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THE INTERVAL [0,1] ADMITS NO FUNCTORIAL EMBEDDING INTO A FINITE-DIMENSIONAL OR METRIZABLE TOPOLOGICAL GROUP

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Abstract. An embedding $X \subset G$ of a topological space X into a topological group G is called *functorial* if every homeomorphism of X extends to a continuous group homomorphism of G. It is shown that the interval $[0,1]$ admits no functorial embedding into a finite-dimensional or metrizable topological group.

Let $\mathcal{A}, \mathcal{B}$ be subcategories of the category of all topological spaces and their continuous maps. A covariant functor $F : \mathcal{A} \to \mathcal{B}$ is called *an embedding functor* provided there exists a class of embeddings $i_X : X \to FX$, $X \in |\mathcal{A}|$, satisfying the naturality conditions: for every morphism $f : X \to Y$ in $\mathcal{A}$ the equality $F(f) \circ i_X = i_Y \circ f$ holds [2]. In this note we are interested in a special case of this notion when the class of objects of a category $\mathcal{A}$ contains only one topological space X and the set of all morphisms of X coincides with the set of all autohomeomorphisms of X, and the category $\mathcal{B}$ is a subcategory of (the underlying spaces of) topological groups and their continuous homomorphisms.

Thus, we come to the following version of the above notion. An embedding $X \subset G$ of a topological space X into a topological group G is called a

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functorial embedding if every homeomorphism f of X extends to a continuous group homomorphism $G(f)$ of G and the correspondence $f \mapsto G(f)$ preserves the composition.

Clearly, the natural embedding of a Tychonoff space X into its free (totally bounded) topological group $F(X)$ is a functorial embedding.

In [2] D. Shakhmatov proved that every zero-dimensional metrizable compact space X admits a functorial embedding into a zero-dimensional compact metrizable topological group $\mathbb{Z}_2^{C(X,\mathbb{Z}_2)}$, where $C(X, \mathbb{Z}_2)$ is the (countable) set of all continuous maps of X into the two-element group $\mathbb{Z}_2$.

On the other hand, using some rather sophisticated arguments, he proved that the closed interval $[0, 1]$ admits no functorial embedding into a finite-dimensional metrizable topological group.

In this note we generalize this result in two directions.

Theorem. *If $[0, 1] \subset G$ is a functorial embedding of the closed interval into a topological group G, then G is infinite-dimensional and non-metrizable.*

Proof. Suppose that $[0, 1] \subset G$ is a functorial embedding into a topological group. First we show that the group G is infinite-dimensional. Denote by $\mathrm{grp}(A)$ the group hull of a subset $A \subset [0, 1]$.

Claim A. *For every closed subset $A \subset [0, 1]$ we have* $\mathrm{grp}(A) \cap [0, 1] = A$.

Indeed, let $f : [0, 1] \to [0, 1]$ be a homeomorphism whose set of fixed points (i.e., the set $\{x \in [0, 1] \mid f(x) = x\}$) coincides with A. Let $G(f)$ be a continuous group homomorphism of G extending the homeomorphism f. Then the set $H = \{g \in G \mid G(f)(g) = g\}$ is a subgroup of G with $A \subset \mathrm{grp}(A) \subset H$ and $A = [0, 1] \cap H \supset \mathrm{grp}(A) \cap [0, 1] \supset A$.

Claim B. *Let $0 < t < 1$ and let $K \subset$ $\mathrm{grp}([0, t])$ be a compactum. Then for any $s \in (t, 1]$ the set* $\mathrm{grp}([0, s])$ *contains a topological copy of the space* $K \times [0, 1]$.

Indeed, consider the multiplication map $m : K \times [t, s] \to G$. Show that this map is an embedding. Suppose, on the contrary, that $m(g_1, h_1) = m(g_2, h_2)$. Without loss of generality, we may assume that $h_1 < h_2$. But then $h_2 = g_2^{-1} g_1 h_1 \in \mathrm{grp}([0, h_1])$ which contradicts to Claim A.

To see that the group G is infinite-dimensional, observe that, by Claim B, the set $\mathrm{grp}([0, 1/2])$ contains an arc. Thus, the set $[0, 1 - 2^{-2}]$ contains a topological copy of the space $[0, 1]^2$, and, similarly, the set $[0, 1 - 2^{-n}])$ contains a topological copy of the space $[0, 1]^n$. Hence, G is infinite dimensional.

To prove the second part of Theorem, suppose on the contrary that the group G is metrizable. Fix any left-invariant metric ρ on G and for every element $g \in G$ let $\|g\| = \rho(g, e)$, where e is the unit of the group G. Without loss of generality, $\rho(0,1) = 1$, where 0 and 1 are the end-points of $[0,1] \subset G$. To get a contradiction, we shall construct two sequences $(a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty \subset G$ such that

- $(a_n)_{n=1}^\infty$ converges to e, while $(b_n)_{n=1}^\infty$ does not;
- $h(a_n) = b_n$, $n \in \mathbb{N}$, for some continuous group homomorphism h of G.

In the construction of the sequences (a_n), (b_n) we shall use the standard Cantor set $C \subset [0,1]$. It is well khown that C is homeomorphic to the Cantor cube $\{0,1\}^{\mathbb{N}}$ via the homeomorphism $f : \{0,1\}^{\mathbb{N}} \to C$,

$$f : (x_n)_{n=1}^\infty \mapsto \sum_{n=1}^\infty \frac{2x_n}{3^n}$$

for $(x_n)_{n=1}^\infty \in \{0,1\}^{\mathbb{N}}$. For every integer $n \geq 0$ consider the subsets

$$\{0,1\}^n_- = \{(x_i) \in \{0,1\}^{\mathbb{N}} : x_i = 0 \text{ for } i > n\},$$

$$\{0,1\}^n_+ = \{(x_i) \in \{0,1\}^{\mathbb{N}} : x_i = 1 \text{ for } i > n\}$$

in $\{0,1\}^{\mathbb{N}}$. Let $C_n^- = f(\{0,1\}^n_-)$, $C_n^+ = f(\{0,1\}^n_+)$ and $C_n = C_n^- \cup C_n^+$.

For every subset $S = \{s_1, \dots, s_n\} \subset [0,1]$, where $s_1 < \dots < s_n$, let

$$\Pi(S) = s_1^{-1} s_2 s_3^{-1} \dots s_n^{(-1)^n}.$$

By induction on n, construct increasing functions $\alpha_n : C_n \to [0,1]$, $\beta_n : C_n \to [0,1]$, $n \geq 0$, such that letting $a_n = \Pi(\alpha_n(C_n))$ and $b_n = \Pi(\beta_n(C_n))$ for $n \geq 0$ we have

(1) $\alpha_0(C_0) = \beta_0(C_0) = \{0,1\}$, $\|b_0\| = \rho(0,1) = 1$;
(2) $\alpha_{n+1}|C_n = \alpha_n$ and $\beta_{n+1}|C_n = \beta_n$;
(3) $[a, a + 2^{-n}] \cap \alpha_n(C_n^+) \neq \emptyset$ for every $a \in \alpha_n(C_n^-)$;
(4) $[b, b + 2^{-n}] \cap \beta_n(C_n^+) \neq \emptyset$ for every $b \in \beta_n(C_n^-)$;
(5) $\|a_n\| \leq 2^{-n}$;
(6) $\|b_n\| > \|b_{n-1}\| - \frac{1}{2^{n+1}} \geq \frac{1}{2} + \frac{1}{2^{n+1}}$.

The conditions (1)–(4) imply the existence of two increasing homeomorphisms α, $\beta : [0,1] \to [0,1]$ such that $\alpha|C_n = \alpha_n$ and $\beta|C_n = \beta_n$ for every $n \in \mathbb{N}$. Let $h : G \to G$ be a continuous group homomorphism extending the homeomorphissm $\beta \circ \alpha^{-1} : [0,1] \to [0,1]$. It is easy to see that $h(a_n) = b_n$ for every n. Since the sequence (a_n) converges to the unity $e \in G$, the sequence (b_n) converges to e too, a contradiction with (6).

Remark 1. It is intersting to compare our Theorem with the classical results of M. I. Graev [3] on extending metrics from a space X omto the free group $F(X)$. It is known that for every Lipschitz map $f : X \to X$ the induced group homomorphism $F(f) : F(X) \to F(X)$ is Lipschitz with respect to the Graev metric on $F(X)$.

Theorem implies that the free group $F(I)$ over the interval $I = [0, 1]$ admits no metrizable group topology such that the natural inclucion $I \subset F(I)$ is a functorial embedding.

It is known [1] that every non-metrizable topological group G which is a k_ω-space contains a closed topological copy of the Frèchet-Urysohn fan, that is the quotient space

$$S_0 \times \mathbb{N}/\{0\} \times \mathbb{N},$$

where $S_0 = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is the convergent sequence.

Problem. *Suppose $I \subset G$ is a functorial embedding of the interval into a topological group G. Does G contain a topological copy of the Frèchet-Urysohn fan? Does the group hull* $\mathrm{grp}(I)$ *of I in G contain a closed topological copy of Frèchet-Urysohn fan?*

Remark 2. Note that Theorem gives a negative answer to the Question 8 from [2].

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