27) follows that as $n \to \infty$ we have $R(n,k) = o(1)$ uniformly for all integer k, which completes the proof of the Lemma.

Lemma 3.3. *Suppose that A) holds. If* $n \to \infty$ *then uniformly for all* $m, k = 0, 1, 2, \ldots$

$$P(\zeta_n = k) - P(\zeta_{n+m} = k) = O\left(\frac{m}{n\sqrt{n}}\right).$$

P r o o f. If $m, n \to \infty$ in a way that $n = O(m)$ the assertion of the Lemma follows directly from (3.2), (3.15) and Lemma 3.1.

Hence it suffices to consider only the case $m = o(n)$.

Since (3.21) still holds as $x/\sigma\sqrt{n} \to 0$ we have

$$\left(\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right)^m = 1 - \frac{x^2}{2}\frac{m}{n}(1 + o(1)).$$

Hence there exists such $\delta_1 > 0$ that when $\frac{|x|}{\sigma\sqrt{n}} < \delta_1$ we have

$$\left|1 - \varphi^m\left(\frac{x}{\sigma\sqrt{n}}\right)\right| < \frac{m}{n}x^2. \tag{3.28}$$

From (3.21) there exists also such $\delta_2 > 0$ that when $\frac{|x|}{\sigma\sqrt{n}} < \delta_2$

$$\left|\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right|^n < e^{-x^2/4}. \tag{3.29}$$

Denote $\delta = \min(\delta_1, \delta_2)$.

We have again for $k, n = 1, 2, 3, \ldots$

$$P(\zeta_n = k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\theta k}\varphi^n(\theta)d\theta.$$

Setting $z = \dfrac{k}{\sigma\sqrt{n}}$ and $x = \theta\sigma\sqrt{n}$ we may write

$$\begin{aligned}
&P(\zeta_n = k) - P(\zeta_{n+m} = k)\\
&\quad = \frac{1}{2\pi\sigma\sqrt{n}} \int_{-\pi\sigma\sqrt{n}}^{\pi\sigma\sqrt{n}} e^{-ixz} \left(\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right)^n \left(1 - \left(\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right)^m\right) dx\\
&\quad = I_1(n, m, k, \delta) + I_2(n, m, k, \delta) \quad \text{, say,}
\end{aligned}$$

where

$$I_1(n, m, k, \delta) = \frac{1}{2\pi\sigma\sqrt{n}} \int\limits_{|x| \le \delta\sigma\sqrt{n}} e^{-ixz} \left(\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right)^n \left(1 - \left(\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right)^m\right) dx$$

$$I_2(n, m, k, \delta) = \frac{1}{2\pi\sigma\sqrt{n}} \int\limits_{\delta\sigma\sqrt{n} < |x| \le \pi\sigma\sqrt{n}} e^{-ixz} \left(\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right)^n \left(1 - \left(\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right)^m\right) dx.$$

We will show that as $n \to \infty$

$$I_1(n, m, k, \delta) = o\left(\frac{m}{n\sqrt{n}}\right) \tag{3.30}$$

and

$$I_2(n, m, k, \delta) = o(1). \tag{3.31}$$

Let $n, m \to \infty$ in a way that $m = o(n)$.
Using (3.28) and (3.29) we get

$$\begin{aligned}
|I_1(n, m, k, \delta)| &< \frac{1}{2\pi\sigma\sqrt{n}} \int_{|x| \le \delta\sigma\sqrt{n}} \frac{m}{n} x^2 e^{-x^2/4} dx\\
&< \frac{m}{n\sqrt{n}} \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} x^2 e^{-x^2/4} dx\\
&= O\left(\frac{m}{n\sqrt{n}}\right),
\end{aligned}$$

hence (3.30) holds.

Since A) holds there exists such q, $0 < q < 1$ that if $\delta < |\theta| < \pi$ then

$$|\varphi(\theta)| < q.$$

Hence

$$\begin{aligned}
|I_2(n,m,k,\delta)| &\le \frac{1}{2\pi\sigma\sqrt{n}} \int\limits_{\delta\sigma\sqrt{n}<|x|\le\pi\sigma\sqrt{n}} \left|\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right|^n |1-(\varphi(\frac{x}{\sigma\sqrt{n}}))^m|dx \\
&\le \frac{1}{2\pi\sigma\sqrt{n}} \int\limits_{\delta\sigma\sqrt{n}<|x|\le\pi\sigma\sqrt{n}} \left|\varphi\left(\frac{x}{\sigma\sqrt{n}}\right)\right|^n dx \\
&\le 2q^n = o(1)
\end{aligned}$$

and (3.31) holds, which completes the proof of Lemma 3.3.

Lemma 3.4. *If $n,k\to\infty$ in a way that $kn^{-1/2}=O(1)$ then*

$$P(\min_{1<j<n} S_j>0, S_n=k) = P(\zeta_n=-k+1)-P(\zeta_n=-k-1)+o\left(\frac{1}{n}\right).$$

P r o o f. Clearly for $n,k=1,2,3,\dots$

$$\begin{aligned}
P(\min_{1<j<n} S_j>0; S_n=k) &= P(S_n=k)-P(\min_{1\le j\le n} S_j\le 0, S_n=k) \\
&= P(S_n=k)-\sum_{i=1}^{n-1} P(\min_{1<j<i} S_j>0; S_i=0)P(\zeta_{n-i}=k),
\end{aligned}$$

and since $\sum\limits_{i=1}^{\infty} P(\min_{1<j<i} S_j>0; S_i=0)=1$ using (3.1) we get

$$\begin{aligned}
(3.32)\quad P(\min_{1<j<n} S_j>0; S_n=k) &= \sum_{i=1}^{\infty} P(\min_{1<j<i} S_j>0; S_i=0)P(S_n=k) \\
&\quad -\sum_{i=1}^{n-1} P(\min_{1<j<i} S_j>0; S_i=0)P(\zeta_{n-i}=k) \\
&= \sum_{i=1}^{\infty} \frac{1}{i}P(\zeta_i=-1)P(\zeta_n=k-1) \\
&\quad -\sum_{i=1}^{n-1} \frac{1}{i}P(\zeta_i=-1)P(\zeta_{n-i}=k) \\
&= S_1(n,k)+S_2(n,k)+S_3(n,k), \text{ say,}
\end{aligned}$$

where

$$S_1(n,k) = \sum_{i=1}^{n-1} \frac{1}{i} P(\zeta_i = -1)(P(\zeta_n = k-1) - P(\zeta_n = k)),$$
$$S_2(n,k) = \sum_{i \geq n} \frac{1}{i} P(\zeta_i = -1) P(\zeta_n = k-1)$$
$$S_3(n,k) = \sum_{i=1}^{n-1} \frac{1}{i} P(\zeta_i = -1)(P(\zeta_n = k) - P(\zeta_{n-i} = k)).$$

Let $n, k \to \infty$ in a way that $kn^{-1/2} = O(1)$.

From Lemma 3.2 follows that as $n \to \infty$ uniformly for all $k = 1, 2, \ldots, n-1$

$$P(\zeta_n = k-1) - P(\zeta_n = k) = P(\zeta_n = -k+1) - P(\zeta_n = -k) + o\left(\frac{1}{n}\right).$$

Hence

$$S_1(n,k) = \sum_{i=1}^{n-1} \frac{1}{i} P(\zeta_i = -1)(P(\zeta_n = -k+1) - P(\zeta_n = -k)) + o\left(\frac{1}{n}\right). \tag{3.33}$$

Next, using (3.2) one could easily obtain that as $n \to \infty$

$$\sum_{i \geq n} \frac{1}{i} P(\zeta_i = -1) = \frac{\sqrt{2}}{\sqrt{\pi \sigma^2 n}} (1 + o(1)).$$

From (3.2) we have also $P(\zeta_n = k-1) = P(\zeta_n = -k+1) + o\left(\frac{1}{\sqrt{n}}\right)$. Therefore

$$S_2(n,k) = \sum_{i \geq n} \frac{1}{i} P(\zeta_i = -1) P(\zeta_n = -k+1) + o\left(\frac{1}{n}\right). \tag{3.34}$$

Consider some $\varepsilon > 0$.

If $n \to \infty$ a combination of (3.2) and Lemma 3.3 produces uniformly for all $m = 1, 2, 3, \ldots,$ $m < n$

$$\begin{aligned}\sum_{i=1}^{m} \frac{1}{i} P(\zeta_i = -1)(P(\zeta_n = k) - P(\zeta_{n-i} = k)) &= \sum_{i=1}^{m} \frac{1}{i} O\Big(\frac{1}{\sqrt{i}}\Big) O\Big(\frac{i}{(n-m)\sqrt{n-m}}\Big) \\ &= \sum_{i=1}^{m} O\left(\frac{1}{(n-m)\sqrt{i(n-m)}}\right).\end{aligned}$$

Hence there exists such $\delta_1 = \delta_1(\varepsilon) > 0$ that

$$\left| \sum_{1 \le i \le \delta_1 n} \frac{1}{i} P(\zeta_i = -1)(P(\zeta_n = k) - P(\zeta_{n-i} = k)) \right| < \frac{\varepsilon}{n}.$$

From (3.2) when $i = O(n)$ we have

$$\frac{1}{i} P(\zeta_i = -1) = O\left(\frac{1}{n\sqrt{n}}\right)$$

and from (3.5) uniformly for all $i = 0, 1, \dots, n-1$

$$P(\zeta_{n-i} = k) = O\left(\frac{n-i}{k^2}\frac{1}{k}\right) = O\left(\frac{1}{\sqrt{n}}\right).$$

Hence there exists such $\delta_2 = \delta_2(\varepsilon) > 0$ that

$$\left| \sum_{n-\delta_2 n \le i < n} \frac{1}{i} P(\zeta_i = -1)(P(\zeta_n = k) - P(\zeta_{n-i} = k)) \right| < \frac{\varepsilon}{n}.$$

Let $\delta = \min(\delta_1, \delta_2)$.
With the help of Lemma 3.2 we obtain

$$\begin{aligned} &\sum_{\delta n < i < (1-\delta)n} \frac{1}{i} P(\zeta_i = -1)(P(\zeta_n = k) - P(\zeta_{n-i} = k)) \\ &\quad = \sum_{\delta n < i < (1-\delta)n} \frac{1}{i} P(\zeta_i = -1)(P(\zeta_n = -k) - P(\zeta_{n-i} = -k)) + o\left(\frac{1}{n}\right). \end{aligned}$$

Hence

$$S_3(n,k) = \sum_{i \ge n} \frac{1}{i} P(\zeta_i = -1)(P(\zeta_n = -k) - P(\zeta_{n-i} = -k)) + o\left(\frac{1}{n}\right). \tag{3.35}$$

Now from (3.32) – (3.35) follows that

$$\begin{aligned} P(S_j > 0, j = \overline{1,n}; S_n = k) &= \sum_{i=1}^{n-1} \frac{1}{i} P(\zeta_i = -1) \times \\ &\qquad \times (P(\zeta_n = -k+1) - P(\zeta_{n-i} = -k)) \\ &\quad + \sum_{i \ge n} \frac{1}{i} P(\zeta_i = -1) P(\zeta_n = -k+1) + o\left(\frac{1}{n}\right) \\ &= P(\zeta_n = -k+1) - P(\zeta_n = -k-1) + o\left(\frac{1}{n}\right), \end{aligned}$$

which completes the proof of the Lemma. □

Lemma 3.5. *If* $k \to \infty$ *then*

$$\lim k \sum_{i \geq k} P(C(k,i)) \leq \frac{\sigma^2}{2}.$$

P r o o f. Evidently for every $k, n = 1, 2, 3, \dots\ , k \leq n$

$$P(A(k,0,n)) = P(N(k) = n) = \frac{k}{n} P(\zeta_n = -k).$$

Moreover for every $s, k, n, n_1 = 1, 2, 3, \dots\ , n_1 < n$ we have

$$\begin{aligned} (3.36)\quad P(A(s+k,0,n)) &= \sum_{i=k}^{n-s} P(A(s,k,n-i))P(C(k,i)) \\ &> \sum_{i<n_1} P(A(s,k,n-i))P(C(k,i)) \\ &= \sum_{i<n_1} (P(\zeta_{n-i} = -s) - P(B(s,k,n-i))P(C(k,i)) \end{aligned}$$

Consider some $\delta > 0$.

For each choice of $s, k, n, i = 1, 2, 3, \dots\ , i < n$ we have

$$\begin{aligned} P(B(s,k,n-i)) &= \sum_{l=1}^{n-i-k-s} P(A(s+k,0,n-i-l))P(\zeta_l = k) \\ &= \sum_{l<\delta k^2} P(A(s+k,0,n-i-l))P(\zeta_l = k) \\ &\quad + \sum_{l \geq \delta k^2} P(A(s+k,0,n-i-l))P(\zeta_l = k) \end{aligned}$$

Let $s, k, n, n_1 \to \infty$ simultaneously in a way that $k = o(s)$, $n = O(s^2)$, $n_1 = o(n)$ and $k n_1^{-1/2} = o(1)$.

From (3.4) and (3.15) follows that

$$\sum_{l<\delta k^2} P(A(s+k,0,n-i-l))P(\zeta_l = k) = \sum_{l<\delta k^2} O\left(\frac{1}{s^2}\right) O\left(\frac{1}{k}\right) = o\left(\frac{1}{s}\right).$$

Next using (3.2) we get

$$\sum_{l\geq \delta k^2} P(A(s+k,0,n-i-l))P(\zeta_l=k) = \\ = \sum_{l\geq \delta k^2} P(A(s+k,0,n-i-l))P(\zeta_l=-k)(1+o(1)).$$

Hence

$$P(B(s,k,n-i)) = \sum_{l\geq \delta k^2} P(A(s+k,0,n-i-l))P(\zeta_l=-k)(1+o(1)) + o\left(\frac{1}{s}\right).$$

Using (3.5) and (3.15) we obtain

$$\sum_{l<\delta k^2} P(A(s+k,0,n-i-l))P(\zeta_l=-k) = \sum_{l<\delta k^2} O\left(\frac{1}{s^2}\right) O\left(\frac{1}{k}\right) = o\left(\frac{1}{s}\right),$$

therefore

$$(3.37)\quad P(B(s,k,n-i)) \\ = \sum_{l=k}^{n-i-k-s} P(A(s+k,0,n-i-l))P(\zeta_l=-k)(1+o(1)) + o\left(\frac{1}{s}\right) \\ = P(\zeta_{n-i}=-s-2k) + o\left(\frac{1}{s}\right).$$

Now from (3.36) we have

$$P(A(s+k,0,n)) \\ > \sum_{i<n_1} \left(P(\zeta_{n-i}=-s) - P(\zeta_{n-i}=-s-2k) + o\left(\frac{1}{s}\right)\right) P(C(k,i)) \\ = 2k(P(\zeta_{n-i}=-s) - P(\zeta_{n-i}=-s-1))(1+o(1)) \sum_{i<n_1} P(C(k,i))$$

hence

$$\sum_{i<n_1} P(C(k,i)) < \frac{P(A(s+k,0,n))}{2k(P(\zeta_n=-s)-P(\zeta_n=-s-1))}(1+o(1)).$$

Finally, from (3.1), (3.2) and Lemma 3.2 we get that as $k\to\infty$

$$\lim \sum_{i<n_1} P(C(k,i)) < \frac{\sigma^2}{2k}$$

which completes the proof of the lemma. □

Lemma 3.6. *If $k \to \infty$ then uniformly for all $n = k, k+1, k+2, \dots$*

$$P(C(k,n)) = O\left(\frac{1}{k^3}\right).$$

P r o o f. Evidently for every $s, n, k = 1, 2, 3, \dots,\quad s < k$

$$P(C(k,n)) = \sum_{i<n} P(D(k-s,s,i))P(C(s,n-i)). \tag{3.38}$$

Let $n, k, s \to \infty$ in a way that $kn^{-1/2} = O(1)$ and $ks^{-1} = 2 + o(1)$.
For every $i = k-s, k-s+1, \dots, n-s$ we have

$$\begin{aligned} P(D(k-s,s,i)) \ &< \ P(\max_{j<i} S_j < 1; S_i = 1-k+s) \\ &= \sum_{l_1,l_2,\dots,l_i=-1}^{i} p_{l_1}p_{l_2}\dots p_{l_i}\times \\ &\quad \times \mathbf{I}(\max_{j<i}(l_1+l_2+\dots+l_j) < 1; l_1+l_2+\dots+l_i = 1-k-s) \\ &= \sum_{l_1,l_2,\dots,l_i=-1}^{i} p_{l_i}p_{l_{i-1}}\dots p_{l_1}\times \\ &\quad \times \mathbf{I}(\min_{j<i}(l_i+l_{i-1}+\dots+l_{i-j+1}) > 1-k-s; l_1+\dots+l_i = 1-k-s) \\ &= P(\min_{j<i} S_j > 1-k+s; S_i = 1-k+s) \end{aligned}$$

and from (3.4) uniformly for all $i = k-s, k-s+1, \dots, n-s$ we get

$$P(D(k-s,s,i)) = O\left(\frac{1}{k^2}\right).$$

Hence from (3.38)

$$P(C(k,n)) = O\left(\frac{1}{k^2}\right)\sum_{i<n} P(C(s,n-i))$$

and from Lemma 3.5 follows that uniformly for all $n \ge k$

$$P(C(k,n)) = O\left(\frac{1}{k^2}\right)O\left(\frac{1}{k}\right) = O\left(\frac{1}{k^3}\right),$$

which completes the proof of the lemma. □

Lemma 3.7. *If $s, n, k \to \infty$ in a way that $kn^{-1/2} = O(1)$ and $ks^{-1}=O(1)$ then*

$$\sum_{l<n}(P(B(s,k,n-l)) - P(B(s+1,k,n-l)))P(C(k,l))$$
$$= \sum_{l<n}(P(\zeta_{n-l} = -s-2k) - P(\zeta_{n-l} = -s-2k-1))P(C(k,l)) + o\left(\frac{1}{k^3}\right).$$

Proof. Write

$$\sum_{l<n}(P(B(s,k,n-l)) - P(B(s+1,k,n-l)))P(C(k,l))$$
$$= \sum_{l<n}\sum_{i=1}^{n-l-s-k}(P(A(s+k,0,n-l-i)) - P(A(s+k+1,0,n-l-i)))\times$$
$$\times P(\zeta_i = k)P(C(k,l))$$

The proof leans on the fact that if $E\xi = 0$ and $i, k \to \infty$ in a way that $ki^{-1/2} = O(1)$ then

$$P(\zeta_i = k) = P(\zeta_i = -k)(1 + o(1)).$$

The parts of the sum for which that can not be applied will be properly estimated.

Consider some δ, $0 < \delta < 1$. Then

$$\sum_{l<n}(P(B(s,k,n-l)) - P(B(s+1,k,n-l)))P(C(k,l)) \tag{3.39}$$
$$= \sum_{l<n}\sum_{i\geq\delta n}(P(A(s+k,0,n-l-i)) - P(A(s+k+1,0,n-l-i)))P(\zeta_i=k)P(C(k,l))$$
$$+ \sum_{l<n}\sum_{i<\delta n}(P(A(s+k,0,n-l-i)) - P(A(s+k+1,0,n-l-i)))P(\zeta_i=k)P(C(k,l))$$

Let $s, n, k \to \infty$ in a way that $kn^{-1/2} = O(1)$, $ks^{-1} = O(1)$.

Using (3.2) we get

$$\sum_{l<n}\sum_{i\geq\delta n}(P(A(s+k,0,n-l-i)) - P(A(s+k+1,0,n-l-i)))P(\zeta_i=k)P(C(k,l)) \tag{3.40}$$
$$= \sum_{l<n}\sum_{i\geq\delta n}(P(A(s+k,0,n-l-i)) - P(A(s+k+1,0,n-l-i)))\times$$
$$\times P(\zeta_i = -k)P(C(k,l))(1 + o(1)).$$

On the other hand from (3.15) uniformly for all $i = 1, 2, 3, \ldots$ we have

$$P(\zeta_i = k) = O\left(\frac{1}{k}\right),$$

and applying (3.15) and Lemma 3.6 we obtain

$$\begin{aligned}
&\sum_{l<n}\sum_{i<\delta n}(P(A(s+k,0,n-l-i))-P(A(s+k+1,0,n-l-i)))P(\zeta_i=k)P(C(k,l)) \\
&= \sum_{i<\delta n}\sum_{i\le l<n}(P(A(s+k,0,n-l))-P(A(s+k+1,0,n-l)))P(\zeta_i=k)P(C(k,l-i)) \\
&< \delta O(\frac{1}{k^2})\sum_{l<n}(P(A(s+k,0,n-l)) - P(A(s+k+1,0,n-l))) \\
&= \delta O(\frac{1}{k^2})\sum_{l<n}(P(A(s+k,0,l)) - P(A(s+k+1,0,l))).
\end{aligned}$$

In order to estimate the sum $\sum_{l<n}(P(A(s+k,0,l)) - P(A(s+k+1,0,l)))$ first note that when $0 < l < n$

$$\frac{1}{P(\zeta_{2n-l} = -k)} = O(k).$$

Hence using Lemma 3.2 we get

$$\begin{aligned}
&\sum_{l<n}(P(A(s+k,0,n-l)) - P(A(s+k+1,0,n-l))) \\
&\quad = O(k)\sum_{l<n}(P(A(s+k,0,n-l)) - P(A(s+k+1,0,n-l)))P(\zeta_{2n-l} = -k) \\
&\quad < O(k)(P(\zeta_{2n} = -s-2k) - P(\zeta_{2n} = -s-2k-1)) \\
&\quad = O\left(\frac{1}{k}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
(3.41)\qquad &\sum_{l<n}\sum_{i<\delta n}(P(A(s+k,0,n-l)) - P(A(s+k+1,0,n-l)))\times \\
&\qquad \times P(\zeta_i = k)P(C(k,l)) < \delta O\left(\frac{1}{k^3}\right)
\end{aligned}$$

In the same way, using this time (3.5) and Lemma 3.6 one gets

$$\sum_{l<n}\sum_{i<\delta n}(P(A(s+k,0,n-l-i))-P(A(s+k+1,0,n-l-i)))P(\zeta_i=-k)P(C(k,l))$$
$$< \delta O\left(\frac{1}{k^2}\right)\sum_{l<n}(P(A(s+k,0,n-l))-P(A(s+k+1,0,n-l))$$
$$< \delta O\left(\frac{1}{k}\right)\sum_{l<n}(P(A(s+k,0,n-l))-P(A(s+k+1,0,n-l))P(\zeta_{2n-l}=-k),$$

i.e.

$$\sum_{l<n}\sum_{i<\delta n}(P(A(s+k,0,n-l))-P(A(s+k+1,0,n-l)))\times \\ \times P(\zeta_i=-k)P(C(k,l)) < \delta O\left(\frac{1}{k^3}\right). \tag{3.42}$$

Now since (3.39) – (3.42) hold for any $\delta \in (0,1)$ we have

$$\sum_{l<n}(P(B(s,k,n-l))-P(B(s+1,k,n-l)))P(C(k,l))$$
$$=\sum_{l<n}(P(\zeta_{n-l}=-s-2k)-P(\zeta_{n-l}=-s-2k-1))P(C(k,l))+o\left(\frac{1}{k^3}\right),$$

which completes the proof of the Lemma. □

Lemma 3.8. *If $m,k\to\infty$ in a way that $m=o(k^2)$ then*

$$\sum_{i<m}(P(\zeta_i=-k+1)-P(\zeta_i=-k-1))=o(1).$$

P r o o f. For $k=1,2,3,\ldots$ write

$$P(\zeta_{k^6}=-k^3-k+1)-P(\zeta_{k^6}=-k^3-k-1)$$
$$=\sum_{i<k^6}P(A(k^3,0,k^6-i))(P(\zeta_i=-k+1)-P(\zeta_i=-k-1))$$
$$>\sum_{i<k^4}P(A(k^3,0,k^6-i))(P(\zeta_i=-k+1)-P(\zeta_i=-k-1)).$$

Applying Lemma 3.2 to the left hand side and using (3.1) and (3.2) to estimate $P(A(k^3, 0, k^6 - i))$ one can easily obtain that

$$\lim_{k\to\infty} \sum_{i>k} (P(\zeta_i = -k+1) - P(\zeta_i = -k-1)) \leq \frac{1}{\sigma^2}. \tag{3.43}$$

Consider some A and B, $0 < A < B < \infty$.

When $k \to \infty$ using Lemma 3.2 we obtain

$$\sum_{Ak^2<i<Bk^2} (P(\zeta_i = -k+1) - P(\zeta_i = -k-1))$$

$$= \frac{1}{\sigma^2\sqrt{2\pi}} \sum_{Ak^2<i<Bk^2} \frac{\sigma^2}{k^2} \left(\frac{k}{\sigma\sqrt{i}}\right)^3 \exp\left\{-\frac{k^2}{2\sigma^2 i}\right\} (1 + o(1))$$

$$= \frac{1}{\sigma^2\sqrt{2\pi}} \int_{1/\sigma\sqrt{B}}^{1/\sigma\sqrt{A}} e^{-x^2/2} dx (1 + o(1)).$$

From this statement and (3.43) we get

$$\sum_{i<Ak^2} (P(\zeta_i = -k+1) - P(\zeta_i = -k-1)) = O\left(\int_{1/\sigma\sqrt{A}}^{\infty} e^{-x^2/2} dx\right)$$

and since A can be arbitrary small then for $m = o(k^2)$ we have

$$\sum_{i<m} (P(\zeta_i = -k+1) - P(\zeta_i = -k-1)) = o(1),$$

which completes the proof of the Lemma. □

Lemma 3.9. *If $k \to \infty$ then*

$$\lim k \sum_{i\geq k} P(C(k,i) = \frac{\sigma^2}{2}.$$

Proof. From Lemma 3.5 follows that it suffices to show that

$$\lim_{k\to\infty} kP(C(k,i)) \geq \frac{\sigma^2}{2}. \tag{3.44}$$

Evidently for every $s, k, n = 1, 2, 3, \ldots$

$$P(A(s+k, 0, n)) = \sum_{i<n} P(A(s,k,i)) P(C(k, n-i)).$$

Consider some $\varepsilon > 0$.

Let $s, k, n \to \infty$ simultaneously in a way that $n = \frac{s^2}{3\sigma^2}(1 + o(1))$ and $k = O(s)$, but $P(A(s,0,n)) - P(A(s+k,0,n)) < \frac{\varepsilon}{n}$.

If $0 < n_1 < n$ from (3.5) and Lemma 3.6

$$\begin{aligned}\sum_{i\le n_1} P(A(s,k,i))P(C(k,n-i)) &= O\left(\frac{1}{k^3}\right)\sum_{i\le n_1} P(A(s,k,i))\\ &\le O\left(\frac{1}{k^3}\right)\sum_{i\le n_1} P(\zeta_i = -s))\\ &= O\left(\frac{n_1}{k^3 s}\right),\end{aligned}$$

hence there exists such $\delta = \delta(\varepsilon) > 0$ that

$$\lim n \sum_{i\le \delta n} P(A(s,k,i))P(C(k,n-i)) < \varepsilon.$$

Therefore

$$\begin{aligned}\text{(3.45)}\ \lim n \sum_{i>\delta n} P(A(s,k,i))P(C(k,n-i)) &\ge \lim nP(A(s+k,0,n)) - \varepsilon\\ &\ge \lim nP(A(s,0,n)) - 2\varepsilon.\end{aligned}$$

We will also show that

$$\text{(3.46)}\ \lim n \sum_{i>\delta n} P(A(s,k,i))P(C(k,n-i)) \le \lim nP(A(s,k,n))\sum_{i\ge k} P(C(k,i)).$$

Without loss of generality we may assume that $\delta n > k$.

For every i, $\delta n < i < n$ we have

$$P(A(s,k,i)) = P(\zeta_i = -s) - P(B(s,k,i))$$

and since (3.37) holds either for $k = O(s)$ and $k = o(s)$ then

$$\begin{aligned}P(A(s,k,i)) &= P(\zeta_i = -s) - P(\zeta_i = -s-2k) + o\left(\frac{1}{s}\right)\\ &= 2k(P(\zeta_i = -s) - P(\zeta_i = -s-1))(1+o(1)).\end{aligned}$$

Now Lemma 3.2 yelds

$$\text{(3.47)}\qquad P(A(s,k,i)) = \frac{2ks}{\sqrt{2\pi}\sigma^3 i\sqrt{i}} e^{-s^2/2\sigma^2 i}(1+o(1))$$

and since the function $g(x) = x^3 x^{-x^2/2}$ reaches its maximum at $x = \sqrt{3}$ then

$$\begin{aligned} P(A(s,k,i)) &= \frac{2k}{s^2}\frac{1}{\sqrt{2\pi}}\frac{s^3}{\sigma^3 i^{3/2}}e^{-s^2/2\sigma^2 i}(1+o(1)) \\ &\le \frac{2k}{s^2}\frac{1}{\sqrt{2\pi}}\frac{s^3}{\sigma^3 n^{3/2}}e^{-s^2/2\sigma^2 n}(1+o(1)) \\ &= P(A(s,k,n))(1+o(1)). \end{aligned}$$

From this inequality and the simple fact that $\sum_{i>\delta n} P(C(k,i)) \le \sum_{i\ge k} P(C(k,i))$ follows that (3.46) holds.

Next, (3.45) and (3.46) imply

$$\sum_{i\ge k} P(C(k,i)) \ge \frac{\lim nP(A(s,0,n)) - 2\varepsilon}{\lim nP(A(s,k,n))}$$

Finally from (3.2) and (3.47) follows that

$$\sum_{i\ge k} P(C(k,i)) \ge \frac{\sigma^2}{2k}\left(1 - \sqrt{\frac{8\pi e^3}{3}\varepsilon}\right),$$

hence (3.44) holds and we are done. □

Lemma 3.10. *If $m, k \to \infty$ in a way that $mk^{-2} = o(1)$ then*

$$\sum_{n<m} P\left(\min_{j<n} S_j > 0; S_n = k\right) = o(1).$$

P r o o f. Evidently for $k = 1, 2, 3, \dots$

$$P(M' > k) = \sum_{n=1}^{\infty} P\left(\min_{j\le n} S_j > 0; S_n = k\right)\sum_{l=k}^{\infty} P(C(k,l)),$$

therefore from (3.3) as $k \to \infty$

$$k\sum_{n=1}^{\infty} P\left(\min_{j\le n} S_j > 0; S_n = k\right)\sum_{l=k}^{\infty} P(C(k,l)) = 1 + o(1).$$

Aplying Lemma 3.9 we get that

$$k\sum_{n=1}^{\infty} P\left(\min_{j\le n} S_j > 0; S_n = k\right)\frac{\sigma^2}{2k} = 1 + o(1),$$

i.e.

$$\sum_{n=1}^{\infty} P\left(\min_{j\le n} S_j > 0; S_n = k\right) = \frac{2}{\sigma^2} + o(1).$$

If $\delta > 0$ using Lemma 3.2 and Lemma 3.4 one can obtain

$$\begin{aligned}
\sum_{n\ge\delta k^2} P\left(\min_{j\le n} S_j > 0; S_n = k\right) &= \sum_{n\ge\delta k^2} (P(\zeta_n = -k+1) - P(\zeta_n = -k-1))(1+o(1)) \\
&= 2\sum_{n\ge\delta k^2} \frac{k}{\sqrt{2\pi}\sigma^3 n\sqrt{n}} \exp\{-k^2/2\sigma^2 n\}(1+o(1)) \\
&= \frac{2}{\sigma^2}\sum_{n\ge\delta k^2} \frac{k}{\sqrt{2\pi\sigma^2}}\frac{1}{n\sqrt{n}} \exp\{-k^2/2\sigma^2 n\}(1+o(1)).
\end{aligned}$$

Noting that as $k \to \infty$

$$\sum_{n>0} \frac{k}{\sqrt{2\pi\sigma^2}}\frac{1}{n\sqrt{n}} \exp\{-k^2/2\sigma^2 n\} = (1+o(1))$$

we conclude that if $m, k \to \infty$, $mk^{-2} = o(1)$

$$\sum_{n<m} P\left(\min_{j<n} S_j > 0; S_n = k\right) = o(1),$$

and the Lemma is proved. □

4. Proof of Theorem 2.1. Consider some $\varepsilon > 0$.

Let $k, n \to \infty$ in a way that $\dfrac{k}{\sigma\sqrt{n}} \to x$, where $0 < x < \infty$.

From Lemma 3.6 and Lemma 3.8 follows the existence of such $r = r(\varepsilon, x)$ that

$$\sum_{i<n} (P(\zeta_i = -(2r+1)k+1) - P(\zeta_i = -(2r+1)k-1))\, P(C(k, n-i)) < \frac{\varepsilon}{3k^3}. \tag{4.1}$$

From Lemma 3.6 and Lemma 3.10 there exists such $\delta_1 = \delta_1(\varepsilon, x) > 0$ that

$$\sum_{i<\delta_1 n} P\left(\min_{j<i} S_j > 0; S_i = k\right) P(C(k, n-i)) < \frac{\varepsilon}{3k^3}. \tag{4.2}$$

Moreover using Lemma 3.6 and Lemma 3.8 one can easily obtain that there exists such $\delta_2 = \delta_2(\varepsilon, x) > 0$ that

$$\sum_{i<\delta_2 n} (P(\zeta_i = -k+1) - P(\zeta_i = -k-1))P(C(k, n-i)) < \frac{\varepsilon}{3k^3}. \tag{4.3}$$

Let $\delta = \min(\delta_1, \delta_2)$.

The idea of the proof is to show that

$$P(M' > k, \tau = n) = \sum_{l=1}^{r} (P(A(2lk-1, 0, n)) - P(A(2lk+1, 0, n))) + o\left(\frac{1}{k^3}\right). \tag{4.4}$$

and use Lemma 3.2 to estimate the terms of the sum.

Write

$$\begin{aligned} P(M^{'} > k, \tau = n) &= \sum_{i=1}^{n-k} P\left(\min_{j<i} S_j > 0; S_i = k\right) P(C(k, n-i)) \\ &= \sum_{i<\delta n} P\left(\min_{j<i} S_j > 0; S_i = k\right) P(C(k, n-i)) \\ &\quad + \sum_{i\geq\delta n} P\left(\min_{j<i} S_j > 0; S_i = k\right) P(C(k, n-i)) \end{aligned} \tag{4.5}$$

Applying Lemma 3.4 and Lemma 3.6 we get

$$\begin{aligned} &\sum_{i\geq\delta n} P\left(\min_{j<i} S_j > 0; S_i = k\right) P(C(k, n-i)) \\ &\quad = \sum_{i\geq\delta n} (P(\zeta_i = -k+1) - P(\zeta_i = -k-1))P(C(k, n-i)) + o\left(\frac{1}{k^3}\right) \\ &\quad = \sum_{i=1}^{n-k} (P(\zeta_i = -k+1) - P(\zeta_i = -k-1))P(C(k, n-i)) \\ &\qquad - \sum_{i<\delta n} (P(\zeta_i = -k+1) - P(\zeta_i = -k-1))P(C(k, n-i)) + o\left(\frac{1}{k^3}\right). \end{aligned}$$

Hence from (4.5)

$$P(M'>k, \tau=n) = \sum_{i=1}^{n-k} (P(\zeta_i = -k+1) - P(\zeta_i = -k-1))P(C(k, n-i)) \tag{4.6}$$

$$+ \sum_{i<\delta n} P\left(\min_{j<i} S_j > 0; S_i = k\right) P(C(k, n-i))$$
$$- \sum_{i<\delta n} (P(\zeta_i = -k+1) - P(\zeta_i = -k-1)) P(C(k, n-i)) + o\left(\frac{1}{k^3}\right).$$

Obviously for each choice of $i, j, k = 1, 2, 3, \dots$ we have

$$P(\zeta_i = -j) = P(A(j, k, i)) + P(B(j, k, i)).$$

Then for every $l = 1, 2, \dots, r$

$$\begin{aligned}
&\sum_{i<n} (P(\zeta_i = -(2l-1)k+1) - P(\zeta_i = -(2l-1)k-1)) P(C(k, n-i)) \\
&\quad = \sum_{i<n} (P(A((2l-1)k-1, k, i)) + P(B((2l-1)k-1, k, i)) \\
&\qquad - P(A((2l-1)k+1, k, i)) - P(B((2l-1)k+1, k, i))) P(C(k, n-i)) \\
&\quad = \sum_{i<n} P(A((2l-1)k-1, k, i)) P(C(k, n-i)) \\
&\qquad - \sum_{i<n} P(A((2l-1)k+1, k, i)) P(C(k, n-i)) \\
&\qquad + \sum_{i<n} (P(B((2l-1)k-1, k, i)) - P(B((2l-1)k+1, k, i))) P(C(k, n-i)) \\
&\quad = P(A(2lk-1, 0, n)) - P(A(2lk+1, 0, n)) \\
&\qquad + \sum_{i<n} (P(B((2l-1)k-1, k, i)) - P(B((2l-1)k+1, k, i))) P(C(k, n-i))
\end{aligned}$$

and with the help of Lemma 3.7 we obtain for $l = 1, 2, 3, \dots, r$

$$\begin{aligned}
&\sum_{i<n} (P(\zeta_i = -(2l-1)k+1) - P(\zeta_i = -(2l-1)k-1)) P(C(k, n-i)) \\
&\quad = P(A(2lk-1, 0, n)) - P(A(2lk+1, 0, n)) \\
&\qquad + \sum_{i<n} (P(\zeta_i = -(2l+1)k+1) - P(\zeta_i = -(2l+1)k-1)) P(C(k, n-i)) + o\left(\frac{1}{k^3}\right).
\end{aligned}$$

Applying consequently this equality to (4.6) with $l = 1, 2, \ldots, r$ one gets

$$
\begin{aligned}
P(M' > k, \tau = n) = & \sum_{l=1}^{r} (P(A(2lk-1, 0, n)) - P(A(2lk+1, 0, n))) \\
& + \sum_{i<n} (P(\zeta_i = -(2r+1)k+1) - P(\zeta_i = -(2r+1)k-1)) \times \\
& \quad \times P(C(k, n-i)) \\
& + \sum_{i<\delta n} P\left(\min_{j<i} S_j > 0; S_i = k\right) P(C(k, n-i)) \\
& - \sum_{i<\delta n} (P(\zeta_i = -k+1) - P(\zeta_i = -k-1)) P(C(k, n-i)) + o\left(\frac{1}{k^3}\right)
\end{aligned}
$$

and from (4.1)-(4.3) we conclude that (4.4) holds.

Finally by using (3.1), (3.2) and Lemma 3.2 we get

$$
\begin{aligned}
P(M' > k, \tau = n) = & \sum_{l=1}^{r} \left(\frac{2lk-1}{n} P(\zeta_n = -2lk+1 \right) \\
& - \frac{2lk+1}{n} P(\zeta_n = -2lk-1)) + o\left(\frac{1}{k^3}\right) \\
= & \sum_{l=1}^{r} \frac{2lk+1}{n} (P(\zeta_n = -2lk+1) - P(\zeta_n = -2lk-1)) \\
& - 2\sum_{l=1}^{r} \frac{1}{n} P(\zeta_n = -2lk+1) + o\left(\frac{1}{k^3}\right) \\
= & 2\sum_{l=1}^{r} (\frac{2lk}{\sigma n})^2 P(\zeta_n = -2lk+1)(1+o(1)) \\
& - 2\sum_{l=1}^{r} \frac{1}{n} P(\zeta_n = -2lk+1) + o\left(\frac{1}{k^3}\right) \\
= & 2\sum_{l=1}^{r} \left(\frac{(2lk)^2}{\sigma^2 n^2} - \frac{1}{n} \right) \frac{1}{\sqrt{2\pi\sigma^2 n}} \exp\left\{ -\frac{(2lk)^2}{2\sigma^2 n} \right\} (1+o(1)) + o\left(\frac{1}{k^3}\right) \\
= & 2\sum_{l=1}^{r} \left(\left(\frac{2lk}{\sigma\sqrt{n}} \right)^2 - 1 \right) \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{n\sqrt{n}} \exp\left\{ -\frac{(2lk)^2}{2\sigma^2 n} \right\} + o\left(\frac{1}{k^3}\right).
\end{aligned}
$$

Noting also that

$$
n\sqrt{n} P(\tau = n) = \frac{1}{\sqrt{2\pi\sigma^2}} + o(1)
$$

we complete the proof of Theorem 2.1. □

5. Proof of Theorem 2.2. For every $n, k = 1, 2, 3, \dots\ , n \geq k$

$$P(\overline{M} > k, \tau = n) \leq P(M' \geq k, \tau = n).$$

Hence it suffices to show that if $n, k \to \infty$ in a way that $kn^{-1/2} = O(1)$ then

$$\lim P(\overline{M} > k, \tau = n) \geq \lim P(M' \geq k, \tau = n). \tag{5.1}$$

Denote for $k = 1, 2, 3, \dots$

$$\mu_k = \begin{cases} \max\{i : S_i = k, i < \tau\} & \text{if} \quad M' \geq k \\ 1 \quad \text{if} \quad M' < k \end{cases}$$

Let us define for $i = -1, -2, -3, \dots$

$$S_i \equiv 1 \ \text{ a.s..}$$

Consider some $\varepsilon > 0$.

From Theorem 2.1 for every $c > 0$ there exists such $A = A(\varepsilon, c) > 0$ that for each choice of $n, k = 1, 2, 3, \dots\ , k > cn^{1/2}$ we have

$$P(\overline{M} > Ak, \tau = n) \leq P(M' > Ak, \tau = n) < \frac{\varepsilon}{6} n^{-3/2}.$$

Then for every $k, n = 1, 2, 3, \dots\ , k > cn^{1/2}$ we get

$$\begin{aligned} (5.2)\ & P(\overline{M} > k - k^{3/4}, \tau = n) \\ &\geq P(Ak + k^{3/4} > \overline{M} > k - k^{3/4}, Ak > M' > k, \tau = n) + \frac{\varepsilon}{6} n^{-3/2} \\ &= P(Ak + k^{3/4} > \overline{M} > k - k^{3/4}, S_{\mu_k} = k, Ak > M', \tau = n) + \frac{\varepsilon}{6} n^{-3/2} \\ &\geq P(S_{\mu_k} = k, Ak > M', \max_{1 \leq j \leq Ak} |S_{\mu_k} - S_{\mu_k - j}| \leq k^{3/4}, \tau = n) + \frac{\varepsilon}{6} n^{-3/2} \\ &\geq P(S_{\mu_k} = k, \max_{1 \leq j \leq Ak} |S_{\mu_k} - S_{\mu_k - j}| \leq k^{3/4}, \tau = n) + \frac{\varepsilon}{3} n^{-3/2} \end{aligned}$$

Let $k, n \to \infty$ in a way that $kn^{-1/2} = O(1)$.

Then there exist such $c > 0$ and $A > 0$ that $k > cn^{1/2}$ and (5.2) holds.

Clearly for every $k, n, n_1 = 1, 2, 3, \dots, n_1 < n$

$$P(S_{\mu_k} = k, \mu_k \leq n_1, \tau = n) = \sum_{i \leq n_1} P(\min_{j<i} S_j > 0, S_i = k) P(C(k, n - i)),$$

therefore from Lemma 3.6 and Lemma 3.10 there exists such δ, $0 < \delta < 1$ that

$$P(S_{\mu_k} = k, \mu_k \le \delta n, \tau = n) < \frac{\varepsilon}{3} n^{-3/2}. \tag{5.3}$$

Hence

$$P(S_{\mu_k} = k, \max_{1\le j\le Ak} |S_{\mu_k} - S_{\mu_k - j}| \le k^{3/4}, \mu_k \le \delta n, \tau = n) < \frac{\varepsilon}{3} n^{-3/2}. \tag{5.4}$$

Without loss of generality we may assume that $\delta n > Ak$.
Then for $k, n = 1, 2, 3, \ldots$, $\delta n > Ak$ we have

$$\begin{aligned} &P\left(S_{\mu_k} = k, \max_{1\le j\le Ak} |S_{\mu_k} - S_{\mu_k - j}| \le k^{3/4}, \mu_k > \delta n, \tau = n\right) \\ &= \sum_{\delta n < i < n-k} \sum_{k-k^{3/4} < l < k+k^{3/4}} P\left(\min_{j<i-Ak} S_j > 0, S_{i-[Ak]} = l\right) \times \\ &\quad \times P\left(\max_{1\le j<Ak} |S_j + l - 1| < k^{3/4}, S_{[Ak]} = k - l + 1\right) P(C(k, n-i)) \\ &= \sum_{\delta n < i < n-k} \sum_{|k-l|<k^{2/3}} P\left(\min_{j<i-Ak} S_j > 0, S_{i-[Ak]} = l\right) \times \\ &\quad \times P\left(\max_{1\le j<Ak} |S_j + l - 1| < k^{3/4}, S_{[Ak]} = k - l + 1\right) P(C(k, n-i)) \\ &+ \sum_{\delta n < i < n-k} \sum_{k^{2/3}\le |k-l| < k^{3/4}} P\left(\min_{j<i-Ak} S_j > 0, S_{i-[Ak]} = l\right) \times \\ &\quad \times P\left(\max_{1\le j<Ak} |S_j + l - 1| < k^{3/4}, S_{[Ak]} = k - l + 1\right) P(C(k, n-i)). \end{aligned} \tag{5.5}$$

From Lemma 3.2 and Lemma 3.4 follows that if $|l - k| < k^{3/4}$ and $\delta n < i < n - k$ then

$$P(\min_{j<i-Ak} S_j > 0, S_{i-[Ak]} = l) = P(\min_{j<i} S_j > 0, S_i = k)(1 + o(1)).$$

Evidently

$$\begin{aligned} &\sum_{k^{2/3}\le |k-l| < k^{3/4}} P(\max_{1\le j<Ak} |S_j + l - 1| < k^{3/4}, S_{[Ak]} = k - l) \\ &< \sum_{k^{2/3}\le |k-l|} P(S_{[Ak]} = k - l) \ = \ o(1) \end{aligned}$$

and therefore

$$(5.6)\qquad \sum_{\delta n<i<n-k}\ \sum_{k^{2/3}\le|k-l|<k^{3/4}} P\left(\min_{j<i-Ak} S_j>0, S_{i-[Ak]}=l\right)\times$$

$$\times P\left(\max_{1\le j<Ak}|S_j+l-1|<k^{3/4}, S_{[Ak]}=k-l+1\right)P(C(k,n-i))=o(n^{-3/2}).$$

It is not difficult to see that

$$\sum_{|k-l|<k^{2/3}} P\left(\max_{1\le j<Ak}|S_j+l-1|<k^{3/4}, S_{[Ak]}=k-l+1\right)$$
$$=\sum_{|k-l|<k^{2/3}} P(S_{[Ak]}=k-l+1)+o(1) \ =\ 1+o(1),$$

hence

$$(5.7)\qquad \sum_{\delta n<i<n-k}\ \sum_{|k-l|<k^{2/3}} P\left(\min_{j<i-Ak} S_j>0, S_{i-[Ak]}=l\right)\times$$
$$\times P\left(\max_{1\le j<Ak}|S_j+l-1|<k^{3/4}, S_{[Ak]}=k-l+1\right)P(C(k,n-i))$$
$$=\sum_{\delta n<i<n-k} P\left(\min_{j<i} S_j>0, S_i=k\right)P(C(k,n-i))(1+o(1)).$$

Now from (5.5), (5.6) and (5.7) we get that if $k,n\to\infty$ in a way that $kn^{-1/2}=O(1)$ then

$$P\left(S_{\mu_k}=k, \max_{1\le j\le Ak}|S_{\mu_k}-S_{\mu_k-j}|\le k^{3/4}, \mu_k>\delta n, \tau=n\right)$$
$$=\sum_{\delta n<i<n-k} P\left(\min_{j<i} S_j>0, S_i=k\right)P(C(k,n-i))(1+o(1))$$
$$=P(S_{\mu_k}=k, \mu_k>\delta n, \tau=n)(1+o(1))$$

and obviously for sufficiently big k and n we have

$$P\left(S_{\mu_k}=k, \max_{1\le j\le Ak}|S_{\mu_k}-S_{\mu_k-j}|\le k^{3/4}, \mu_k>\delta n, \tau=n\right)$$
$$>P(S_{\mu_k}=k, \mu_k>\delta n, \tau=n)+\frac{\varepsilon}{3}n^{-3/2}.$$

Combining this statement with (5.2), (5.3) and (5.4) we obtain

$$P(\overline{M} > k - k^{3/4}, \tau = n) > P(S_{\mu_k} = k, \tau = n) + \varepsilon n^{-3/2},$$

hence

$$\lim P(\overline{M} > k - k^{3/4}, \tau = n) \geq \lim P(M' \geq k, \tau = n)$$

and (5.1) holds.

Theorem 2.2 is proved. □

Kennedy [11] has shown that if $a = \sum_{i=1}^{\infty} ip_i < \infty$ and the distribution $p_0, p_1, p_2, \ldots$ satisfies the condition B) we can define a new process $\{\tilde{Z}_t\}_{t=0}^{\infty}$ with offspring distribution $\tilde{p}_k = \alpha^k p_k / f(\alpha)$, $k = 0, 1, 2, \ldots$ which is a critical one. For each $n \geq 1$, $j \geq 1$, $0 \leq k_1, \ldots, k_j \leq n$ and all choices of $r_1, \ldots, r_j \geq 1$ we have

$$P(Z_{k_1} = r_1, \ldots, Z_{k_j} = r_j | N = n) = P(\tilde{Z}_{k_1} = r_1, \ldots, \tilde{Z}_{k_j} = r_j | \tilde{N} = n),$$

where $\tilde{N}$ denotes the total progeny of $\{\tilde{Z}_t\}$.

From this construction and Theorem 2.2 it is clear that Theorem 2.3 holds.

6. An Application for Random Trees. Consider a random labelled rooted tree T_n with n nodes. For each node the number of arcs, connected to it, excluding the one that belongs to the path, leading to the root, is called a number of its direct successors. Let $Z_t(r, T_n)$ be the number of the nodes in the tree with height t and exactly r direct successors, $n = 1, 2, 3, \ldots$; $r = 0, 1, \ldots, n - t$.

Let G be a critical BGW process with a Poisson offspring distribution of one particle with parameter 1. Let us denote the number of particles in the t-th generation of G with exactly r direct successors by $Z_t(r, G)$, $t, r = 0, 1, 2, \ldots$. Let $N(G)$ denote the total progeny of the process.

Using Theorem 2.2 and the arguments of Kolchin [13, Th.7] one can prove Corollary 2.1.

Acknowledgment. The author wishes to thank Dr. L. Mutafchiev, G. Yanev and Dr. N. Yanev for their help.

REFERENCES

[1] S. R. Adke. The maximum population size in the first N generations of a branching process. *Biometrics* **20** (1964), 649-651.

[2] K. B. Athreya. On the maximum sequences in a critical branching process. *Ann. Probab.* **16** (1988), 502-507.

[3] K. A. Borovkov, V. A. Vatutin. On distribution tails and expectations of maxima in critical branching process. *J. Appl. Probab.* **33** (1996), 614-622.

[4] J. Bishir. Maximum population size in a branching process. *Biometrics* **18** (1962), 394-403.

[5] K. L. Chung. Maxima in Brownian excursions. *Bull. Amer. Math. Soc.* **81** (1975), 742-745.

[6] R. Durrett, D. Iglehart. Functionals of Brownian meander and Brownian excursion. *Ann. Prob.* **5** (1977), 130-135.

[7] M. Dwass. The total progeny in a branching process. *J. Appl. Prob.* **6** (1966), 682-686.

[8] W. D. Kaigh. An invariance principle for random walk conditioned on late return to zero. *Ann. Prob.* **4** (1976), 115-121.

[9] W. D. Kaigh. An elmentary derivation of the distribution of the maxima of Brownian meander and Brownian excursion. *Rocky Mountain J. Math.* **8** (1978), 641-645.

[10] K. Kämmerle, H. J. Schuh. The maximum in critical Galton-Watson and birth and death processes. *J. Appl. Probab.* **23** (1986), 601-613.

[11] D. P. Kennedy. The Galton-Watson process conditioned on the total progeny. *J.Appl. Prob.* **12** (1975), 800-806.

[12] Tz. B. Kerbashev. A Refinment of a local limit theorem for branching process, conditioned on the total progeny. Preprint No. 6, Institute of Mathematics and Informatics, Sofia, 1994.

[13] V. F. Kolchin. Branching processes, random trees and the generalized scheme of allocating particles. *Math. Notes* **21**, *5* (1977), 691-705 (in Russian).

[14] V. F. Kolchin. Random mappings. Optimization Software, Inc. Publication Division, New York, 1986.

[15] T. Lindvall. On the maximum of a branching process. *Scand. J. Statist.* **3** (1976), 209-214.

[16] A. G. PAKES. Remarks on the maxima of a martingale sequence with applications to the simple critical branching process. *J. Appl. Probab.* **24** (1987), 768-772.

[17] L. SMITH, D. DIACONIS. Honest Bernoulli excursions. *J. Appl. Prob.* **25** (1988), 464-477.

[18] A. SPĂTARU. Equations concerning maxima of critical Galton-Watson process. *Rev. Roumaine Math. Puras Appl.* **37**, *10* (1992), 935-937.

[19] V. A. VATUTIN. The distribution of the distance to the root and the minimal subtree containing all vertices of a given height. *Theory Probab. Appl.* **38**, *2* (1993), 273-287 (in Russian).

[20] V. A. VATUTIN. On the maximum of a simple random walk. *Theory Probab. Appl.* **40**, *2* (1995), 412-417 (in Russian).

[21] V. A. VATUTIN,V. A. TOPCHII. The maximum of critical Galton-Watson processes and the left-continuous random walk. *Theory Probab. Appl.* **42**, *1* (1997), 21-34 (in Russian).

[22] H. WEINER. Moments of the maximum in a critical branching process. *J. Appl. Probab.* **21** (1984), 920-923.

P.O. Box 256
7201 Razgrad
Bulgaria
e-mail: tkerbashev@yahoo.com

Received December 17, 1998