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A CAUCHY INTEGRAL RELATED TO A ROBOT-SAFETY DEVICE SYSTEM

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Abstract. We introduce a robot-safety device system attended by two different repairmen. The twin system is characterized by the natural feature of cold standby and by an admissible "risky" state. In order to analyse the random behaviour of the entire system (robot, safety device, repair facility) we employ a stochastic process endowed with probability measures satisfying general Hokstad-type differential equations. The solution procedure is based on the theory of sectionally holomorphic functions, characterized by a Cauchy-type integral defined as a Cauchy principal value in double sense. An application of the Sokhotski-Plemelj formulae determines the long-run availability of the robot-safety device. Finally, we consider the particular but important case of deterministic repair.

1. Introduction. Innovations in the field of microelectronics and micromechanics have enhanced the involvement of "smart" robots in all kind of advanced technical systems [2].

Unfortunately, no robot is completely reliable. Therefore, up-to-date robots are often connected with a safety device [3]-[5]. Such a device prevents possible damage, caused by a robot failure, in the robot's neighbouring environment.

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However, the random behaviour of the entire system (robot, safety device, repair facility) could jeopardize some prescribed safety requirements. For instance, if we allow the robot to operate during the repair time of the failed safety device. Such a "risky" state is called admissible, if the associate event: "The robot is operative but the safety device is under repair", constitutes a *rare* event. Therefore, an appropriate statistical analysis of robot-safety device systems is quite indispensable to support the system designer in problems of risk acceptance and safety assessments.

In order to avoid undesirable delays in repairing failed units, we introduce a robot-safety device attended by *two* different repairmen (henceforth called a **T**-system). The **T**-system satisfies the usual conditions, i.e. independent identically distributed random variables and perfect repair [6].

Each repairman has his own particular task. Repairman $\mathcal{S}$ is skilled in repairing the safety unit, whereas repairman $\mathcal{R}$ is an expert in repairing robots. Both repairmen are jointly busy if, and only if, both units (robot + safety device) are down. In the other case, at least one repairman is idle. In any case, the safety device always waits (in cold standby [1]) until the repair of the robot has been completed.

In order to analyse the random behaviour of the **T**-system, we introduce a stochastic process endowed with probability measures satisfying general Hokstad-type differential equations. The solution procedure is based on the theory of sectionally holomorphic functions [7], characterized by a Cauchy-type integral defined as a Cauchy principal value in double sense. An application of the Sokhotski-Plemelj formulae determines the long-run availability of the robot-safety device.

Finally, we consider the particular but important case of deterministic repair.

2. Formulation. Consider a **T**-system satisfying the usual conditions.

The robot has a constant failure rate $\lambda > 0$ and a general repair time distribution $R(\bullet), R(0) = 0$ with mean ρ and variance σ^2.

The *operative* safety device has a constant failure rate $\lambda_s > 0$ but a zero failure rate in standby (the so-called cold standby state) and a general repair time distribution $R_s(\bullet), R_s(0) = 0$ with mean ρ_s and variance σ_s^2. The corresponding repair times are denoted by r and r_s.

Characteristic functions (and their duals) are formulated in terms of a complex transform variable. For instance,

$$\mathbf{E}e^{i\omega r} = \int_0^\infty e^{i\omega x} dR(x), \mathrm{Im}\ \omega \geq 0.$$

Note that

$$\mathbf{E}e^{-i\omega r} = \int_0^\infty e^{-i\omega x} dR(x) = \int_{-\infty}^0 e^{i\omega x} d\left(1 - R((-x)-)\right), \mathrm{Im}\ \omega \leq 0.$$

The corresponding Fourier-Stieltjes transforms are called *dual* transforms.

Without loss of generality (see our forthcoming remarks) we may assume that both repair time distributions have bounded densities (in the Radon-Nikodym sense) defined on [0,∞).

In order to analyse the random behaviour of the **T**-system, we introduce a stochastic process $\{N_t, t \geq 0\}$ with arbitrary discrete state space $\{A, B, C, D\} \subset [0, \infty)$, characterized by the following events:

$\{N_t = A\}$: "Both units are operating in parallel at time t."

$\{N_t = B\}$: "The robot is operative but the safety device is under repair at time t." State B is the so-called risky state.

$\{N_t = C\}$: "The safety device is in cold standby and the robot is under repair at time t."

$\{N_t = D\}$: "Both units are simultaneously down at time t."

The following Figure 1 shows a functional block-diagram of the **T**-system operating in states A, B, C and D.

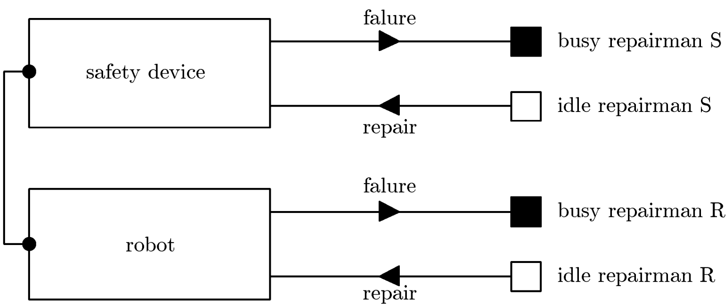

Fig. 1. Functional block-diagram of the **T**-system operating in states A, B, C, D.

A Markov characterization of the process $\{N_t\}$ is piecewise and conditionally defined by:

$\{N_t\}$, if $N_t = A$ (i.e. if the event $\{N_t = A\}$ occurs).
$\{N_t, X_t\}$, if $N_t = B$, where X_t denotes the *remaining* repair time of the safety device in progressive repair at time t.
$\{N_t, Y_t\}$, if $N_t = C$, where Y_t denotes the *remaining* repair time of the robot in progressive repair at time t.
$\{(N_t, X_t, Y_t)\}$, if $N_t = D$.

The state space of the underlying Markov process is given by

$$\{A\} \cup \{(B,x); x \geq 0\} \cup \{(C,y); y \geq 0\} \cup \{(D,x,y); x \geq 0, y \geq 0\}\,.$$

Next, we consider the **T**-system in stationary state (the so-called ergodic state) with *invariant* measure $\{p_K; K = A, B, C, D\}$, $\sum_K p_K = 1$, where

$$p_K := \mathbf{P}\{N = K\} := \lim_{t\to\infty} \mathbf{P}\{N_t = K | N_0 = A\}\,.$$

Finally, we introduce the measures

$$\begin{aligned}
\varphi_B(x)dx &:= \mathbf{P}\{N = B, X \in dx\} := \lim_{t\to\infty} \mathbf{P}\{N_t = B, X_t \in dx | N_0 = A\}\,,\\
\varphi_C(y)dy &:= \mathbf{P}\{N = C, Y \in dy\} := \lim_{t\to\infty} \mathbf{P}\{N_t = C, Y_t \in dy | N_0 = A\}\,,\\
\varphi_D(x,y)dxdy &:= \mathbf{P}\{N = D, X \in dx, Y \in dy\}\\
&:= \lim_{t\to\infty} \mathbf{P}\{N_t = D, X_t \in dx, Y_t \in dy | N_0 = A\}\,.
\end{aligned}$$

Notations

The robot and the safety device are only jointly available (operative) in state A. Therefore, the long-run availability of the robot-safety device, denoted by $\mathcal{A}$, is given by p_A.
The real line and the complex plane are denoted by $\mathbf{R}$ and $\mathbf{C}$ with obvious superscript notations $\mathbf{C}^+$, $\mathbf{C}^-$, $\mathbf{C}^+ \cup \mathbf{R}$, $\mathbf{C}^- \cup \mathbf{R}$. For instance, $\mathbf{C}^+ := \{\omega \in \mathbf{C} : \mathrm{Im}\,\omega > 0\}$.
The indicator of an event $\mathcal{E}$ is denoted by

$$\mathbb{1}(\mathcal{E}) := \begin{cases} 1, & \text{if } \mathcal{E} \text{ occurs,}\\ 0, & \text{otherwise.}\end{cases}$$

Note that, for instance,

$$\mathbf{E}\left\{e^{i\omega X} e^{i\eta Y} \mathbb{1}(N = D)\right\} = \int_0^\infty \int_0^\infty e^{i\omega x} e^{i\eta y} \varphi_D(x,y)dxdy, \mathrm{Im}\ \omega \geq 0, \mathrm{Im}\ \eta \geq 0.$$

So that,

$$p_D = \int_0^\infty \int_0^\infty p_D(x,y)dxdy.$$

Finally, we propose the following risk-criterion: State B is admissible if p_B satisfies the relation $p_B < \delta << 1$ for some $\delta > 0$, called the security level.

3. Differential equations. In order to determine the φ-functions, we first construct a system of steady-state Hokstad-type differential equations based on a time independent version of Hokstad's supplementary variable technique (see e.g. [8]).

Proposition 3.1. *The φ-functions satisfy the following set of steady-state Hokstad-type differential equations.*
For $x > 0, y > 0$,

$$\begin{aligned}
(\lambda_s + \lambda)p_A &= \varphi_B(0) + \varphi_C(0), \\
\left(\lambda - \frac{d}{dx}\right)\varphi_B(x) &= \varphi_D(x,0) + \lambda_s p_A \frac{d}{dx} R_s(x), \\
-\frac{d}{dy}\varphi_C(y) &= \varphi_D(0,y) + \lambda p_A \frac{d}{dy} R(y), \\
\left(-\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\varphi_D(x,y) &= \lambda \varphi_B(x) \frac{d}{dy} R(y).
\end{aligned}$$

P r o o f. The above differential equations are similar to the equations derived in [8]. We refer to Ref [8, page 526] for further technical details.

4. Solution procedure. Note that our equations are well adapted to an integral transformation. The integrability of the φ-functions and their corresponding derivatives implies that each φ-function vanishes at infinity *irrespective* of the asymptotic behaviour of the underlying repair time densities! Applying a routine Fourier transform technique to the equations and invoking the boundary condition $(\lambda_s + \lambda)p_A = \varphi_B(0) + \varphi_C(0)$, reveals that

$$\begin{gathered}
i(\omega + \eta)\mathbf{E}\left\{e^{i\omega X}e^{i\eta Y}\mathbb{1}(N = D)\right\} + \left(\lambda(1 - \mathbf{E}e^{i\eta r}) + i\omega\right)\mathbf{E}\left\{e^{i\omega X}\mathbb{1}(N = B)\right\} + \\
i\eta\mathbf{E}\left\{e^{i\eta Y}\mathbb{1}(N = C)\right\} + \lambda p_A(1 - \mathbf{E}e^{i\eta r}) + \lambda_s p_A(1 - \mathbf{E}e^{i\omega r_s}) = 0.
\end{gathered} \tag{1}$$

Observe that Eq (1) holds for any pair $(\omega, \eta) \in \mathbf{C}\times\mathbf{C}$: $\operatorname{Im}\omega \geq 0$, $\operatorname{Im}\eta \geq 0$. Therefore, substituting $\omega = t$, $\eta = -t(t \in \mathbf{R})$ into Eq (1), yields the functional equation

$$\psi^+(t) - \psi^-(t) = \varphi(t), \tag{2}$$

where

$$\psi^+(t) := p_A^{-1}\mathbf{E}\left\{e^{itX}1\!\!1(N = B)\right\},$$

$$\psi^-(t) := \frac{1}{1+\lambda\rho\varphi^-(t)}p_A^{-1}\mathbf{E}\left\{e^{-itY}1\!\!1(N = C)\right\} - \frac{\lambda\rho\varphi^-(t)}{1+\lambda\rho\varphi^-(t)},$$

$$\varphi^-(t) := \frac{1-\mathbf{E}e^{-itr}}{it\rho}, \quad \varphi^-(0) := 1, \quad \varphi^+(t) := \frac{\mathbf{E}e^{itr_s}-1}{it\rho_s},$$

$$\varphi^+(0) := 1, \quad \varphi(t) := \frac{\lambda_s\rho_s\varphi^+(t)}{1+\lambda\rho\varphi^-(t)}.$$

Eq (2) constitutes a Plemelj boundary value problem on the real line which can be solved by the theory of sectionally holomorphic functions. First, we need the following

Definition 4.1. *Let $f(t), t \in \mathbf{R}$ be a bounded and continuous function. f is called Γ-integrable, if*

$$\lim_{\substack{T\to\infty \\ \varepsilon\downarrow 0}} \int_{L_{T,\varepsilon}} f(t)\frac{dt}{t-u}, u \in \mathbf{R},$$

exists, where $L_{T,\varepsilon} := (-T, u-\varepsilon] \cup [u+\varepsilon, T)$. The corresponding integral, denoted by

$$\frac{1}{2\pi i}\int_\Gamma f(t)\frac{dt}{t-u},$$

is called a Cauchy principal value in double sense.

Proposition 4.1. *The Cauchy-type integral*

$$\frac{1}{2\pi i}\int_\Gamma \varphi(\tau)\frac{d\tau}{\tau-\omega},$$

exists for all $\omega \in \mathbf{C}$ (real or complex).

P r o o f. φ is (uniformly) Lipschitz continuous on $\mathbf{R}$. (Indeed, $|\varphi'(t)|$ is bounded on $\mathbf{R}$. Therefore, our assertion follows from the mean value theorem). Finally, φ is Hölder continuous at infinity, i.e. $|\varphi(t)| = O(|t|^{-1})$, if $|t| \to \infty$.

Consequently, the Cauchy-type integral exists for *all* $\omega \in \mathbf{C}$ (real or complex) and defines a sectionally holomorphic function.

Proposition 4.2. *For the* $\mathbf{T}$*-system, satisfying the usual conditions, the long-run availability of the robot-safety device is given by*

$$\mathcal{A} = \frac{1}{(1+\lambda\rho)(1+\psi^{+}(0))},$$

where

$$\psi^{+}(0) = \frac{1}{2}\varphi(0) + \frac{1}{2\pi i}\int_{\Gamma}\varphi(\tau)\frac{d\tau}{\tau}.$$

P r o o f. An application of Rouché's theorem reveals that the function $1+\lambda\rho\varphi^{-}(\omega)$, Im $\omega \leq 0$, has no zeros in $\mathbf{C}^{-} \cup \mathbf{R}$. Consequently, the function $\psi^{-}(\omega)$, Im $\omega \leq 0$, is analytic in $\mathbf{C}^{-}$, bounded and continuous on $\mathbf{C}^{-} \cup \mathbf{R}$, whereas $\psi^{+}(\omega)$, Im $\omega \geq 0$, is analytic in $\mathbf{C}^{+}$, bounded and continuous on $\mathbf{C}^{+} \cup \mathbf{R}$ and

$$\lim_{\substack{|\omega|\to\infty \\ \pi\leq\arg\omega\leq 2\pi}} \psi^{-}(\omega) = \lim_{\substack{|\omega|\to\infty \\ 0\leq\arg\omega\leq\pi}} \psi^{+}(\omega) = 0.$$

A straightforward application of the Cauchy formulae for the regions $\mathbf{C}^{+}$ and $\mathbf{C}^{-}$, entails that

$$\mathbf{E}\left\{e^{i\omega X}\mathbb{1}(N=B)\right\} = p_A\frac{1}{2\pi i}\int_{\Gamma}\varphi(\tau)\frac{d\tau}{\tau-\omega}, \omega \in \mathbf{C}^{+},$$

$$\mathbf{E}\left\{e^{-i\omega Y}\mathbb{1}(N=C)\right\} =$$

$$= p_A\left\{\left(1+\lambda\rho\varphi^{-}(\omega)\right)\frac{1}{2\pi i}\int_{\Gamma}\varphi(\tau)\frac{d\tau}{\tau-\omega} + \lambda\rho\varphi^{-}(\omega)\right\}, \omega \in \mathbf{C}^{-}.$$

In particular, $p_B = p_A\psi^{+}(0)$, where

$$\psi^{+}(0) = \lim_{\substack{\omega\to 0 \\ \omega\in\mathbf{C}^{+}}} \frac{1}{2\pi i}\int_{\Gamma}\varphi(\tau)\frac{d\tau}{\tau-\omega}.$$

Note that by the Sokhotski-Plemelj formulae,

$$\psi^+(0) = \frac{1}{2}\varphi(0) + \frac{1}{2\pi i}\int_\Gamma \varphi(\tau)\frac{d\tau}{\tau}.$$

Moreover,

$$p_D = \lim_{\substack{\eta\to 0\\ \eta\in\mathbf{C}^+}} \lim_{\substack{\omega\to 0\\ \omega\in\mathbf{C}^+}} \mathbf{E}\left\{e^{i\omega X}e^{i\eta Y}\mathbb{1}(N=D)\right\}.$$

Applying the limit procedure to Eq (1) and invoking the condition $p_A + p_B + p_C + p_D = 1$, yields the additional relations

$$p_A + p_B = \frac{1}{1+\lambda\rho}\ ;\ p_C + p_D = \frac{\lambda\rho}{1+\lambda\rho}\ ;\ p_D + p_B = \lambda_s\rho_s p_A.$$

Substituting $p_B = p_A\psi^+(0)$ into the first relation yields $p_A = [(1+\lambda\rho)(1+\psi^+(0))]^{-1}$. Observe that we have completely determined the invariant measure simply and solely depending upon $\psi^+(0)$.

Remarks. Note that the kernel $\varphi(t), t\in\mathbf{R}$, preserves all the relevant properties to ensure the existence of the Cauchy integral for *arbitrary* repair time distributions with finite mean and variance. First of all, the order relation $|\varphi(t)| = O(|t|^{-1})$, $|t|\to\infty$, also holds for arbitrary characteristic functions. Moreover, the H-continuity of φ on $\mathbf{R}$ does not depend on the canonical structure (decomposition) of R or R_s. For instance, the Hölder inequality

$$\left|\mathbf{E}e^{it_2 r} - \mathbf{E}e^{it_1 r}\right| \le \rho\left|t_2 - t_1\right|, (t_1, t_2\in\mathbf{R}),$$

always holds for *any* r with mean ρ.
The requirement of finite variances is extremely mild since the current probability distributions employed to model repair times [1] even have moments of all orders!

Consequently, our initial assumptions concerning the existence of repair time densities are totally superfluous to ensure the existence of an invariant measure.

5. Example. As an example, we consider the case of deterministic repair, i.e., for $t_0 > 0$ and $\theta < 1$, let

$$R(x) = \begin{cases} 1, & \text{if } x \ge t_0,\\ 0, & \text{if } x < t_0,\end{cases} \qquad R_s(x) = \begin{cases} 1, & \text{if } x \ge \theta t_0,\\ 0, & \text{if } x < \theta t_0.\end{cases}$$

Theorem 5.1. *For a* $\mathbf{T}$*-system with deterministic repair, the long-run availability of the robot-safety device is given by*

$$\mathcal{A} = \frac{1}{(1 + \lambda^{-1}\lambda_s(1 - e^{-\lambda t_0}))\,(1 + \lambda\theta t_0)}.$$

P r o o f. Clearly, $\mathbf{E}e^{itr} = e^{itt_0}$, $\rho = t_0, \sigma^2 = 0$ and $\mathbf{E}e^{-itr_s} = e^{-it\theta t_0}$, $\rho_s = \theta t_0, \sigma_s^2 = 0$.

From the identity

$$\frac{1}{2\pi i}\int_\Gamma \varphi(t)\frac{dt}{t-\omega} = \begin{cases} \dfrac{i\lambda_s(e^{-\lambda t_0} - e^{i\omega t_0})}{\omega - i\lambda}, \text{ if } \operatorname{Im}\omega > 0, \omega \neq i\lambda, \\ \\ \lambda_s t_0 e^{-\lambda t_0}, \text{ if } \omega = i\lambda, \end{cases}$$

we obtain $\psi^+(0) = \lambda^{-1}\lambda_s(1 - e^{-\lambda t_0})$.

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