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NULL CONDITION FOR SEMILINEAR WAVE EQUATION WITH VARIABLE COEFFICIENTS

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Abstract. In this work we analyse the nonlinear Cauchy problem

$$(\partial_{tt}-\Delta)u(t,x)=\left\langle\left(\lambda g+O\left(\frac{1}{(1+t+|x|)^{a}}\right)\right)\nabla_{t,x}u(t,x),\nabla_{t,x}u(t,x)\right\rangle,$$

whit initial data $u(0,x)=\epsilon\, u_0(x), u_t(0,x)=\epsilon\, u_1(x)$. We assume $a\geq 1$, $x\in\mathbb{R}^n$ $(n\geq 3)$ and g the matrix related to the Minkowski space. It can be considerated a pertubation of the case when the quadratic term has constant coefficient λg (see Klainerman [6])
We prove a global existence and uniqueness theorem for very regular initial data. The proof avoids a direct application of Klainermann method (Null condition, energy conformal method), because the result is obtained by a combination beetwen the energy estimate (norm L^2) and the decay estimate (norm L^∞).

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Introduction. In this work we study the nonlinear Cauchy problem for the wave equation

$$(1)\qquad \begin{aligned} (\partial_{tt}-\Delta)u(t,x) &= F(t,x,\nabla_{t,x}u(t,x)),\\ u(0,x) &= \epsilon\, u_0(x),\\ u_t(0,x) &= \epsilon\, u_1(x). \end{aligned}$$

Here $x \in \mathbb{R}^n$, the space dimension is $n \geq 3$ and the nonlinear term $F(t,x,w)$ is assumed to be a quadratic function in w with variable coefficients depending on t, x. This quadratic form we can represent as follows

$$F(t,x,w) = \langle Q(t,x)w, w\rangle,$$

where $w \in \mathbb{R}^{n+1}$ and $\langle \cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^{n+1}$.

For the matrix $Q(t,x)$ we assume

$$Q(t,x) = \lambda g + O\left(\frac{1}{(1+t+|x|)^a}\right),$$

where $a \geq 1$, a is an integer, and g is the matrix related to the Minkowski space $\mathbb{R}^{n+1}$

$$g = \begin{pmatrix} -1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The constant coefficient case, that is $Q(t,x) = \lambda g$, was studied by S. Klainerman:

Theorem [S.Klainerman, 1984]**.** *If $u_0(x), u_1(x) \in C_0^\infty(\mathbb{R}^n)$ and the nonlinear term $F(t,x,\nabla_{t,x}u(t,x))$ satisfies the "Null Condition", there is ϵ_0, sufficiently small, so that, for all $0 < \epsilon \leq \epsilon_0$, the solution of the Cauchy problem (1) with $Q(t,x) = \lambda g$ exists in the class $C^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$.*

Remark 1. The choice $Q(t,x) = \lambda g$ in the above theorem of Klainerman is closely connected with the null condition definition (see [4, 6]). Recall that the quadratic form

$$\langle Qw, w\rangle$$

(Q being a constant symmetric matrix) satisfies the null condition, if the condition $\langle gw, w\rangle = 0$ implies that $\langle Qw, w\rangle = 0$. Therefore, for any null co-vector w (i.e. on the surface $\langle gw, w\rangle = 0$) we have $\langle Q\,w, w\rangle = 0$.

Our goal is to study the case when $Q(t,x)$ has additional perturbation term of type

$$O\left(\frac{1}{(1+t+|x|)^a}\right).$$

Under the assumption $a \geq 1$, a an integer, we aim at proving the existence of global solution for the Cauchy problem (1) with small initial data $\epsilon u_0(x), \epsilon u_1(x) \in C_0^\infty(\mathbb{R}^n)$.

It seems that the case of quadratic form with variable coefficients needs a specific approach, since the method of Klainerman (see [6]) can not be applied directly for this case. More precisely, we need a suitable apriori estimate for quadratic forms with variable coefficients. Nevertheless, exploiting the fact that the decay rate of the perturbed coefficients is $a \geq 1$ we are able to adapt the method of Klainerman for this problem.

Our main result is the following.

Theorem 1. *Let $n \geq 3$, $u_0(x), u_1(x) \in C_0^\infty(\mathbb{R}^n)$ and $s \geq 2([n/2]+2)$. There is $\epsilon_0 = \epsilon_0(\|u_0\|_{H^s}, \|u_1\|_{H^{s-1}})$, sufficiently small, so that, for all $0 < \epsilon \leq \epsilon_0$, there exists a unique solution to the Cauchy problem* (1)

$$u(t,x) \in \bigcap_{k=0,\dots,s} C^{s-k}([0,+\infty[;H^k(\mathbb{R}^n)).$$

Moreover we have the decay estimate

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{(1+t)^{\frac{n-1}{2}}}.$$

The main idea of the proof is based on a combination between energy inequality and decay estimates.

A direct application of Klainerman method is complicated due to the fact we have variable coefficients.

Moreover, modifiyng the pointwise estimates from Klainerman it is possible to obtein a suitable estimate for

$$\|F(t,x,\nabla_{t,x}u(t,x))\|_{L^\infty_x}$$

On the basis of a direct application of Fourier transform we establish conformal energy estimate for the solution and its higher derivatives.

A combination beetwen this energy estimate and decay estimate for wave equation leads to global existence result and gives also informations about the decay of solution of nonlinear wave equation.

The plan of the work is the following. In Section 1 we present the differential operators of Poincarè group. In Section 2 we give some L^∞ – estimates about the nonlinear term $F(t, x, \nabla_{t,x}u(t, x))$. In Section 3 we prove a generalized energy estimate and, finally, in Section 4 we show the global existence of the solution to the problem (1).

Remark 2. It's possible to improve the estimates presented in this work. In fact, in Theorem 1 we need $s \geq 2([n/2] + 2)$; this follows because we use a subset (the Poincarè group and the radial scalar field (Scaling)) of the conformal group.

1. The generators of the Poincarè group. We consider the Minkowski space $\mathbb{R} \times \mathbb{R}^n$ with $x_0 = t$, $x = (x_1, \ldots, x_n)$ and

$$g = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

If $\overline{x} = (x_0, x) \in \mathbb{R} \times \mathbb{R}^n$, the metrics in the Minkowski space is defined by

$$[\overline{x}]^2 = -g^{ab}x_a x_b = t^2 - |x|^2, \tag{2}$$

where $|x|$ is the euclidean metrics.

Set $\partial_0 = -\partial_t$ and $\partial_i = \partial_{x_i}$, for $i = 1, \ldots, n$.

We have

$$(\partial_{tt} - \Delta) = -g^{ab}\partial_a\partial_b, \tag{3}$$

with $a, b = 0, 1, \ldots, n$.

Let's introduce the following differential operators:

– the classical partial derivates ∂_a, for $a = 0, 1, \ldots, n$,

– the spatial rotations $\Omega_{ij} = x_i\partial_j - x_j\partial_i$, for $i, j = 1, \dots, n$,
– the spatial-time rotations $\Omega_{0i} = t\partial_i + x_i\partial_t$, for $i = 1, \dots, n$,
– the radial vector field (Scaling) $S = t\partial_t + x_1\partial_1 + \cdots + x_n\partial_n$.

It's clear that these operators satisfy the standard commutative properties:

a) $[\Omega_{ab}, (\partial_{tt} - \Delta)] = 0$, for $a, b = 0, 1, \dots, n$,

b) $[S, (\partial_{tt} - \Delta)] = -2(\partial_{tt} - \Delta)$,

c) $[S, \Omega_{ab}] = 0$, for $a, b = 0, 1, \dots, n$,

d) $[S, \partial_a] = -\partial_a$, for $a = 0, 1, \dots, n$.

Therefore,we can choose $\mathcal{A}$ as the family of operators whose elements are:

$$\Omega = \{\Omega_{ab}\}_{a,b=0,\dots,n},$$

$$\partial = \{\partial_a\}_{a=0,\dots,n},$$

and the scalar field S.

In this way, $\mathcal{A}$ results a subset of the conformal group.

Let's now denote $z = \{z_k\}_{k=1,\dots,p} \subset \mathcal{A}$.

We are able to introduce a generalized Sobolev norm, as follows:

$$\|u(t,\cdot)\|_{z,s}^2 = \sum_{|\alpha| \le s} \|z^\alpha u(t,\cdot)\|_{L^2(R^n)}^2, \tag{4}$$

where

$$z^\alpha = \prod_{k=1,\dots,p} {z_k}^{\alpha_k}. \tag{5}$$

2. Preliminary results. Let us consider the Cauchy problem

$$\begin{aligned} (\partial_{tt} - \Delta)u(t,x) &= \left\langle \left(\lambda g + o\left(\frac{1}{(1+t+|x|)^a} \right) \right) \nabla_{t,x}u(t,x), \nabla_{t,x}u(t,x) \right\rangle, \\ u(0,x) &= \epsilon u_0(x), \\ u_t(0,x) &= \epsilon u_1(x). \end{aligned} \tag{6}$$

In order to obtain a-priori estimate for the quadratic term

$$\langle \lambda g \nabla_{t,x} u(t,x)\,, \nabla_{t,x} u(t,x) \rangle = \lambda \left(-|\partial_t u(t,x)|^2 + |\nabla_x u(t,x)|^2 \right) \tag{7}$$

we use the generator

$$\Omega_{0i} = t\partial_i + x_i\partial_t\,, \tag{8}$$

and we express the usual partial derivatives as follows

$$\partial_i = \frac{\Omega_{0i}}{t} - \frac{x_i\partial_t}{t}\,. \tag{9}$$

Substituting in (7), we get

$$|\langle \lambda g\,\nabla_{t,x}u, \nabla_{t,x}u\rangle| =$$

$$= |\lambda| \left| -\frac{\partial_t u}{t}\left(t\partial_t u + \sum_{1\le i\le n} x_i\partial_i u \right) + \sum_{1\le i\le n} \frac{\Omega_{0i}u}{t}\partial_i u \right| =$$

$$= |\lambda| \left| -\frac{\partial_t u}{t} Su + \sum_{1\le i\le n} \frac{\Omega_{0i}u}{t}\partial_i u \right|.$$

Therefore, for $t > 0$ and $\Omega_0 = (\Omega_{01}, \ldots, \Omega_{0n})$, we have the estimate

$$|\langle \lambda g \nabla_{t,x}u(t,x)\,, \nabla_{t,x}u(t,x)\rangle| \le \frac{C(|\lambda|)}{t}\,|(S,\Omega_0)\,u(t,x)|\;|\nabla_{t,x}u(t,x)|\,. \tag{10}$$

Remark 3. For $t = 0$ the estimate (10) is not valid: in fact, the identity (9) works well only for $t \neq 0$.

On the other hand, for $t + |x| \to +\infty$,

$$\left| \left\langle O\left(\frac{1}{(1{+}t{+}|x|)^a}\right) \nabla_{t,x}u(t,x), \nabla_{t,x}u(t,x) \right\rangle \right| \le \frac{1}{(1{+}t{+}|x|)^a}\,|\nabla_{t,x}u(t,x)|^2\,. \tag{11}$$

Remark 4. By the Huygens principle (the support of $u(t,x)$ travels in this way: $|x| \le R + t$), we find

$$\frac{1}{(1+t+|x|)^a} \simeq \frac{1}{(1+t)^a}\,. \tag{12}$$

If we want to give similar estimates for t near 0,we have to modify (9). For this, we proceed as follows.

We divide (8) for $(1 + t)$:

$$\frac{\Omega_{0i}}{1+t} = \frac{t\partial_i}{1+t} + \frac{x_i\partial_t}{1+t}\,, \tag{13}$$

and we express the partial derivatives as follows

$$\partial_i = \frac{\partial_i}{1+t} + \frac{\Omega_{0i}}{1+t} - \frac{x_i \partial_t}{1+t}\,. \tag{14}$$

By a straightforward calculation, we get

$$|\,\langle \lambda g\, \nabla_{t,x} u(t,x), \nabla_{t,x} u(t,x)\rangle\,| \leq$$

$$\leq \left|\left\langle \lambda g \left[\left(\partial_t u, -\frac{x_1\partial_t u}{1{+}t}, \ldots, -\frac{x_n \partial_t u}{1{+}t}\right) + \left(0, \frac{\Omega_{01} u}{1{+}t}, \ldots, \frac{\Omega_{0n} u}{1{+}t}\right)\right], (\partial_t u, \partial_1 u, \ldots, \partial_n u)\right\rangle\right| +$$

$$+ \left|\left\langle \lambda g \left(0, \frac{\partial_1 u}{1+t}, \ldots, \frac{\partial_n u}{1+t}\right), (\partial_t u, \partial_1 u, \ldots, \partial_n u)\right\rangle\right|. \tag{14}$$

Now, the first term in the right side of the above expression is equal to

$$|\lambda| \left| -(\partial_t u)^2 - \frac{x_1 \partial_t u}{1+t}\partial_1 u - \cdots - \frac{x_n \partial_t u}{1+t}\partial_n u + \frac{\Omega_{01} u}{1+t}\partial_1 u + \cdots + \frac{\Omega_{0n} u}{1+t}\partial_n u \right| =$$

$$= |\lambda| \left| -\frac{\partial_t u}{1+t}\left((1+t)\partial_t u + \sum_{1\leq i\leq n} x_i \partial_i u\right) + \sum_{1\leq i\leq n} \frac{\Omega_{0i} u}{1+t}\partial_i u \right| \leq$$

$$\leq |\lambda| \left| -\frac{\partial_t u}{1+t}\left(t\partial_t u + \sum_{1\leq i\leq n} x_i \partial_i u\right) + \sum_{1\leq i\leq n} \frac{\Omega_{0i} u}{1+t}\partial_i u \right| + |\lambda| \left| -\frac{\partial_t u}{1+t}\partial_t u \right|.$$

From this estimates it follows that this term is not greater than

$$\frac{C(|\lambda|)}{1+t}|(S,\Omega_0)u||\nabla_{t,x} u| + \frac{C(|\lambda|)}{1+t}|\partial_t u||\partial_t u|\,.$$

For the second term in the right side of (14) we have

$$\left|\left\langle \lambda g \left(0, \frac{\partial_1 u}{1+t}, \ldots, \frac{\partial_n u}{1+t}\right), (\partial_t u, \partial_1 u, \ldots, \partial_n u)\right\rangle\right| \leq \frac{C(|\lambda|)}{1+t}|\nabla_x u|^2. \tag{15}$$

Therefore, for any $t \geq 0$, we give the estimate

$$|\,\langle \lambda g\, \nabla_{t,x} u(t,x), , \nabla_{t,x} u(t,x)\rangle\,| \leq \frac{C(|\lambda|)}{1+t}|(S,\Omega_0,\Omega,\partial_t,\partial_x)u(t,x)||\nabla_{t,x} u(t,x)|. \tag{16}$$

So, by (11) and (16), we obtein a suitable estimate for

$$F(t,x,\nabla_{t,x}u(t,x)) = \langle Q(t,x)\,\nabla_{t,x}u(t,x), \nabla_{t,x}u(t,x)\rangle\,,$$

where

$$Q(t,x) = \lambda g + O\left(\frac{1}{(1+t+|x|)^a}\right).$$

This is our main a priori estimate for the quadratic non-linearity.

3. Generalized energy estimate. The classical energy estimate for

$$\begin{aligned}(\partial_{tt}-\Delta)u(t,x) &= F(t,x),\\ u(0,x) &= 0,\\ u_t(0,x) &= 0.\end{aligned} \tag{17}$$

gives

$$\|\nabla_{t,x}u(t,\cdot)\|_{L^2} \le C\int_0^t \mathrm{d}s \|F(s,\cdot)\|_{L^2}\,.$$

To control the L^2- norm of the solution $u(t,x)$ to problem (17), one can use the conformal energy method developed by Klainerman (see [6]). But, in this section, our approach is more direct and simplified. In fact we shall use Fourier transform

$$\hat{u}(t,\xi) = \int \mathrm{d}x e^{-ix\xi}u(t,x)$$

in combination with the Hardy inequality.

In particular, we prove

Lemma 1. *If* $\mathrm{supp}(F(s,y)) \subseteq \{|y| \le s+R\}$, *we can establish for problem* (17) *the estimate*

$$\|u(t,\cdot)\|_{L^2} \le C\int_0^t \mathrm{d}s(R+s)\|F(s,\cdot)\|_{L^2}\,.$$

To prove Lemma 1, we cite

Lemma 2. *Let* $\chi(t,\xi)$ *be an integrable function with respect to the time and space. Then*

$$\left\|\int_0^t \mathrm{d}s\,\chi(s,\cdot)\right\|_{L^2_\xi} \le \int_0^t \mathrm{d}s\,\|\chi(s,\cdot)\|_{L^2_\xi}$$

P r o o f o f L e m m a 2. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^n}\mathrm{d}\xi\left(\int_0^t \mathrm{d}s\,\chi(s,\xi)\right)^2\right) =$$

$$= 2\int_{\mathbb{R}^n}\mathrm{d}\xi\left(\chi(t,\xi)\int_0^t \mathrm{d}s\,\chi(s,\xi)\right);$$

By Scwhartz inequality, the last term is less than

$$2\left\|\int_0^t \mathrm{d}s\,\chi(s,\cdot)\right\|_{L^2_\xi}\|\chi(t,\cdot)\|_{L^2_\xi};$$

By integrating with respect to t, we prove Lemma 2. □

P r o o f o f L e m m a 1. By using the Fourier transform, problem (17) becomes

$$\begin{aligned}\partial_{tt}\hat{u}(t,\xi)+|\xi|^2\hat{u}(t,\xi) &= \hat{F}(t,\xi),\\ \hat{u}(0,\xi) &= 0,\\ \hat{u}_t(0,\xi) &= 0.\end{aligned}$$

The Duhamels principle gives

$$\hat{u}(t,\xi)=\int_0^t \mathrm{d}s\frac{\sin\left((t-s)|\xi|\right)}{|\xi|}\hat{F}(s,\xi).$$

In a standard way, we find

$$|\hat{u}(t,\xi)|\le\int_0^t \mathrm{d}s\left|\frac{\sin\left((t-s)|\xi|\right)}{|\xi|}\hat{F}(s,\xi)\right|\le\int_0^t \mathrm{d}s\frac{|\hat{F}(s,\xi)|}{|\xi|}$$

and

$$\left(\int_{R^n}\mathrm{d}\xi|\hat{u}(t,\xi)|^2\right)^{\frac{1}{2}}\le\left(\int_{R^n}\mathrm{d}\xi\left(\int_0^t \mathrm{d}s\frac{|\hat{F}(s,\xi)|}{|\xi|}\right)^2\right)^{\frac{1}{2}}.$$

The Plancharel identity

$$\|u(t,\cdot)\|_{L^2_x}=(2\pi)^{-n/2}\|\hat{u}(t,\cdot)\|_{L^2_\xi}$$

in combination with Lemma 2, give

$$\|u(t,\cdot)\|_{L^2_x} \le C \int_0^t \mathrm{d}s \left\| \frac{\hat{F}(s,\xi)}{|\xi|} \right\|_{L^2_\xi}.$$

Now, by the Huygens principle, it's clear that, for every fixed s, $F(s,x) \in C_0^\infty(\mathbb{R}^n)$. For this, its Fourier transform with respect to the space decay very rapidly at infinity.

We apply the Hardy inequality (we are working in the case $n = 3$) and we get:

$$\int_0^t \mathrm{d}s \left\| \frac{\hat{F}(s,\xi)}{|\xi|} \right\|_{L^2_\xi} \le C \int_0^t \mathrm{d}s \|\nabla_\xi \hat{F}(s,\xi)\|_{L^2_\xi}.$$

Since $\nabla_\xi \hat{F}(s,\xi) = -i\,(\widehat{xF})(\xi)$, we finally prove that

$$\|u(t,\cdot)\|_{L^2_x} \le C \int_0^t \mathrm{d}s \|xF(s,x)\|_{L^2_x} \le$$

$$\le C \int_0^t \mathrm{d}s (R+s) \|F(s,x)\|_{L^2_{\{|x|\le R+s\}}}. \qquad \square$$

Our next step is to obtein higher order derivative estimates involving the operators in the family $\mathcal{A}$. This complete the proof of the global existence and uniqueness to the solution of problem (1).

Let's denote, for semplicity, $z = (\Omega, \Omega_0, S, \partial_t, \partial_x)$.

Proposition 1. *We have the following energy estimate:*

(18)
$$\sum_{0\le|\alpha|\le s} \|z^\alpha u(t,\cdot)\|_{L^2(R^n)} \le C \sum_{0\le|\alpha|\le s} \|z^\alpha u(0,\cdot)\|_{L^2(R^n)} +$$

$$+ C \sum_{0\le|\alpha|\le s} \int_0^t \mathrm{d}s (1+s) \|z^\alpha F(s,\cdot)\|_{L^2(R^n)}.$$

Let's recall, for the moment, the commutative properties

1) $[(\partial_{tt} - \Delta), \Omega_{ij}] = 0$, for $i,j = 1,\dots,n$,

2) $[(\partial_{tt} - \Delta), \Omega_{0i}] = 0$, for $i = 1,\dots,n$,

3) $[(\partial_{tt} - \Delta), S] = 2(\partial_{tt} - \Delta)$.

P r o o f o f P r o p o s i t i o n 1. In general, for the problem

$$(19)\qquad \begin{aligned} (\partial_{tt}-\Delta)u(t,x) &= F(t,x),\\ u(0,x) &= u_0(x),\\ u_t(0,x) &= u_1(x), \end{aligned}$$

we know the standard estimate for the solution $u(t,x)$:

$$(20)\qquad \|u(t,\cdot)\|_{L^2} \le C\|u(0,\cdot)\|_{L^2} + C\int_0^t \mathrm{d}s \||y|F(s,\cdot)\|_{L^2}\,.$$

Remark 5. As we said before, S. Klainerman proved similar energy estimates by using the energy conformal method. Here, instead, we have initial data with compact support and $F(t,x,\nabla_{t,x}u(t,x))$ is a quadratic term in $\nabla_{t,x}u(t,x)$. We can therefore only work with Fourier transform.

We proceed in the proof of Proposition 1. We analyse two different cases:
1) If $\Gamma = \partial_x^\alpha$ or ∂_t^k or Ω or Ω_0, then $[(\partial_{tt}-\Delta),\Gamma]=0$ and

$$(21)\qquad (\partial_{tt}-\Delta)\,\Gamma^\alpha u(t,x) = \Gamma^\alpha F(t,x),$$

with initial data $\Gamma^\alpha u(0,x)$ and $\Gamma^\alpha u_t(0,x)$;

2) If, instead, we have the operator S, then

$$(22)\qquad (\partial_{tt}-\Delta)\,Su(t,x) = (2+S)F(t,x),$$

with initial data $Su(0,x)$ and $Su_t(0,x)$.

So, in general, for $z=(\Omega,\Omega_0,S,\partial_t,\partial_x)$,

$$(23)\qquad (\partial_{tt}-\Delta)\,z^\alpha u(t,x) = \sum_{|\beta|\le|\alpha|} C_\beta^\alpha z^\beta F(t,x),$$

with initial data $z^\alpha u(0,x)$ e $z^\alpha u_t(0,x)$.

By denoting

$$u^{(\alpha)}(t,x) = z^\alpha u(t,x)$$

and

$$F^{(\alpha)}(t,x) = \sum_{|\beta|\le|\alpha|} C_\beta^\alpha z^\beta F(t,x),$$

we find (see (20))

$$\|u^{(\alpha)}(t,\cdot)\|_{L^2} \le C\|u^{(\alpha)}(0,\cdot)\|_{L^2} + C\int_0^t \mathrm{d}s\, \||y|F^{(\alpha)}(s,\cdot)\|_{L^2}\,. \tag{24}$$

The Huygens principle states that

$$\|z^\alpha u(t,\cdot)\|_{L^2} \le C\|z^\alpha u(0,\cdot)\|_{L^2} + C\int_0^t \mathrm{d}s\,(1+s)\left\|\sum_{|\beta|\le|\alpha|} C^\alpha_\beta z^\beta F(s,\cdot)\right\|_{L^2} \tag{25}$$

or, equivalently,

$$\|z^\alpha u(t,\cdot)\|_{L^2} \le C\|z^\alpha u(0,\cdot)\|_{L^2} + C\int_0^t \mathrm{d}s\,(1+s)\sum_{|\beta|\le|\alpha|} C^\alpha_\beta \left\|z^\beta F(s,\cdot)\right\|_{L^2}\,.$$

Finally

$$\sum_{0\le|\alpha|\le s}\|z^\alpha u(t,\cdot)\|_{L^2} \le C\sum_{0\le|\alpha|\le s}\|z^\alpha u(0,\cdot)\|_{L^2} + C\sum_{0\le|\alpha|\le s}\int_0^t \mathrm{d}s\,(1+s)\|z^\alpha F(s,\cdot)\|_{L^2}.$$

We recall that

$$F(t,x,\nabla_{t,x}u) = \left\langle \left(\lambda g + o\left(\frac{1}{(1+t+|x|)^a}\right)\right)\nabla_{t,x}u(t,x)\,,\nabla_{t,x}u(t,x)\right\rangle$$

and that

$$|\left\langle \lambda g\,\nabla_{t,x}u(t,x)\,,\nabla_{t,x}u(t,x)\right\rangle| \le \frac{C(|\lambda|)}{1+t}|(S,\Omega_0,\Omega,\partial_t,\partial_x)u(t,x)||\nabla_{t,x}u(t,x)|.$$

Moreover, if we consider the higher derivatives, we find

$$z^\alpha\left\langle \lambda g\nabla_{t,x}u\,,\nabla_{t,x}u\right\rangle = \lambda\sum_{\beta+\gamma+\delta=\alpha} C^\alpha_{\beta,\gamma,\delta}\left\langle z^\beta g\, z^\gamma\nabla_{t,x}u\,, z^\delta\nabla_{t,x}u\right\rangle = \tag{26}$$

$$= \lambda\sum_{\gamma+\delta=\alpha} C^\alpha_{\gamma,\delta}\left\langle g\, z^\gamma\nabla_{t,x}u\,, z^\delta\nabla_{t,x}u\right\rangle.$$

Before getting on in the proof of Proposition 1, we need to prove the identity

$$z^\alpha\nabla_{t,x}u = \sum_{\beta\le\alpha} C_\beta\nabla_{t,x}z^\beta u \tag{27}$$

P r o o f o f i d e n t i t y (27). For simplicity, we show that

$$[\nabla_{t,x}, (\Omega, \Omega_0, S, \partial_t, \partial_x)] = \sum_{i=0,\dots,n} c_i\, \partial_i\,.$$

In fact it's sufficient to show that

$$[\partial_i, \Omega_{km}] = [\partial_i, x_k\partial_m] - [\partial_i, x_m\partial_k] = \delta_{ik}\partial_m - \delta_{im}\partial_k\,.$$

with c_i the Kronecker deltas. □

By using (27) we write (26) in the equivalent expression

$$\lambda \sum_{\gamma+\delta\le\alpha} C^{\alpha}_{\gamma,\delta} \left\langle g\, \nabla_{t,x} z^{\gamma} u\,, \nabla_{t,x} z^{\delta} u \right\rangle\,. \tag{28}$$

So

$$|\lambda| \left|\left\langle g\nabla_{t,x}z^{\gamma}u, \nabla_{t,x}z^{\delta}u\right\rangle\right| \le \frac{C(|\lambda|)}{1+t}|(S,\Omega_0,\Omega,\partial_t,\partial_x)z^{\gamma}u(t,x)||\nabla_{t,x}z^{\delta}u(t,x)|. \tag{29}$$

and, in L^2 norm,

$$|\lambda|\ \| \left\langle g\, \nabla_{t,x} z^{\gamma} u\,, \nabla_{t,x} z^{\delta} u \right\rangle \|_{L^2} \le \tag{30}$$

$$\le \frac{C(|\lambda|)}{1+t}\|(S,\Omega_0,\Omega,\partial_t,\partial_x)z^{\gamma}u(t,\cdot)\|_{L^2}\|\nabla_{t,x}z^{\delta}u(t,\cdot)\|_{L^\infty}.$$

On the other hand,

$$\left\|\left\langle o\left(\frac{1}{(1+t+|x|)^a}\right)\nabla_{t,x}u(t,\cdot)\,, \nabla_{t,x}u(t,\cdot)\right\rangle\right\|_{L^2} \le$$

$$\le \frac{1}{(1+t+|x|)^a}\|\nabla_{t,x}u(t,\cdot)\|_{L^\infty}\|\nabla_{t,x}u(t,\cdot)\|_{L^2}$$

and, for generical derivatives z^{α} we proceed as follows.

First, we consider

$$z^{\alpha}\left\langle o\left(\frac{1}{(1+t+|x|)^a}\right)\nabla_{t,x}u(t,x)\,, \nabla_{t,x}u(t,x)\right\rangle = \tag{31}$$

$$= \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\left(z^{\alpha_1}\frac{1}{(1+t+|x|)^a}\right)\left(z^{\alpha_2}\nabla_{t,x}u(t,x)\right)\left(z^{\alpha_3}\nabla_{t,x}u(t,x)\right)\,;$$

then, we exchange, by (27), z^α with $\nabla_{t,x}$. Finally we prove:

$$\left| z^\beta \frac{1}{(1+t+|x|)^a} \right| \leq \frac{C}{(1+t+|x|)^a}. \tag{32}$$

for every z^β.

P r o o f o f i d e n t i t y (32). We consider, for semplicity, $z = \Omega_{0i} = t\partial_i + x_i\partial_t$; in this case we get

$$\left| (t\partial_i + x_i\partial_t) \frac{1}{(1+t+|x|)^a} \right| \leq \frac{|a|t}{(1+t+|x|)^{a+1}} + \frac{|a||x|}{(1+t+|x|)^{a+1}}\,.$$

Since $\operatorname{supp}(u(t,x)) \subseteq \{|x| \leq 1+t\}$, for $t+|x| \to +\infty$, the expression above decay as

$$\frac{C}{(1+t)^a}\,. \qquad \square$$

Therefore, for $n = 3$,

$$\sum_{0\leq|\alpha|\leq s} \left\| z^\alpha \left\langle o\left(\frac{1}{(1+t+|x|)^a}\right) \nabla_{t,x} u(t,\cdot)\,, \nabla_{t,x} u(t,\cdot) \right\rangle \right\|_{L^2} \tag{33}$$

$$\leq \sum_{0\leq|\alpha|\leq s} \sum_{\gamma+\delta\leq\alpha} C^\alpha_{\gamma,\delta} \frac{1}{(1+t)^a} \|\nabla_{t,x} z^\gamma u(t,\cdot)\|_{L^2} \|\nabla_{t,x} z^\delta u(t,\cdot)\|_{L^\infty},$$

where we can also assume that $|\gamma| \leq s/2$.

This complete the proof of Proposition 1. $\square$

4. The global existence. In this last section we prove the global existence of the solution to the problem (1). For this purpose we use the continuation principle of differential equations.

Let's define

$$f(t) = \sum_{0\leq|\alpha|\leq s} \|z^\alpha u(t,\cdot)\|_{L^2} \tag{34}$$

and

$$g(t) = \sum_{0\leq|\alpha|\leq s/2} (1+t)\|z^\alpha u(t,\cdot)\|_{L^\infty}. \tag{35}$$

We first prove:

$$f(t) \leq C\epsilon + \int_0^t \mathrm{d}\tau \frac{f(\tau)\,g(\tau)}{(1+\tau)} \tag{36}$$

$$g(t) \le C\epsilon + \sup_{0\le\tau\le t} \frac{f(\tau)\, g(\tau)}{(1+\tau)^{-\delta+\frac{1}{2}}} \tag{37}$$

with δ a sufficiently small positive constant(we assume that $0 < \delta < 1/2$).

(36) and (37) imlpy that there exists two constants $C \ge 0$, $\sigma \ge 0$ so that

$$\begin{cases} f(t) \le C\,\epsilon\,(1+t)^{\sigma}, \\ g(t) \le C\,\epsilon\,. \end{cases}$$

P r o o f o f (36). Let us recall formula (18). Its first member is $f(t)$, while our hypothesis on the initial data assure that

$$\sum_{0\le|\alpha|\le s} \|z^{\alpha} u(0,\cdot)\|_{L^2(R^n)} \le C\epsilon\,. \tag{38}$$

On the other and,

$$\sum_{0\le|\alpha|\le s} \left\| z^{\alpha} \left\langle \left(\lambda g + o\left(\frac{1}{(1+t+|x|)^{a}} \right) \right) \nabla_{t,x} u(t,\cdot)\,, \nabla_{t,x} u(t,\cdot) \right\rangle \right\|_{L^2} \le$$

$$\le \sum_{0\le|\alpha|\le s} \|z^{\alpha} \left\langle \lambda g\, \nabla_{t,x} u(t,\cdot)\,, \nabla_{t,x} u(t,\cdot) \right\rangle \|_{L^2} +$$

$$+ \sum_{0\le|\alpha|\le s} \left\| z^{\alpha} \left\langle o\left(\frac{1}{(1+t+|x|)^{a}} \right)\, \nabla_{t,x} u(t,\cdot)\,, \nabla_{t,x} u(t,\cdot) \right\rangle \right\|_{L^2}.$$

If we recall (30) and (33) and if we notice that, for $t \to +\infty$, ($a \ge 1$)

$$\frac{1}{(1+t)^{a}} \le \frac{1}{(1+t)}, \tag{39}$$

we get

$$\sum_{|\alpha|\le s} \int_0^t \mathrm{d}s(1+s) \|z^{\alpha} F(s,\cdot)\|_{L^2(R^n)} \le \int_0^t \mathrm{d}s(1+s) \frac{C}{(1+s)^2} f(s)\, g(s)\,, \tag{40}$$

and so (36). $\square$

To prove (37) we cite from [2] the following

Lemma 3. *Let $u(t,x)$ be a solution to the problem* (17), *with supp*$(F(t,x)) \subseteq \{|x| \le 1+t\}$.

Let's denote

$$|u(t,\cdot)|_k = \sum_{0\le|\alpha|\le k} \|z^\alpha u(t,\cdot)\|_{L^\infty}\,, \tag{41}$$

where $z \in (\Omega, \Omega_0, \partial_t, \partial_x)$ *(we are excluding the Scaling operator).*

By [2] *we have, for any* $t \ge 0$,

$$|u(t,\cdot)|_k \le \frac{C}{(1+t)} \left(\sum_{j=0,\dots,+\infty} \sup_{s\in I_j\cap[0,t]} 2^{3j/2} \|F(s,\cdot)\|_{k+3} \right)\,, \tag{42}$$

where $I_j = [2^{j-1}, 2^{j+1}]$, *for* $j > 0$, *and* $I_0 = [0,2]$ *are the supports of the functions related to the Littlewood-Paley decomposition.*

Proof of (37). By (35)

$$g(t) = (1+t)|u(t,\cdot)|_{s/2}\,. \tag{43}$$

By calling $\psi_j(s)$ the Littlewood-Paley functions, we give an estimate equivalent to (42)

$$|u(t,\cdot)|_{s/2} \le \frac{C}{(1+t)} \left(\sum_{j=0,\dots,+\infty} \sup_{\tau\in I_j\cap[0,t]} 2^{3j/2} \psi_j(\tau) \|F(\tau,\cdot)\|_{(s/2)+3} \right)\,. \tag{44}$$

In (44) initial data are 0. But, if it's not the case,the Von Wahl estimates give:

$$g(t) \le C\epsilon + C \sum_{j=0,\dots,+\infty} \sup_{\tau\in I_j\cap[0,t]} 2^{3j/2} \psi_j(\tau) \|F(\tau,\cdot)\|_{(s/2)+3}\,. \tag{45}$$

Since, in the Littlewood-Paley decomposition, $2^{3j/2}$ is equivalent to $\tau^{3/2}$, we find

$$g(t) \le C\epsilon + \tag{46}$$

$$\sum_{j=0}^{+\infty} \sup_{\tau\in I_j\cap[0,t]} \tau^{3/2}\psi_j(\tau) \sum_{|\alpha|\le(s/2)+3} \left\| z^\alpha \left\langle \left(\lambda g + o\left(\frac{1}{(1+\tau+|x|)^a} \right) \right) \nabla_{t,x} u, \nabla_{t,x} u \right\rangle \right\|_{L^2}.$$

If we choose $s \ge (s/2)+3$ or, equivalently, $s \ge 6$ we get

$$\sum_{|\alpha|\le(s/2)+3} \left\| z^\alpha \left\langle \left(\lambda g + o\left(\frac{1}{(1+\tau+|x|)^a} \right) \right) \nabla_{t,x} u, \nabla_{t,x} u \right\rangle \right\|_{L^2} \le \frac{C}{(1+\tau)^2} f(\tau) g(\tau), \tag{47}$$

and so

$$g(t) \leq C\epsilon + \sum_{j=0,\dots,+\infty} \sup_{\tau\in I_j\cap[0,t]} \tau^{3/2}\psi_j(\tau)\frac{C}{(1+\tau)^2} f(\tau)\, g(\tau)\,. \tag{48}$$

But $\forall j = 0,\dots,+\infty\,,\ \exists\, C \geq 0$ so that $0 \leq \psi_j(\tau) \leq C$;if we multiply and divide for $(1+\tau)^\delta$, with $0<\delta<1/2$ we find

$$g(t) \leq C\epsilon + \sum_{j=0,\dots,+\infty} \sup_{\tau\in I_j\cap[0,t]} \frac{1}{(1+\tau)^\delta}\frac{C}{(1+\tau)^{\frac{1}{2}-\delta}} f(\tau)\, g(\tau)\,.$$

Moreover $(1+\tau)^\delta \simeq \left(2^{(j-1)}\right)^\delta$ and so

$$g(t) \leq C\epsilon + \sup_{\tau\in[0,t]} \frac{C}{(1+\tau)^{\frac{1}{2}-\delta}} f(\tau)\, g(\tau) \sum_{j=0,\dots,+\infty} \frac{1}{\left(2^{(j-1)}\right)^\delta}.$$

The convergence of the series prove (37).

Proposition 2. *We are now able to show our main result: there exists two constants $C \geq 0$, $\sigma > 0$ so that*

$$\begin{cases} f(t) \leq C\,\epsilon\,(1+t)^\sigma, \\ g(t) \leq C\,\epsilon\,. \end{cases}$$

Proof. Proof of Proposition 2. Let

$$h(t) = f(t)\,(1+t)^{-\frac{1}{2}+\delta}; \tag{49}$$

by (36) it's clear that

$$h(t) \leq C\epsilon(1+t)^{-\frac{1}{2}+\delta} + (1+t)^{-\frac{1}{2}+\delta}\int_0^t \mathrm{d}\tau(1+\tau)^{-\frac{1}{2}-\delta}(1+\tau)^{-\frac{1}{2}+\delta} f(\tau)\, g(\tau) =$$

$$= C\epsilon(1+t)^{-\frac{1}{2}+\delta} + (1+t)^{-\frac{1}{2}+\delta}\int_0^t \mathrm{d}\tau(1+\tau)^{-\frac{1}{2}-\delta} h(\tau)\, g(\tau),$$

while

$$g(t) \leq C\epsilon + \sup_{0\leq\tau\leq t} h(\tau)\, g(\tau)\,.$$

The corresponding monotone functions

$$H(t) = \sup_{0\leq\tau\leq t} h(\tau) \tag{50}$$

and

$$G(t) = \sup_{0 \leq \tau \leq t} g(\tau). \tag{51}$$

verify:

$$H(t) \leq C\epsilon(1+t)^{-\frac{1}{2}+\delta} + (1+t)^{-\frac{1}{2}+\delta} H(t)\, G(t) \int_0^t \mathrm{d}\tau (1+\tau)^{-\frac{1}{2}-\delta}$$

and

$$G(t) \leq C\epsilon + H(t)\, G(t)\,.$$

By solving the integral in the time, we get

$$H(t) \leq C\epsilon(1+t)^{-\frac{1}{2}+\delta} + \frac{1}{\frac{1}{2}-\delta} H(t)\, G(t) - \frac{(1+t)^{-\frac{1}{2}+\delta}}{\frac{1}{2}-\delta} H(t)\, G(t)\,. \tag{52}$$

It follows:

$$\begin{aligned} H(t) &\leq C\epsilon + \frac{1}{\frac{1}{2}-\delta} H(t)\, G(t), \\ G(t) &\leq C\epsilon + H(t)\, G(t). \end{aligned} \tag{53}$$

We rewrite (53) in the form

$$\begin{aligned} H(t) &\leq C\,\epsilon + \frac{1}{\frac{1}{2}-\delta} \Big((H(t))^2 + (G(t))^2 \Big)\,, \\ G(t) &\leq C\epsilon + (H(t))^2 + (G(t))^2. \end{aligned} \tag{54}$$

In this way, if

$$Y(t) = (H(t),\, G(t)) \in \mathbb{R}^2\,, \tag{55}$$

has norm

$$\|Y(t)\|^2_{\mathbb{R}^2} = (H(t))^2 + (G(t))^2\,, \tag{56}$$

system (54) results equivalent to

$$\|Y(t)\|_{\mathbb{R}^2} \leq C\epsilon + C\|Y(t)\|^2_{\mathbb{R}^2}\,. \tag{57}$$

Initial data sufficiently small gives the stability of the system (57): in fact, $H(0) \leq C\epsilon$ and $G(0) \leq C\epsilon$ imply

$$\|Y(0)\|_{\mathbb{R}^2} \leq C\epsilon \tag{58}$$

and, by (57), $\forall t \geq 0$,

(59) $$\|Y(t)\|_{\mathbb{R}^2} \leq C\epsilon\,.$$

Therefore, we obtein

(60) $$\begin{aligned} H(t) &\leq C\epsilon, \\ G(t) &\leq C\epsilon\,, \end{aligned}$$

and so

(61) $$\begin{aligned} h(t) &\leq C\epsilon\,, \\ g(t) &\leq C\epsilon. \end{aligned}$$

(61) and the continuation principle of differential equations (see [3, 9]) say that there $\exists\, C > 0$, $\sigma = \frac{1}{2} - \delta > 0$ so that, $\forall t \geq 0$,

$$\begin{cases} f(t) \leq C\epsilon(1+t)^{\frac{1}{2}-\delta}, \\ g(t) \leq C\epsilon. \end{cases}$$ □

Remark 6. In particular we have obtained:

1) $u(t,x)$ and its derivatives are estimated in the L^∞ norm by $\dfrac{C\epsilon}{(1+t)}$;

2) $u(t,x)$ and its derivatives are estimated in the L^2 norm by $C\epsilon(1+t)^{\frac{1}{2}-\delta}$.

Remark 7. We have proved our result in the case $n = 3$. The same works well for $n > 3$.

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