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ON AVERAGING NULL SEQUENCES OF REAL-VALUED FUNCTIONS

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Abstract. If ξ is a countable ordinal and (f_k) a sequence of real-valued functions we define the repeated averages of order ξ of (f_k). By using a partition theorem of Nash-Williams for families of finite subsets of positive integers it is proved that if ξ is a countable ordinal then every sequence (f_k) of real-valued functions has a subsequence (f'_k) such that either every sequence of repeated averages of order ξ of (f'_k) converges uniformly to zero or no sequence of repeated averages of order ξ of (f'_k) converges uniformly to zero. By the aid of this result we obtain some results stronger than Mazur's theorem.

Introduction. Argyros, Mercourakis and Tsarpalias in [3] introduced for every $\xi < \omega_1$ the summability methods (ξ_n^L), where $L \in [\mathsf{N}]$.

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In this paper we define the families $\mathcal{M}_\xi[N]$, $\xi < \omega_1$, where $N = (n_k)$ a strictly increasing sequence of positive integers, as follows: We set $\mathcal{M}_0[N] = \{\{n_k\} : k = 1, 2, \ldots\}$. If $\mathcal{M}_\xi[N]$ has been defined then we set

$$\mathcal{M}_{\xi+1}[N] = \bigcup_{k=1}^{\infty} \{\cup_{i=1}^{n_k} A_i : A_1, \ldots, A_{n_k} \in \mathcal{M}_\xi[N] \text{ with } A_1 < \ldots < A_{n_k} \text{ and } \min A_1 = n_k\}.$$

If ξ is a limit ordinal and (ζ_k) be the strictly increasing sequence of successor ordinals with $\sup_k \zeta_k = \xi$ that defines the sequence (ξ_n^L) for every $L \in [N]$ then we set

$$\mathcal{M}_\xi[N] = \bigcup_{k=1}^{\infty} \{A \in \mathcal{M}_{\zeta_{n_k}}[N] : \min A = n_k\}.$$

If (f_k) is a sequence of real-valued functions defined on a set X, $\xi < \omega_1$ and $H \in \mathcal{M}_\xi[\mathsf{N}]$ we define the function $a^\xi((f_k); H)$ called repeated average of order ξ of sequence (f_k) (cf. Definition 1.7).

By using a well-known result of Nash-Williams in [16] and a method created by Prof. Negrepontis and author (cf. [12] or [15, Def. 3.6, Lemma 3.7]) we prove that if $n \in \mathsf{N}$, $\xi_1, \ldots, \xi_n < \omega_1$, $N \in [\mathsf{N}]$ and $\{\mathcal{P}_1, \mathcal{P}_2\}$ a partition of $[N]^{<\omega}$ then there exist $N' \in [N]$ and $j \in \{1, 2\}$ such that $\cup_{i=1}^n E_i \in \mathcal{P}_j$ for every $E_1 \in \mathcal{M}_{\xi_1}[N'], \ldots, E_n \in \mathcal{M}_{\xi_n}[N']$ with $E_1 < \ldots < E_n$ (cf. Theorem 2.1).

By using the above result we prove the following dichotomy: If (f_k) is a sequence of real-valued functions defined on a set X, $M \in [N]$ and $\xi < \omega_1$ then there exists $N \in [M]$ such that

either (1) for every strictly increasing sequence (H_n) of $\mathcal{M}_\xi[N]$ the sequence $g_n = a^\xi(f_k); H_n)$, $n \in \mathsf{N}$ converges uniformly to zero;

or (2) does not exist a strictly increasing sequence (H_n) of $\mathcal{M}_\xi[N]$ such that the sequence $g_n = a^\xi((f_k); H_n)$, $n \in \mathsf{N}$ converges uniformly to zero

(cf. Theorem 2.5). This result is analogous with a dichotomy theorem of Erdos and Magidor in [7] for regular methods of summability.

Kechris and Louveau in [9] defined the convergence index "$\gamma((f_k))$" of a sequence (f_k) of continuous real-valued functions. We prove that if K is a compact metric space, (f_k) a uniformly bounded sequence of continuous real-valued functions on K and $\xi < \omega_1$ with $\gamma((f_k)) \leq \omega^\xi$ then for every $M \in [\mathsf{N}]$ there exists $N \in [M]$ such that for every strictly increasing sequence (H_n) of members of $\mathcal{M}_\xi[N]$ the sequence $g_n = a^\xi((f_{2k+1} - f_{2k}); H_n)$, $n \in \mathsf{N}$ converges uniformly to zero (cf. Proposition 2.6).

Also we prove that if K is a compact metric space, (f_k) a sequence of continuous real-valued functions and $1 \leq \xi < \omega_1$ such that for every subsequence (f'_k) of (f_k) there exists a strictly increasing sequence (H_n) of members of $\mathcal{M}_\xi[\mathsf{N}]$ such that the sequence $g_n = a^\xi((f'_{2k+1} - f'_{2k}); H_n)$, $n \in \mathsf{N}$ converges uniformly to zero then there exists a subsequence (f'_k) of (f_k) with $\gamma((f'_k)) \leq \omega^\xi$ (cf. Proposition 2.7).

Kechris and Louveau in [9] defined the oscillation index "$\beta(f)$" of a real-valued function f. We prove that if K is a compact metric space, $1 \leq \xi < \omega_1$ and (f_k) a sequence of continuous real-valued functions on K pointwise converging to f then the following hold:

(i) If for every subsequence (f'_k) of (f_k) there exists a strictly increasing sequence (H_n) of members of $\mathcal{M}_\xi[\mathsf{N}]$ such that the sequence $g_n = a^\xi((f'_{2k+1} - f'_{2k}); H_n)$, $n \in \mathsf{N}$ converges uniformly to zero then $\beta(f) \leq \omega^\xi$.

(ii) If f is bounded and $\beta(f) \leq \omega^\xi$ then there exists a sequence (h_k) of convex blocks of (f_k) (i.e., $h_k \in \operatorname{conv}((f_p)_{p \geq k})$ for all k) such that for every $M \in [\mathsf{N}]$ there exists $N \in [M]$ such that for every strictly increasing sequence (H_n) of members of $\mathcal{M}_\xi[N]$ the sequence $g_n = a^\xi((h_{2k+1} - h_{2k}); H_n)$, $n \in \mathsf{N}$ converges uniformly to zero. (Here $\operatorname{conv}((\phi_k))$ denotes the set of all combinations of the ϕ_k's) (cf. Corollary 2.8).

Also we obtain the following result: If K is a compact metric space and $1 \leq \xi < \omega_1$ such that for every sequence (f_k) of continuous real-valued functions on K pointwise converging to zero there exists a strictly increasing sequence (H_n) of members of $\mathcal{M}_\xi[\mathsf{N}]$ such that the sequence $g_n = a^\xi((f_k); H_n)$, $n \in \mathsf{N}$ converges uniformly to zero. Then $\beta(f) \leq \omega^\xi$ for every Baire-1 function f on K (cf. Corollary 2.9).

Finally, we prove that if X is a pseudompact topological space (i.e., if (U_n) is a decreasing sequence of non-empty open subsets of X then $\bigcap_{n=1}^\infty \operatorname{cl} U_n \neq \emptyset$) and (f_k) a uniformly bounded sequence of continuous real-valued functions on X pointwise converging to zero with $\inf_k \|f_k\|_\infty > 0$ then there exists $1 \leq \xi < \omega_1$ such that for every $M \in [\mathsf{N}]$ there exists $N \in [M]$ such that for every strictly increasing sequence (H_n) of members of $\mathcal{M}_\xi[N]$ the sequence $g_n = a^\xi((f_k); H_n)$, $n \in \mathsf{N}$ converges uniformly to zero (cf. Proposition 2.12).

1. Preliminaries. This section contains definitions, combinatorial lemmas and known results which we shall use for the proof of main results in the section 2.

By N we mean the set of all positive integers, by ω we mean the first infinite ordinal (i.e., $\omega := \{0, 1, 2, \ldots\}$) and by ω_1 we mean the first uncountable ordinal. For any set M, the set of all finite subsets of M and the set of all infinite subsets of M will be denoted by $[M]^{<\omega}$ and $[M]$ respectively. A family $\mathcal{F}$ of finite subsets of N is said to be *hereditary* if all subsets of members of $\mathcal{F}$ belong to $\mathcal{F}$. A family $\mathcal{F}$ of finite subsets of N is said to be *compact* if the set of all characteristic functions χ_F, where $F \in \mathcal{F}$, is compact subspace of $\{0,1\}^\omega$ with the product topology. A family $\mathcal{F}$ of finite subsets of N is said to be *adequate* if it is hereditary and compact.

For $\mathcal{F} \subseteq [\mathsf{N}]^{<\omega}$ and $M \in [\mathsf{N}]$ the set $\mathcal{F} \cap [M]^{<\omega}$ denoted by $\mathcal{F}[M]$.

Generalized Schreier families.

Definition 1.1. *Let* $F, H \in [\mathsf{N}]^{<\omega}$ *and* $n \in \mathsf{N}$. *We write* $F < H$ *if either set is empty or if* $\max F < \min H$, *and* $n \leq F$ *iff* $n = \min F$ *or* $\{n\} < F$.

Alspach and Argyros in [1] introduced some families called generalized Schreier families. We can define these families as follows:

Let $\mathcal{S}_\xi$, be an family of finite subsets of N for each $\xi < \omega_1$. The families $\{\mathcal{S}_\xi\}_{\xi<\omega_1}$ will be said to have the *generalized Schreier property* if

(i) $\mathcal{S}_0 = \{\emptyset\} \cup \{\{n\} : n \in \mathsf{N}\}$;

(ii) $\mathcal{S}_1$ is the Schreier family, i.e., $\mathcal{S}_1 = \{F \subset \mathsf{N} : |F| \leq \min F\}$ (cf. [18]);

(iii) $\mathcal{S}_{\xi+1} = \bigcup_{n=1}^\infty \{\cup_{i=1}^n F_i : n \leq F_1 < \ldots < F_n, F_i \in \mathcal{S}_\xi \text{ for } i = 1, \ldots n\}$ for $1 \leq \xi < \omega_1$;

(iv) for every limit ordinal $\xi < \omega_1$, there exists a strictly increasing sequence of ordinals (ξ_n) such that $\xi = \sup\{\xi_n : n \in \mathsf{N}\}$ and $\mathcal{S}_\xi = \bigcup_{n=1}^\infty \{F \in \mathcal{S}_{\xi_n} : n \leq F\}$.

It can be noticed that for each $m < \omega$ there is an unique $\mathcal{S}_m$. The families $(\mathcal{S}_m)_{m<\omega}$ appeared for the first time in an example constructed by Alspach and Odell [2]. For every $\zeta \geq \omega$ there are infinitely many families $\mathcal{S}_\zeta$ such that the families $\{\mathcal{S}_\xi\}_{\xi<\omega_1}$ have the generalized Schreier property.

Summability methods. The following definition was given by Argyros, Mercourakis and Tsarpalias in [3].

Definition 1.2. *We denote by* S_{l^1} *the positive part of the unit sphere of* $l^1(\mathsf{N})$. *For* $A = (\alpha_n)$ *in* $S_{l^1}^+$ *we set* $\operatorname{supp} A = \{n \in \mathsf{N} : \alpha_n \neq 0\}$.

If $F=(x_n)$ is a sequence in a Banach space X and $A=(\alpha_n)$ in $S^+_{l^1}$ with $|\operatorname{supp} A|<+\infty$ we denote by $A\cdot F$ the usual matrices product, that is:

$$A\cdot F=\sum_{n=1}^{\infty}\alpha_n x_n.$$

For $A=(a_n)\in l^1(\mathsf{N})$ and $F\in[\mathsf{N}]^{<\omega}$ we denote by $\langle A,F\rangle$ the quantity $\sum\limits_{n\in F}a_n$.

For $M\in[\mathsf{N}]$ an M-summability method is a sequence $(A_n)\subseteq S^+_{l^1}$ with $\operatorname{supp} A_n<\operatorname{supp} A_{n+1}$ for each n and $M=\cup_{n=1}^{\infty}\operatorname{supp} A_n$.

For each $M\in[\mathsf{N}]$ and $\xi<\omega_1$ the M-summability method (ξ_n^M) is defined, inductively, as it follows:

(i) For $\xi=0$, $M=(m_n)$ we set $\xi_n^M=e_{m_n}$.

(ii) If $\xi=\zeta+1$, $M\in[\mathsf{N}]$ and (ζ_n^M) has been defined then we, inductively, define (ξ_n^M) as it follows. We set $k_1=0$, $s_1=\min\operatorname{supp}\zeta_1^M$, and

$$\xi_1^M=\frac{\zeta_1^M+\ldots+\zeta_{s_1}^M}{s_1}.$$

Suppose that for $j=1,2,\ldots,n-1$, k_j, s_j have been defined and

$$\xi_j^M=\frac{\zeta_{k_j+1}^M+\ldots+\zeta_{k_j+s_j}^M}{s_j}.$$

Then we set,

$$k_n=k_{n-1}+s_{n-1},\quad s_n=\min\operatorname{supp}\zeta_{k_n+1}^M\quad\text{and}$$

$$\xi_n^M=\frac{\zeta_{k_n+1}^M+\ldots+\zeta_{k_n+s_n}^M}{s_n}.$$

This completes the definition for successor ordinals.

(iii) If ξ is a limit ordinal and if we suppose that for every $\zeta<\xi$, $M\in[\mathsf{N}]$ the sequence (ζ_n^M) has been defined, then we define (ξ_n^M) as it follows:

Let (ζ_n) be a strictly increasing sequence of successor ordinals with $\sup_n\zeta_n=\xi$. For $M\in[\mathsf{N}]$, $M=(m_k)$ we inductively define $M_1=M$, $n_1=m_1$, $M_2=\{m_k: m_k\notin\operatorname{supp}[\zeta_{n_1}]_1^{M_1}$, $n_2=\min M_2$, $M_3=\{m_k: m_k\notin\operatorname{supp}[\zeta_{n_2}]_1^{M_2}\}$, and $n_3=\min M_3$, and so on. We set

$$\xi_1^M=[\zeta_{n_1}]_1^{M_1},\quad \xi_2^M=[\zeta_{n_2}]_1^{M_2},\ldots,\xi_k^M=[\zeta_{n_k}]_1^{M_k},\ldots$$

Hence (ξ_n^n) has been defined.

From Theorem 2.2.6 and Proposition 2.3.2 of [3] we get the next theorem:

Theorem 1.3. *Assume that the families $\mathcal{S}_\xi$, $\xi<\omega_1$ of finite subsets* *of* N *have the generalized Schreier property. If* $\mathcal{F}$ *is an adequate family of finite subsets of* N*,* $M \in [\mathsf{N}]$*,* $\xi < \omega_1$ *and* $\epsilon > 0$ *such that* $\sup_{F \in \mathcal{F}} \langle \xi_n^n, F \rangle > \epsilon$ *for every* $N \in [M]$*,* $n \in \mathsf{N}$*, then there exists a strictly increasing sequence* (m_k) *of elements of* M *such that*

$$\{m_j : j \in F\} \in \mathcal{F} \quad \textit{for} \quad \textit{all} \quad F \in \mathcal{S}_\xi.$$

Definition 1.4. *Let* $N = (n_k)$ *be a strictly increasing sequence of positive integers. We define the families* $\mathcal{M}_\xi[N]$*,* $\xi < \omega_1$ *as follows: We set*

$$\mathcal{M}_0[N] = \{\{n_k\} : k = 1, 2, \ldots\}.$$

If $\mathcal{M}_\xi[N]$ *has been defined then we set*

$$\mathcal{M}_{\xi+1}[N] = \bigcup_{k=1}^{\infty} \{\cup_{i=1}^{n_k} A_i : \quad A_1, \ldots, A_{n_k} \in \mathcal{M}_\xi[N] \quad \textit{with} \quad A_1 < \ldots < A_{n_k}$$

$$\textit{and} \quad \min A_1 = n_k\}.$$

If ξ *is a limit ordinal and* (ζ_n) *be the strictly increasing sequence of successor ordinals with* $\sup_n \zeta_n = \xi$ *that defines the sequence* (ξ_n^L) *for every* $L \in [N]$ *then we set*

$$\mathcal{M}_\xi[N] = \bigcup_{k=1}^{\infty} \{A \in \mathcal{M}_{\zeta_{n_k}}[N] : \min A = n_k\}.$$

Proposition 1.5. *For every* $N \in [\mathsf{N}]$ *and* $\xi < \omega_1$ *holds*

$$\mathcal{M}_\xi[N] = \{\operatorname{supp} \xi_k^L : \quad L \in [N], \quad k = 1, 2, \ldots\}.$$

Proof. Let $\xi < \omega_1$ and $N \in [\mathsf{N}]$.

Step 1: $\mathcal{M}_\xi[N] \subseteq \{\operatorname{supp} \xi_k^L : \quad L \in [N], \quad k = 1, 2, \ldots\}$.

Claim. *For every* $F \in \mathcal{M}_\xi[N]$ *it holds* $F = \operatorname{supp} \xi_1^{N_F}$*, where* $N_F = F \cup \{m \in N : m > \max F\}$.

Proof of Claim. We proceed by induction on $\xi < \omega_1$. Let $\xi = 0$ and let $F \in \mathcal{M}_0[N]$. Then $F = \{n\}$ for some $n \in N$. Therefore, $N_F = \{m \in N : m \geq n\}$ and $\operatorname{supp} \xi_1^{N_F} = \{n\} = F$.

Let $1 \leq \xi < \omega_1$ such that $F = \operatorname{supp} \zeta_1^{N_F}$ for every $\zeta < \xi$, $N \in [\mathsf{N}]$ and $F \in \mathcal{M}_\zeta[N]$. We shall prove that $F = \operatorname{supp} \xi_1^{N_F}$ for every $N \in [\mathsf{N}]$ and $F \in \mathcal{M}_\xi[N]$.

Case 1: $\xi = \eta + 1$, where $\eta < \omega_1$. Clearly $\eta < \xi$. Now let $E \in \mathcal{M}_\xi[N]$. Then there exist $E_1, \dots, E_n \in \mathcal{M}_\eta[N]$ with $E_1 < \dots < E_n$, $\min E_1 = n$ and $E = E_1 \cup \dots \cup E_n$. We set $L = N_E = E \cup \{m \in N : m > \max E\}$. For every $i = 1, \dots, n$, by the inductive assumption, we have $\operatorname{supp} \eta_1^{L_{E_i}} = E_i$, where $L_{E_i} = E_i \cup \{m \in L : m > \max E_i\} = E_i \cup \dots \cup E_n \cup \{m \in N : m > \max E_n\}$. It is easy to see that $\eta_1^L = \eta_1^{L_{E_1}}, \eta_2^L = \eta_2^{L_{E_2}}, \dots, \eta_n^L = \eta_1^{L_{E_n}}$, $\operatorname{supp} \eta_i^L = E_i$ for every $i = 1, \dots, n$ and $\min L = \min E = \min E_1 = n$. Also $\xi_1^{N_E} = [\eta + 1]_1^L = \dfrac{\eta_1^L + \dots + \eta_n^L}{n}$. Hence $\operatorname{supp} \xi_1^{N_E} = \cup_{i=1}^n \operatorname{supp} \eta_i^L = \cup_{i=1}^n E_i = E$.

Case 2: ξ is a limit ordinal. Let (ζ_k) be the strictly increasing sequence of successor ordinals that defines the sequence (ξ_n^L) for every $L \in [N]$. Let also $F \in \mathcal{M}_\xi[N]$. Then $F \in \mathcal{M}_{\zeta_n}[N]$, where $n = \min F$. By the inductive assumption we have $F = \operatorname{supp}[\zeta_n]_1^{N_F}$, where $N_F = F \cup \{m \in N : m > \max F\}$. Since $n = \min N_F$ we have $[\zeta_n]_1^{N_F} = \xi_1^{N_F}$. Hence $\operatorname{supp} \xi_1^{N_F} = \operatorname{supp}[\zeta_n]_1^{N_F} = F$. $\square$

Step 2: $\{\operatorname{supp} \xi_k^L : L \in [N], k = 1, 2, \dots\} \subseteq \mathcal{M}_\xi[N]$.

We proceed by induction on $\xi < \omega_1$. Let $\xi = 0$ and $L \in [N]$ with $L = \{l_1 < \dots < l_k < \dots\}$. Then $\operatorname{supp} \xi_k^L = \operatorname{supp} 0_k^L = \{l_k\} \in \mathcal{M}_0[N]$ for every $k = 1, 2, \dots$.

Now let $1 \leq \xi < \omega_1$ such that $\{\operatorname{supp} \zeta_k^L : L \in [N], k = 1, 2, \dots\} \subseteq \mathcal{M}_\zeta[N]$ for every $\zeta < \xi$. By Definition 1.2 (ii) and (iii) we easily prove that $\{\operatorname{supp} \xi_k^L : L \in [N], k = 1, 2, \dots\} \subseteq \mathcal{M}_\xi[N]$. $\square$

Repeated averages.

Definition 1.6. *Let $N \in [\mathsf{N}]$ and F be a finite subset of N. For every $n \in N$ we define $a_F^0(\{n\}) = \chi_F(n)$, that is, $a_F^0(\{n\}) = 1$ if $n \in F$ and $a_F^0(\{n\}) = 0$ if $n \notin F$. For every $H = \{n_1 < \dots < n_m\} \in \mathcal{M}_1[N]$ we set*

$$a_F^1(H) = \frac{1}{m} \sum_{i=1}^m a_F^0(\{n_i\}) = \frac{|F \cap H|}{|H|}.$$

Let $\xi < \omega_1$ be an ordinal such that the numbers $a_F^\xi(H)$ have been defined for every $H \in \mathcal{M}_\xi[N]$. Then for every $H \in \mathcal{M}_{\xi+1}[N]$ there exist unique $H_1, \dots, H_m \in \mathcal{M}_\xi[N]$ such that $m = \min H_1$, $H_1 < \dots < H_m$ and $H = H_1 \cup \dots \cup H_m$ (cf. Def. 1.4, Def. 2.2.1 and Lemma 2.2.3). We set

$$a_F^{\xi+1}(H) = \frac{1}{m} \sum_{i=1}^m a_F^\xi(H_i).$$

Let ξ be a limit ordinal such that $a_F^\zeta(H)$ has been defined for every $\zeta < \xi$ and $H \in \mathcal{M}_\zeta[N]$. Let (ξ_n) be the strictly increasing sequence of successor ordinals with $\sup_n \xi_n = \xi$ that defines the family $\mathcal{M}_\xi[N]$. For every $H \in \mathcal{M}_\xi[N]$ we set

$$a_F^\xi(H) = a_F^{\xi_n}(H), \quad \text{where} \ \ n = \min H.$$

(By induction on ξ, it is easy to remark that $a_F^\xi(H)$ is well-defined for all ξ, H and F, i.e., if $\zeta < \xi$ and $H \in \mathcal{M}_\zeta[N] \cap \mathcal{M}_\xi[N]$ then $a_F^\zeta(H) = a_F^\xi(H)$ for every finite subset F of N.)

Definition 1.7. *Let (f_k) be a sequence of real-valued functions defined on a set X. For any ordinal $\xi < \omega_1$, we define the function $a^\xi((f_k); H)$, where $H \in \mathcal{M}_\xi[N]$, called* repeated average of order ξ of the sequence (f_k), *as follows:*

For each $n \in N$ we define

$$a^0((f_k); \{n\}) = f_n.$$

For $H = \{n_1, \ldots, n_m\} \in \mathcal{M}_1[N]$ with $n_1 < \ldots < n_m$ we define

$$a^1((f_k); H) = \frac{1}{m} \sum_{i=1}^{m} f_{n_i}.$$

For any $\xi < \omega_1$ and $H = H_1 \cup \ldots \cup H_m \in \mathcal{M}_{\xi+1}[N]$, where $m = \min H_1$ and $H_1, \ldots, H_m \in \mathcal{M}_\xi[N]$ with $H_1 < \ldots < H_m$, we define

$$a^{\xi+1}((f_k); H) = \frac{1}{m} \sum_{i=1}^{m} a^\xi((f_k); H_i).$$

Let ξ be a limit ordinal and (ξ_n) be the strictly increasing sequence of successor ordinals with $\sup_n \xi_n = \xi$ that defines the family $\mathcal{M}_\xi[N]$. Then for every $H \in \mathcal{M}_\xi[N]$ we set

$$a^\xi((f_k); H) = a^{\xi_n}((f_k); H) \quad \textit{where} \ \ n = \min H.$$

It can be noticed that repeated averages of order m, where $m < \omega$, was introduced by Alspach and Odell in [2] by using other notations.

Remarks 1.8. Let $N \in [\mathsf{N}]$. By induction on $\xi < \omega_1$ it is easy to show that

(i) If $L \in [N]$, $n \in L$ and $F \in [\mathsf{N}]^{<\omega}$ it holds

$$\langle \xi_n^L, F \rangle = a_F^\xi(H) \ \ \text{where} \ \ H = \operatorname{supp} \xi_n^L.$$

(ii) If $A = (f_k)$, $L \in [N]$ and $n \in L$ it holds

$$\xi_n^L \cdot A = a^\xi((f_k); H) \ \ \text{where} \ \ H = \operatorname{supp} \xi_n^L.$$

Definition 1.9 (cf. [16]). *If $F \subseteq \mathsf{N}$ and $I = \{x \in F : x < n\}$ for some $n \in \mathsf{N}$ we shall call I an* initial segment *of F.*

A family $\mathcal{C}$ of finite subsets of N is said to be thin if there do not exist $A, B \in \mathcal{C}$ such that A is an initial segment of B and $A \neq B$.

Theorem 1.10 (cf. [16]). *If $M \in [\mathsf{N}]$ and $\mathcal{C}$ is thin family of finite subsets of M then for every partition $\{\mathcal{C}_1, \mathcal{C}_2\}$ of $\mathcal{C}$ there exists $N \in [M]$ such that $\mathcal{C} \cap [N]^{<\omega} \subseteq \mathcal{C}_1$ or $\mathcal{C} \cap [N]^{<\omega} \subseteq \mathcal{C}_2$.*

Trees.

Definition 1.11 (cf. [4]). *Let X be a set.*

(i) *A* tree *on X will be a subset of $\bigcup_{n=1}^{\infty} X^n$ with the property that $(x_1, \ldots, x_n) \in T$ whenever $(x_1, \ldots, x_n, x_{n+1}) \in T$.*

(ii) *A tree T on X is* well-founded *if there is no sequence (x_n) in X satisfying $(x_1, \ldots, x_n) \in T$ for each $n \in \mathsf{N}$.*

(iii) *Proceeding by induction we associate to each ordinal ξ a new tree T^ξ such that: Take $T^0 = T$.*

If T^ξ is obtained, let

$$T^{\xi+1} = \bigcup_{n=1}^{\infty} \{(x_1, \ldots, x_n) \in X^n : (x_1, \ldots, x_n, x) \in T^\xi \ \ \textit{for} \ \ \textit{some} \ \ x \in X\}.$$

If ξ is a limit ordinal, define $T^\xi = \bigcap_{\zeta<\xi} T^\zeta$.

Proposition 1.12 (cf. [4, 5, 6]). *If T is a well-founded tree on N then there is $\xi < \omega_1$ such that $T^\xi = \emptyset$.*

Convergence index, oscillation index. A real-valued function f defined on a set X is bounded if $\|f\|_\infty := \sup_{x\in X} |f(x)| < +\infty$. A sequence (f_k) of real-valued functions defined on a set X is uniformly bounded if $\sup_k \|f_k\|_\infty < +\infty$.

Let X be a topological space and $C(X)$ the set of continuous real-valued functions on X. By $\mathbf{R}$ we mean the set of all real numbers. A function $f : X \to \mathbf{R}$ is Baire-1 if there exists a sequence (f_k) in $C(X)$ that converges pointwise to f.

Definition 1.13 (cf. [8, 9]). *Let K be a compact metric space, $f : K \to \mathbf{R}$ a function, $P \subseteq K$ and $\epsilon > 0$. Let $P^0_{\epsilon,f} = P$ and for any ordinal α let $P^{\alpha+1}_{\epsilon,f}$ be the set of those $x \in P^{\alpha}_{\epsilon,f}$ such that for every open set U arround x there are two points x_1 and x_2 in $P^{\alpha}_{\epsilon,f} \cap U$ such that $|f(x_1) - f(x_2)| \geq \epsilon$. At a limit ordinal α we set $P^{\alpha}_{\epsilon,f} = \bigcap_{\beta<\alpha} P^{\beta}_{\epsilon,f}$. Let $\beta(f, \epsilon)$ be the least α with $K^{\alpha} = \emptyset$ if such an α exists, and $\beta(f, \epsilon) = \omega_1$, otherwise. Define the oscillation index $\beta(f)$ of f by*

$$\beta(f) = \sup\{\beta(f, \epsilon) : \epsilon > 0\}.$$

The complexity of pointwise convergent sequence of continuous real-valued functions defined on a compact metric space is described by a countable ordinal index "γ" which is defined in the following way.

Definition 1.14 (cf. [9]). *Let K be a compact metric space, (f_k) a sequence of continuous real-valued functions defined on K, $P \subseteq K$ and $\epsilon > 0$. Let $P^0_{\epsilon,(f_k)} = P$ and for any ordinal α let $P^{\alpha+1}_{\epsilon,(f_k)}$ be the set of those $x \in P^{\alpha}_{\epsilon,(f_k)}$ such that for every open set U around x and for every $p \in \mathsf{N}$ there are $m, n \in \mathsf{N}$ with $m > n > p$ and a point x' in $P^{\alpha}_{\epsilon,(f_k)} \cap U$ such that $|f_m(x') - f_n(x')| \geq \epsilon$.*

At a limit ordinal α we set $P^{\alpha}_{\epsilon,(f_k)} = \bigcap_{\beta<\alpha} P^{\beta}_{\epsilon,(f_k)}$. (It can be noticed that $P^{\alpha}_{\epsilon,(f_k)}$ is a closed subset of P with the relative topology in P.) Let $\gamma((f_k), \epsilon)$ be the least α with $K^{\alpha}_{\epsilon,(f_k)} = \emptyset$ if such an α exists, and $\gamma((f_k), \epsilon) = \omega_1$, otherwise. Define the convergence index $\gamma((f_k))$ of (f_k) by

$$\gamma((f_k)) = \sup\{\beta((f_k), \epsilon) : \epsilon > 0\}.$$

In [9] it is proved that $\gamma((f_k)) < \omega_1$ iff (f_k) is pointwise converging. Also in [9] it is proved that if the sequence (f_k) of continuous real-valued functions on K converges pointwise to f then $\beta(f) \leq \gamma((f_k))$.

By Lemma 3.3.3 and Definition 3.3.1 of [10] we get the following proposition.

Proposition 1.15. *Asumme that the families $\mathcal{S}_{\xi}$, $\xi < \omega_1$ of finite subsets of N have the genaralized Schreier property. Let K be a compact metric space, $\xi < \omega_1$, $(f_k) \subseteq C(K)$ and $\epsilon > 0$ such that for every $E = \{k_1 < \ldots < k_{\lambda}\} \in \mathcal{S}_{\xi}$ (where $\lambda \in \mathsf{N}$) there is $x_E \in K$ with $|f_{2k_j+1}(x_E) - f_{2k_j}(x_E)| > \epsilon$ for all $1 \leq j \leq \lambda$. Then there exists $x \in K$ such that $x \in K^{\omega^{\xi}}_{\epsilon,(f_k)}$.*

Proposition 1.16 (cf. [10, Prop. 3.2 and Th. 3.3(i) $\Rightarrow$ (iii)]). *Assume that the families $\mathcal{S}_{\xi}$, $\xi < \omega_1$ of finite subsets of N have the generalized Schreier* *property. Let K be a compact metric space, $1 \leq \xi < \omega_1$ $(f_k) \subseteq C(K)$ and $\epsilon > 0$ such that $\gamma((f_{n_k}), \epsilon) > \omega^\xi$ for every strictly increasing sequence (n_k) of positive integers. Then there exists a subsequence (f'_k) of (f_k) such that for every $F = \{k_1 < \ldots < k_\lambda\} \in \mathcal{S}_\xi$ there is $x_F \in K$ such that $|f'_{2k_j+1}(x_F) - f'_{2k_j}(x_F)| > \frac{\epsilon}{4}$ for all $1 \leq j \leq \lambda$.*

2. Main results.

Theorem 2.1. *Let $n \in \mathsf{N}$, $\xi_1, \ldots, \xi_n < \omega_1$, $N \in [\mathsf{N}]$ and $\{\mathcal{P}_1, \mathcal{P}_2\}$ a partition of $[N]^{<\omega}$. Then there exists $N' \in [N]$ such that*

either $\cup_{i=1}^n E_i \in \mathcal{P}_1$ for every $E_1 \in \mathcal{M}_{\xi_1}[N'], \ldots, E_n \in \mathcal{M}_{\xi_n}[N']$ with $E_1 < \ldots < E_n$;

or $\cup_{i=1}^n E_i \in \mathcal{P}_2$ for every $E_1 \in \mathcal{M}_{\xi_1}[N'], \ldots, E_n \in \mathcal{M}_{\xi_n}[N']$ with $E_1 < \ldots < E_n$.

For the proof of the above theorem we shall use the next proposition.

Proposition 2.2. *For every $n \in \mathsf{N}$, $\xi_1, \ldots, \xi_n < \omega_1$ and for every $N \in [\mathsf{N}]$ the family*

$$\mathcal{M}_{\xi_1,\ldots,\xi_n}[N] = \{\cup_{i=1}^n E_i : E_1 \in \mathcal{M}_{\xi_1}[N], \ldots, E_n \in \mathcal{M}_{\xi_n}[N] \ \ \text{with} \ \ E_1 < \ldots < E_n\}$$

is thin subset of $[N]^{<\omega}$.

For the proof of this proposition we shall use a method created by Prof. Negrepontis and the author (cf. [12] or [15, Def. 3.6, Lemma 3.7]). This method consists in a double induction. More precisely, we give the next definition and we prove Lemmas 2.2.2 and 2.2.3.

Definition 2.2.1. *For any $n \in \mathsf{N}$ and $\xi_1, \ldots, \xi_n < \omega_1$ we say that the n-tuple $(\xi_1, \ldots, \xi_n)$ has property (T) if whenever $N \in [\mathsf{N}]$ and $E_1, F_1 \in \mathcal{M}_{\xi_1}[N], \ldots, E_n, F_n \in \mathcal{M}_{\xi_n}[N]$ with $E_1 < \ldots < E_n$ and $F_1 < \ldots < F_n$ such that $E_1 \cup \ldots \cup E_n$ is an initial segment of $F_1 \cup \ldots \cup F_n$ then $E_i = F_i$ for every $i = 1, \ldots, n$.*

Lemma 2.2.2. *If $(\xi_1, \ldots, \xi_n)$ has property (T) then $(\xi, \xi_1, \ldots, \xi_n)$ has property (T) for every $\xi < \omega_1$.*

Proof. We proceed by induction on $\xi < \omega_1$.

Step 1: $\xi = 0$.
Assume that $(\xi_1, \ldots, \xi_n)$ has property (T) and we shall prove that $(0, \xi_1, \ldots, \xi_n)$ has property (T). Indeed, let $N \in [\mathsf{N}]$, $m_1, m_2 \in N$, $E_1, F_1 \in \mathcal{M}_{\xi_1}[N], \ldots, E_n, F_n \in \mathcal{M}_{\xi_n}[N]$ with $m_1 < E_1 < \ldots < E_n$, $m_2 < F_1 < \ldots < F_n$ and $\{m_1\} \cup E_1 \cup \ldots \cup E_n$ is an initial segment of $\{m_2\} \cup F_1 \cup \ldots \cup F_n$. Then $m_1 = m_2$ and $E_1 \cup \ldots \cup E_n$ is an initial segment of $F_1 \cup \ldots \cup F_n$. Since $(\xi_1, \ldots, \xi_n)$ has property (T) we have $E_i = F_i$ for every $i = 1, \ldots, n$.

Step 2: Suppose that $1 \leq \xi < \omega_1$ such that the conclusion holds for every $\zeta < \xi$ and we shall prove that it holds for ξ.
Assume that the n-tuple $(\xi_1, \ldots, \xi_n)$ has property (T) and we shall show that $(\xi, \xi_1, \ldots, \xi_n)$ has property (T). Indeed, let $N \in [\mathsf{N}]$, $H, G \in \mathcal{M}_\xi[N]$, $E_1, F_1 \in \mathcal{M}_{\xi_1}[N], \ldots, E_n, F_n \in \mathcal{M}_{\xi_n}[N]$ with $H < E_1 < \ldots < E_n$, $G < F_1 < \ldots < F_n$ and $H \cup E_1 \cup \ldots \cup E_n$ is initial segment of $G \cup F_1 \cup \ldots \cup F_n$. Then $\min H = \min G$. We set $m = \min H = \min G$. Consider these two cases:

Case 1: $\xi = \zeta + 1$, where $\zeta < \omega_1$. Then $H = H_1 \cup \ldots \cup H_m$, where $m = \min H_1$, $H_1 < \ldots < H_m$ and and $H_j \in \mathcal{M}_\zeta[N]$ for every $j = 1, 2, \ldots, m$. Also $G = G_1 \cup \ldots \cup G_m$, where $m = \min G_1$, $G_1 < \ldots < G_m$ and $G_j \in \mathcal{M}_\zeta[N]$ for every $j = 1, \ldots, m$. Clearly $H_1 \cup \ldots \cup H_m \cup E_1 \cup \ldots \cup E_n$ is initial segment of $G_1 \cup \ldots \cup G_m \cup F_1 \cup \ldots \cup F_n$ and since $\zeta < \xi$ we have that $(\underbrace{\zeta, \ldots, \zeta}_{m-\text{times}}, \xi_1, \ldots, \xi_n)$ has property (T) by the inductive assumption, whence $H_j = G_j$ for $j = 1, \ldots, m$ and $E_i = F_i$ for $i = 1, \ldots, n$.

Case 2: ξ is a limit ordinal. Let (ζ_k) be the strictly increasing sequence of successor ordinals with $\sup_k \zeta_k = \xi$ that defines the family $\mathcal{M}_\xi[N]$. Since $m = \min H = \min G = m$ we have $H, G \in \mathcal{M}_{\zeta_m}[N]$. Since $\zeta_m < \xi$ the $(n+1)-$tuple $(\zeta_m, \xi_1, \ldots, \xi_n)$ has property (T), by the inductive assumption. Therefore $H = G$ and $E_i = F_i$ for $i = 1, \ldots, n$. The proof is complete. □

Lemma 2.2.3. *The n-tuple* $(\xi_1, \ldots, \xi_n)$ *has property* (T) *for every* $n \in \mathsf{N}$ *and* $\xi_1, \ldots, \xi_n < \omega_1$.

P r o o f. By induction on $\xi < \omega_1$ we prove that (ξ) has property (T). For $\xi = 0$ is trivial. Now, let $1 \leq \xi < \omega_1$ such that the 1-tuple (ζ) has property (T) for every $\zeta < \xi$.

If $\xi = \zeta + 1$, where $\zeta < \omega_1$ then (ζ) has property (T) and therefore, by Lemma 2.2.2, $(\underbrace{\zeta, \ldots, \zeta}_{j-\text{times}})$ has property (T) for every $j \in \mathsf{N}$. By using the definition of the property (T) we prove that (ξ) has property (T).

If ξ is a limit ordinal and (ζ_k) a strictly increasing sequence of ordinals with $\sup_k \zeta_k = \xi$ then the 1-tuple (ζ_j) has the property (T) for every $j \in \mathsf{N}$.

By using the definition of the property (T) we obtain that the 1-tuple (ξ) has property (T).

Therefore, by Lemma 2.2.2, $(\xi_1, \dots, \xi_n)$ has property (T) for every $\xi_1, \dots, \xi_n < \omega_1$. $\square$

Proof of Theorem 2.1. It is immediate by Proposition 2.2 and Theorem 1.10. $\square$

The following theorem is an other form of Theorem 1.3. By the proof of Theorem 2.3 we obtain an alternative proof of Theorem 1.3.

Theorem 2.3. *Assume that the families $\mathcal{S}_\xi$, $\xi < \omega_1$ of finite subsets of N have the generalized Schreier property. Let $\mathcal{F}$ be an hereditary family of finite subsets of N, $\xi < \omega_1$, $N \in [\mathsf{N}]$ and δ a positive real number such that for every $N' \in [N]$ there exist $H \in \mathcal{M}_\xi[N']$ and $F \in \mathcal{F}$ with $a_F^\xi(H) \geq \delta$.*

Then there exists a strictly increasing sequence (n_k) of elements of N such that

$$\{m_j : j \in E\} \in \mathcal{F} \quad \textit{for all} \quad E \in \mathcal{S}_\xi.$$

The proof of the above theorem requires the following lemmas.

Lemma 2.3.1. *Let $\zeta < \omega_1$, $N \in [\mathsf{N}]$, $\delta > 0$ and $0 < \delta' < \delta$. Then for every ordinal ξ with $\zeta < \xi < \omega_1$ and $L \in [N]$ there exists $L_\xi \in [L]$ satisfying the following property:*

For every $H \in \mathcal{M}_\xi[L_\xi]$ and $F \in [\mathsf{N}]^{<\omega}$ with $a_F^\xi(H) \geq \delta$ there exists $H' \in \mathcal{M}_\zeta[L_\xi]$ such that $H' \subseteq H$ and $a_F^\xi(H) \geq \delta'$.

Proof. Fix $\zeta < \omega_1$, $N \in [\mathsf{N}]$. We shall prove it by induction for ξ greater than ζ, every $L \in [N]$, $\delta > 0$ and $0 < \delta' < \delta$. Consider these next cases:

Case 1: $\xi = \eta + 1$, where $\eta < \omega_1$. Indeed, if $N \in [\mathsf{N}]$, $L \in [N]$, $\delta > 0$ and $0 < \delta' < \delta$ then there exists $L_\eta \in [L]$ satisfying the conclusion for the ordinal η. We set $L_\xi = L_\eta$ and it is obvious that for every $H = H_1 \cup \ldots \cup H_m \in \mathcal{M}_\xi[L_\xi]$, where $H_1, \dots, H_m \in \mathcal{M}_\eta[L_\xi]$ with $\min H_1 = m$, $H_1 < \ldots < H_m$ and $F \in [\mathsf{N}]^{<\omega}$ with $a_F^\xi(H) \geq \delta$ there exists $i_0 \in \{1, \dots, m\}$ such that $a_F^\eta(H_{i_0}) \geq \delta$ and so, by the inductive assumption, there exists $H' \in \mathcal{M}_\zeta[L_\xi]$ such that $H' \subseteq H_{i_0}$ and $a_F^\zeta(H') \geq \delta'$.

Case 2: ξ is a limit ordinal. Fix the strictly increasing sequence (ξ_n) of successor ordinals such that $\sup_n \xi_n = \xi$ that defines the family $\mathcal{M}_\xi[N]$. Since each ξ_n is successor ordinal it has the form $\xi_n = \zeta_n + 1$.

Choose $L_0 \in [N]$ with $\min L_0 = m_1$, where $m_1 \in N$ with $m_1 \geq 2$, $\zeta_{m_1} > \zeta$ and $\dfrac{1}{m_1} < \dfrac{\delta - \delta'}{2}$. We inductively choose $L_0 \supseteq L_1 \supseteq \ldots \supseteq L_k \supseteq \ldots$ such that if $m_k = \min L_{k-1}$ then (m_k) is strictly increasing and $L_k = L_{\zeta_{m_k}}$ for $N = L_{k-1}$, $\delta = \dfrac{\delta + \delta'}{2}$ and δ'.

Claim. *The set $\{m_k : k = 1, 2, \ldots\}$ is the desired L_ξ.*

Indeed, let $H \in \mathcal{M}_\xi[L_\xi]$ and $F \in [N]^{<\omega}$ with $a_F^\xi(H) \geq \delta$. Then $H \in \mathcal{M}_{\xi_{m_n}}[L_\xi]$, where $m_n = \min H$. Since $\xi_{m_n} = \zeta_{m_n} + 1$ there are $H_1, \ldots, H_{m_n} \in \mathcal{M}_{\zeta_{m_n}}[L_\xi]$ such that $\min H_1 = m_n$, $H_1 < \ldots < H_m$ and $H = H_1 \cup \ldots \cup H_{m_n}$. Then

$$\delta \leq a_F^\xi(H) = a_F^{\xi_{m_n}}(H) = \frac{1}{m_n}\left(\sum_{i=1}^{m_n} a_F^{\zeta_{m_n}}(H_i)\right)$$

and hence $\dfrac{1}{m_n}\sum_{i=2}^{m_n} a_F^{\zeta_{m_n}}(H_i) \geq \delta - \dfrac{1}{m_n} > \dfrac{\delta + \delta'}{2}$. Then there exists $2 \leq i_0 \leq m_n$ such that $a_F^{\zeta_{m_n}}(H_{i_0}) \geq \dfrac{\delta + \delta'}{2}$. It is clear that $\min H_{i_0} > m_n$ and $\dfrac{\delta + \delta'}{2} > \delta'$. Hence $H_{i_0} \in \mathcal{M}_{\zeta_{m_n}}[L_n]$ and hence there exists $H' \in \mathcal{M}_\zeta[L_n]$ such that $a_F^\zeta(H') \geq \delta'$ and $H' \subseteq H_{i_0}$. This completes the proof of Lemma. □

The next Definition and Lemmas 2.3.3, 2.3.4 are based in the method of double induction created by Prof. Negrepontis and the author (cf. [12] or [15, Def. 3.6, Lemma 3.7]).

Definition 2.3.2. *Assume that the families $\mathcal{S}_\xi$, $\xi < \omega_1$ of finite subsets of N have the generalized Schreier property. For $n \in \mathsf{N}$ and $\xi_1, \ldots, \xi_n < \omega_1$ we say that the n-tuple $(\xi_1, \ldots, \xi_n)$ has property $(*)$ if whenever $\mathcal{F}$ is a hereditary family of finite subsets of N, $N \in [\mathsf{N}]$ and δ a positive real number such that for every $H_1 \in \mathcal{M}_{\xi_1}[N], \ldots, H_n \in \mathcal{M}_{\xi_n}[N]$ with $H_1 < \ldots < H_n$ there exists $F \in \mathcal{F}$ with $a_F^{\xi_i}(H_i) \geq \delta$ for every $i = 1, \ldots, n$ then there exists a strictly increasing sequence (m_k) of elements of N such that $\{m_j : j \in \cup_{i=1}^n E_i\} \in \mathcal{F}$ for all $E_1 \in \mathcal{S}_\xi, \ldots, E_n \in \mathcal{S}_{\xi_n}$ with $E_1 < \ldots < E_n$.*

Lemma 2.3.3. *If $(\xi_1, \ldots, \xi_n)$ has property $(*)$ then $(\xi, \xi_1, \ldots, \xi_n)$ has property $(*)$ for every $\xi < \omega_1$.*

Proof. We proceed by induction on $\xi < \omega_1$.

Step 1: $\xi = 0$. Assume that $(\xi_1, \ldots, \xi_n)$ has property $(*)$ and we shall show that $(0, \xi_1, \ldots, \xi_n)$ has property $(*)$. Indeed, let $\mathcal{F}$ be a hereditary family of finite subsets of N, $N \in [\mathsf{N}]$ and δ a positive real number such that for every $m \in N$, $H_1 \in \mathcal{M}_{\xi_1}[N], \ldots, H_n \in \mathcal{M}_{\xi_n}[N]$ with $m < H_1 < \ldots < H_n$ there exists $F \in \mathcal{F}$ with $a_F^0(\{m\}) \geq \delta$ and $a_F^{\xi_i}(H_i) \geq \delta$ for every $i = 1, \ldots, n$.

We set $m_1 = \min N$, $N_1 = (m_k^1) = N$, $N_1' = \{m \in N : m > m_1\}$ and $\mathcal{G}_1 = \{F \in \mathcal{F} : \min F > m_1, \{m_1\} \cup F \in \mathcal{F}\}$. By the hypothesis, for every $H_1 \in \mathcal{M}_{\xi_1}[N], \ldots, H_n \in \mathcal{M}_{\xi_n}[N]$ with $m_1 < H_1 < \ldots < H_n$ there exists $F \in \mathcal{F}$ such that $a_F^0(\{m_1\}) \geq \delta$ and $a_F^{\xi_i}(H_i) \geq \delta$ for every $i = 1, \ldots, n$. Since $a_F^0(\{m_1\}) = \chi_F(m_1) \geq \delta$ it follows $m_1 \in F$. Also the set $G = F \cap N_1'$ belongs to $\mathcal{G}_1$ and $a_G^{\xi_i}(H_i) = a_F^{\xi_i}(H_i)$ for every $i = 1, \ldots, n$. Because $(\xi_1, \ldots, \xi_n)$ has property $(*)$ there exists a strictly increasing sequence $N_2 = (m_k^2)$ of elements of N_1' such that

$$\{m_j^2 : j \in \cup_{i=1}^n E_i\} \in \mathcal{G}_1$$

for every $E_1 \in \mathcal{S}_{\xi_1}, \ldots, E_n \in \mathcal{S}_{\xi_n}$ with $E_1 < \ldots < E_n$.

By induction on $j \geq 1$, we can find a strictly increasing sequence $N_j = (m_k^j)$ of elements of N such that N_{j+1} is subsequence of N_j and setting $m_j = m_j^j$ and $\mathcal{G}_j = \{F \in \mathcal{F} : \min F > m_j, \{m_j\} \cup F \in \mathcal{F}\}$ we get

$$\{m_k^{j+1} : k \in \cup_{i=1}^n E_i\} \in \mathcal{G}_j$$

for every $E_1 \in \mathcal{S}_{\xi_1}, \ldots, E_n \in \mathcal{S}_{\xi_n}$ with $E_1 < \ldots < E_n$. By [3, Lemma 2.1.8(b)], if $\{p_1, \ldots, p_\lambda\} \in \mathcal{S}_\alpha$, where $\alpha < \omega_1$, and $q_1 \geq p_1, \ldots q_\lambda \geq p_\lambda$ then $\{q_1, \ldots, q_\lambda\} \in \mathcal{S}_\alpha$. The proof of Step 1 can be finished by taking the sequence (m_k) and using the above fact.

Step 2: Let $1 \leq \xi < \omega_1$ such that the conclusion of Lemma holds for every $\zeta < \xi$. We shall prove that it holds for ξ.

Indeed, we assume that $(\xi_1, \ldots, \xi_n)$ has property $(*)$ and we shall prove that $(\xi, \xi_1, \ldots, \xi_n)$ has property $(*)$. Let $\mathcal{F}$ be a hereditary family of finite subsets of N, $N \in [\mathsf{N}]$ and $\delta > 0$ such that for every $H \in \mathcal{M}_\xi[N], H_1 \in \mathcal{M}_{\xi_1}[N], \ldots, H_n \in \mathcal{M}_{\xi_n}[N]$ with $H < H_1 < \ldots < H_n$ there exists $F \in \mathcal{F}$ such that $a_F^\xi(H) \geq \delta$ and $a_F^{\xi_i}(H_i) \geq \delta$ for every $i = 1, \ldots, n$. Let δ' with $0 < \delta' < \delta$.

Consider the next cases:

Case 1: $\xi = \zeta + 1$, where $\zeta < \omega_1$. Then $\zeta < \xi$ and so, by the inductive assumption, $(\underbrace{\zeta, \ldots, \zeta}_{j-\text{times}}, \xi_1, \ldots, \xi_n)$ has property $(*)$ for every $j \in \mathsf{N}$.

Claim. *For every $j \in \mathsf{N}$ and for every $N' \in [N]$ there exists $N'' \in [N']$ such that for every $G_1, \ldots, G_j \in \mathcal{M}_\zeta[N''], H_1 \in \mathcal{M}_{\xi_1}[N''], \ldots, H_n \in \mathcal{M}_{\xi_n}[N'']$ with $G_1 < \ldots < G_j < H_1 < \ldots < H_n$ there exists $F \in \mathcal{F}$ such that $a_F^\zeta(G_\lambda) \geq \delta'$*

for $\lambda = 1, \ldots, j$ *and* $a_F^{\xi_i}(H_i) \geq \delta'$ *for* $i = 1, \ldots, n$.

Proof of Claim. Let $j \in \mathsf{N}$ and $N' \in [N]$. We set

$$\mathcal{P}_1 = \{\cup_{\lambda=1}^{j} G_\lambda \cup \cup_{i=1}^{n} H_i : G_1, \ldots, G_j \in \mathcal{M}_\zeta[N'], H_1 \in \mathcal{M}_{\xi_1}[N'], \ldots, H_n$$

$$\in \mathcal{M}_{\xi_n}[N'] \text{ with } G_1 < \ldots < G_j < H_1 < \ldots < H_n \text{ such that there exists}$$

$$F \in \mathcal{F} \text{ with } a_F^{\zeta}(G_\lambda) \geq \delta' \text{ and } a_F^{\xi_i}(H_i) \geq \delta' \text{ for every } \lambda = 1, \ldots, j, \ i = 1, \ldots, n\}$$

and $\mathcal{P}_2 = [N']^{<\omega} \setminus \mathcal{P}_1$. $\{\mathcal{P}_1, \mathcal{P}_2\}$ is a partition of $[N']^{<\omega}$. By Theorem 2.1, it is enough to show that for every $N'' \in [N']$ there exist $G_1, \ldots, G_j \in \mathcal{M}_\zeta[N'']$, $H_1 \in \mathcal{M}_{\xi_1}[N''], \ldots, H_n \in \mathcal{M}_{\xi_n}[N'']$ with $G_1 < \ldots < G_j < H_1 < \ldots < H_n$ such that there exists $F \in \mathcal{F}$ with

$$a_F^{\zeta}(G_\lambda) \geq \delta' \text{ for } \lambda = 1, \ldots, j \text{ and } a_F^{\xi_i}(H_i) \geq \delta' \text{ for } i = 1, \ldots, n.$$

Indeed, let $N'' \in [N']$. We choose $k_j \in N''$ with $\dfrac{j}{k_j} < \delta - \delta'$. Let $G = G_1 \cup \ldots \cup G_{k_j} \in \mathcal{M}_\xi[N'']$, where $\min G_1 = k_j$, $G_1, \ldots, G_{k_j} \in \mathcal{M}_\zeta[N'']$ and $G_1 < \ldots < G_{k_j}$, and let $H_1 \in \mathcal{M}_{\xi_1}[N''], \ldots, H_n \in \mathcal{M}_{\xi_n}[N'']$ with $G < H_1 < \ldots < H_n$. Then there exists $F \in \mathcal{F}$ such that

$$a_F^{\xi}(G) \geq \delta \text{ and } a_F^{\xi_i}(H_i) \geq \delta \text{ for every } i = 1, \ldots, n.$$

We claim that $|\{1 \leq \lambda \leq k_j : a_F^{\zeta}(G_\lambda) \geq \delta'\}| \geq j$. Indeed, we assume that $|\{1 \leq \lambda \leq k_j : a_F^{\zeta}(G_\lambda) \geq \delta'\}| < j$. Then

$$\delta \leq a_F^{\xi}(G) = \frac{1}{k_j} \sum_{\lambda=1}^{k_j} a_F^{\zeta}(G_\lambda) < \frac{j}{k_j} + \frac{k_j - j}{k_j}\delta' < \delta - \delta' + \delta' = \delta,$$

a contradiction. This completes the proof of Claim. □

By using that $(\underbrace{\zeta, \ldots, \zeta}_{j-\text{times}}, \xi_1, \ldots, \xi_n)$ has the property $(*)$ for every $j \in \mathsf{N}$ and using the claim we can find strictly increasing sequences $N_j = (m_k^j)$, $j \in \mathsf{N}$ of elements of N such that for every $j \in \mathsf{N}$, N_{j+1} is subsequence of N_j and $\{m_k^j : k \in \cup_{\lambda=1}^{j} E_\lambda \cup \cup_{i=1}^{n} F_i\} \in \mathcal{F}$ for every $E_1, \ldots, E_j \in \mathcal{S}_\zeta$, $F_1 \in \mathcal{S}_{\xi_1}, \ldots, F_n \in \mathcal{S}_{\xi_n}$ with $E_1 < \ldots < E_j < F_1 < \ldots < F_n$. The proof of Case 1 can be finished by taking the diagonal sequence $\{m_k^k : k = 1, 2, \ldots\}$ and using [3, Lemma 2.1.8(b)].

Case 2: ξ is a limit ordinal. Let (ζ_k) be a strictly increasing sequence of ordinals with $\sup_k \zeta_k = \xi$. By the inductive assumption, $(\zeta_k, \xi_1, \ldots, \xi_n)$ has property $(*)$ for every $k \in \mathsf{N}$.

Claim. *For* $j \in \mathsf{N}$, $N' \in [N]$ *there exists* $N'' \in [N']$ *such that for every*

$H \in \mathcal{M}_{\zeta_j}[N''], H_1 \in \mathcal{M}_{\xi_1}[N''], \ldots, H_n \in \mathcal{M}_{\xi_n}[N'']$ *with* $H < H_1 < \ldots < H_n$ *there exists* $F \in \mathcal{F}$ *with*

$$a_F^{\zeta_j}(H) \geq \delta' \quad and \quad a_F^{\xi_i}(H_i) \geq \delta' \quad for \quad every \quad i = 1, \ldots, n.$$

P r o o f o f C l a i m. Let $j \in \mathsf{N}$ and $N' \in [N]$. By Theorem 2.1, it is enough to show that for every $N'' \in [N']$ there exist $H \in \mathcal{M}_{\zeta_j}[N''], H_1 \in \mathcal{M}_{\xi_1}[N''], \ldots, H_n \in \mathcal{M}_{\xi_n}[N'']$ with $H < H_1 < \ldots < H_n$ and there exists $F \in \mathcal{F}$ with $a_F^{\zeta_j}(H) \geq \delta'$ and $a_F^{\xi_i}(H_i) \geq \delta'$ for every $i = 1, \ldots, n$.

Indeed, let $N'' \in [N']$. By Lemma 2.3.1, there exists $L \in [N'']$ such that for every $H \in \mathcal{M}_\xi[L]$ and $F \in \mathcal{F}$ with $a_F^\xi(H) \geq \delta$ there exists $H' \in \mathcal{M}_{\zeta_j}[L]$ such that $H' \subseteq H$ and $a_F^{\zeta_j}(H') \geq \delta'$. Choose $H \in \mathcal{M}_\xi[L]$, $H_1 \in \mathcal{M}_{\xi_1}[L], \ldots, H_n \in \mathcal{M}_{\xi_n}[L]$ with $H < H_1 < \ldots < H_n$. By the hypothesis there exists $F \in \mathcal{F}$ such that $a_F^\xi(H) \geq \delta$ and $a_F^{\xi_i}(H_i) \geq \delta$ for every $i = 1, \ldots, n$. For H fix $H' \in \mathcal{M}_{\zeta_j}[L]$ such that $H' \subseteq H$ and $a_F^{\zeta_j}(H') \geq \delta'$. Since $H' \subseteq H$ and $H < H_1 < \ldots < H_n$ we have $H' < H_1 < \ldots < H_n$. This completes the proof of Claim. □

We set $M_0 = M$. By using that for every $j \geq 1$, $(\zeta_j, \xi_1, \ldots, \xi_n)$ has property $(*)$ and using the claim we can find a subsequence $M_j = (m_k^j)$ of M_{j-1} such that $\{m_k^j : k \in E \cup \cup_{i=1}^n F_i\} \in \mathcal{F}$ for all $E \in \mathcal{S}_{\zeta_j}, F_1 \in \mathcal{S}_{\xi_1}, \ldots, F_n \in \mathcal{S}_{\xi_n}$ with $E < F_1 < \ldots < F_n$. The proof of Case 2 can be finished by taking the diagonal sequence $\{m_k^k : k = 1, 2, \ldots\}$ and using again [3, Lemma 2.1.8(b)].

Lemma 2.3.4. *The* n*-tuple* $(\xi_1, \ldots, \xi_n)$ *has the property* $(*)$ *for every* $n \in \mathsf{N}$ *and* $\xi_1, \ldots, \xi_n < \omega_1$.

P r o o f. By induction on $\xi < \omega_1$, we prove that (ξ) has property $(*)$. For $\xi = 0$ is trivial. Now, let $1 \leq \xi < \omega_1$ such that the 1-tuple (ζ) has property $(*)$ for every $\zeta < \xi$.

If $\xi = \zeta + 1$, where $\zeta < \omega_1$ then (ζ) has the property $(*)$ and therefore, by Lemma 2.3.3, $(\underbrace{\zeta, \ldots, \zeta}_{j-\text{times}})$ has property $(*)$ for every $j \in \mathsf{N}$. By using the definition of the property $(*)$ and a diagonal argument we prove that (ξ) has property $(*)$.

If ξ is a limit ordinal and (ζ_k) a strictly increasing sequence of ordinals with $\sup_k \zeta_k = \xi$ then the 1-tuple (ζ_j) has the property $(*)$ for every $j \in \mathsf{N}$. Using the definition of the property $(*)$ and a diagonal argument we obtain that the 1-tuple (ξ) has the property $(*)$. Therefore, by Lemma 2.3.3, $(\xi_1, \ldots, \xi_n)$ has property $(*)$ for every $\xi_1, \ldots, \xi_n < \omega_1$. □

P r o o f o f T h e o r e m 2.3. We set

$$\mathcal{P}_1 = \{H \in \mathcal{M}_\xi[N] : \text{ there exists } F \in \mathcal{F} \text{ such that } a_F^\xi(H) \geq \delta\}$$

and $\mathcal{P}_2 = \mathcal{M}_\xi[N] \backslash \mathcal{P}_1$. $\{\mathcal{P}_1, \mathcal{P}_2\}$ is a partition of the set $\mathcal{M}_\xi[N]$. By the assumption $\mathcal{P}_1 \cap \mathcal{M}_\xi[N']$ is nonempty for every $N' \in [N]$. Therefore, by Theorem 2.1, there exists $N' \in [N]$ such that for every $H \in \mathcal{M}_\xi[N']$ there exists $F \in \mathcal{F}$ such that $a_F^\xi(H) \geq \delta$. Also the 1-tuple (ξ) has property $(*)$ by Lemma 2.3.4. Hence there exists a strictly increasing sequence (m_k) of elements of N' such that $\{m_j : j \in F\} \in \mathcal{F}$ for every $F \in \mathcal{S}_\xi$. □

Proposition 2.4. *Let (f_k) be a sequence of real-valued functions defined on a set X, δ a positive real number and $\xi < \omega_1$. Then for every $N \in [\mathsf{N}]$ there exists $N' \in [N]$ such that*

either $\|a^\xi((f_k); H)\|_\infty < \delta$ for every $H \in \mathcal{M}_\xi[N']$;

or $\|a^\xi((f_k); H)\|_\infty \geq \delta$ for every $H \in \mathcal{M}_\xi[N']$.

P r o o f. We set

$$\mathcal{P}_1 = \{H \in \mathcal{M}_\xi[N] : \|a^\xi((f_k); H)\|_\infty < \delta\}$$

and $\mathcal{P}_2 = [N]^{<\omega} \setminus \mathcal{P}_1$. By Theorem 2.1, there exists $N' \in [N]$ such that either $\mathcal{M}_\xi[N'] \subseteq \mathcal{P}_1$ or $\mathcal{M}_\xi[N'] \subseteq \mathcal{P}_2$. The proof is complete. □

Theorem 2.5. *Let (f_k) be a sequence of real-valued functions defined on a set X, $M \in [\mathsf{N}]$ and $\xi < \omega_1$. Then there exists $N \in [M]$ such that*

either (1) *for every strictly increasing sequence (H_n) of members of $\mathcal{M}_\xi[N]$ the sequence $g_n = a^\xi((f_k); H_n)$, $n \in \mathsf{N}$ converges uniformly to zero;*

or (2) *does not exist a strictly increasing sequence (H_n) of members of $\mathcal{M}_\xi[N]$ such that the sequence $g_n = a^\xi((f_k); H_n)$, $n \in \mathsf{N}$ converges uniformly to zero.*

Before to prove this theorem we shall use a method created by Mercourakis in [14]. This method consists in two next lemmas.

Lemma 2.5.1. *Let (f_k) be a sequence of real-valued functions on a set X, $N_0 \in [\mathsf{N}]$ and $\xi < \omega_1$. Suppose that every $N \in [N_0]$ has the following property:*

(**) for each $\delta > 0$ there exists $N_\delta \in [N]$ such that $\|a^\xi((f_k); H)\|_\infty < \delta$ for all $H \in \mathcal{M}_\xi[N_\delta]$.

Then there exists $N \in [N_\delta]$ such that for every strictly increasing sequence (H_n) of members of $\mathcal{M}_\xi[N]$ the sequence $g_n = a^\xi((f_k); H_n)$, $n \in \mathsf{N}$ converges uniformly to zero.

P r o o f. By our assumption applied to $N = N_0$ and $\delta = 1$, there exists $N_1 \in [N_0]$ such that $\|a^\xi((f_k);H)\|_\infty < 1$ for all $H \in \mathcal{M}_\xi[N_1]$. We proceed inductively and find a decreasing sequence of infinite subsets of N_0 $N_1 \supseteq N_2 \supseteq \ldots \supseteq N_n \supseteq \ldots$ such that for every $n = 1, 2, \ldots$ we have,

$$\|a^\xi((f_k);H)\|_\infty < \frac{1}{n} \text{ for all } H \in \mathcal{M}_\xi[N_n].$$

We choose a strictly increasing sequence $m_1 < m_2 < \ldots < m_n < \ldots$ of positive integers such that $m_n \in N_n$ for all $n = 1, 2, \ldots$. We claim that $N = \{m_n : n = 1, 2, \ldots\}$ is the desired set. Indeed, let (H_n) be a strictly increasing sequence of elements of $\mathcal{M}_\xi[N]$, $\delta > 0$ and $n_0 \in \mathsf{N}$ such that $\frac{1}{n_0} < \delta$. Then there exists $n_1 \in \mathsf{N}$ such that $H_n \in \mathcal{M}_\xi[N_{n_0}]$ for every $n \geq n_1$ and hence $\|a^\xi((f_k);H_n)\|_\infty < \frac{1}{n_0} < \delta$ for every $n \geq n_1$. The proof of our Lemma is complete. □

Lemma 2.5.2. *Let (f_k) be a sequence of real-valued functions on a set X, $M \in [\mathsf{N}]$ and $\xi < \omega_1$. Suppose that there exists $N_0 \in [M]$ not having property (**) (stated in the previous lemma). Then there exists $N \in [M]$ and $\delta > 0$ such that $\|a^\xi((f_k);H)\|_\infty \geq \delta$ for every every $H \in \mathcal{M}_\xi[N]$.*

P r o o f. Since N_0 does not have property (**) there exists $\delta > 0$ such that for every $N \in [N_0]$ there exists $H \in \mathcal{M}_\xi[N]$ such that $\|a^\xi((f_k);H)\|_\infty \geq \delta$. Then from Proposition 2.4, there exists $N \in [N_0]$ such that $\|a^\xi((f_k);H)\|_\infty \geq \delta$ for every $H \in \mathcal{M}_\xi[N]$, which finishes the proof of the lemma. □

P r o o f o f T h e o r e m 2.5. Suppose that every $N \in [M]$ has property $(**)$ of Lemma 2.5.1, then (according to this lemma) there exists $N \in [M]$ such that for every strictly increasing sequence (H_n) of elements of $\mathcal{M}_\xi[N]$ the sequence $g_n = a^\xi((f_k);H_n)$, $n \in \mathsf{N}$ converges uniformly to zero, namely we get (1). If there exists $N_0 \in [M]$ not having $(**)$ then we get (2) by Lemma 2.5.2. The proof of Theorem 2.5 is complete. □

Proposition 2.6. *Let K be a compact metric space, (f_k) a uniformly bounded sequence of continuous real-valued functions defined on K and $\xi < \omega_1$ with $\gamma((f_k)) \leq \omega^\xi$. Then for every $M \in [\mathsf{N}]$ there exists $N \in [M]$ such that for every strictly increasing sequence (H_n) of members of $\mathcal{M}_\xi[N]$ the sequence $g_n = \alpha^\xi((f_{2k+1} - f_{2k});H_n)$, $n \in \mathsf{N}$ converges uniformly to zero.*

P r o o f. Let $M \in [\mathsf{N}]$. By Lemma 2.5.1, it is enough to prove that for every $\delta > 0$ and $N \in [M]$ there exists $N_\delta \in [N]$ such that $\|a^\xi((f_{2k+1} - f_{2k});H)\|_\infty < \delta$ for all $H \in \mathcal{M}_\xi[N_\delta]$. Indeed, we assume that there exists $\delta > 0$ such that for every $N' \in [N]$ there exists $H \in \mathcal{M}_\xi[N']$ such that $\|a^\xi((f_{2k+1} - f_{2k});H)\|_\infty \geq \delta$.

For every $x \in K$ let $F_x = \left\{k \in \mathsf{N} : |f_{2k+1}(x) - f_{2k}(x)| > \frac{\delta}{2}\right\}$. Then for every $H \in \mathcal{M}_\xi[N]$ and $x \in K$ we have

$$|a^\xi((f_{2k+1} - f_{2k}); H)(x)| \leq \frac{\delta}{2} + a^\xi_{F_x}(H)(\sup_k \|f_k\|_\infty).$$

So, for every $N' \in [N]$ there exist $H \in \mathcal{M}_\xi[N']$ and $x \in K$ such that $a^\xi_{F_x}(H) \geq \dfrac{\delta}{2 \sup_k \|f_k\|_\infty}$.

Therefore, by Theorem 2.3, there exists a strictly increasing sequence (m_k) of elements of N such that for every $F \in \mathcal{S}_\xi$ there exists $x \in K$ with $|f_{2m_k+1}(x) - f_{2m_k}(x)| \geq \frac{\delta}{2}$ for every $k \in F$.

We set $n_1 = 1$ and $n_{2k} = 2m_k$, $n_{2k+1} = 2m_k+1$ for every $k \in \mathsf{N}$. Therefore, by Proposition 1.15, $K^{\omega^\xi}_{\frac{\delta}{2},(f_{n_k})} \neq \emptyset$ and hence $\gamma((f_k)) > \omega^\xi$. $\square$

Proposition 2.7. *Let K be a compact metric space, (f_k) a sequence of continuous real-valued functions and $1 \leq \xi < \omega_1$ such that for every subsequence (f'_k) of (f_k) there exists a strictly increasing sequence (H_n) of members of $\mathcal{M}_\xi[\mathsf{N}]$ such that the sequence $g_n = a^\xi((f'_{2k+1} - f'_{2k}); H_n)$, $n \in \mathsf{N}$ converges uniformly to zero. Then there exists a subsequence (f'_k) of (f_k) such that $\gamma((f'_k)) \leq \omega^\xi$.*

Proof. Assume that $\gamma((f'_k)) > \omega^\xi$ for every subsequence (f'_k) of (f_k).

Claim. *There exist $\epsilon > 0$ and a subsequence (f'_k) of (f_k) such that $\gamma((f''_k), \epsilon) > \omega^\xi$ for every subsequence (f''_k) of (f'_k).*

Proof of Claim. Assume the contrary. Then for every $\epsilon > 0$ and for every subsequence (f'_k) of (f_k) there exists a subsequence (f''_k) of (f'_k) such that $\gamma((f''_k), \epsilon) \leq \omega^\xi$. We set $M_0 = \mathsf{N}$. So, by induction on $m \in \mathsf{N}$, there exists a subsequence $M_m = (n^m_k)$ of M_{m-1} such that $\gamma((f_{n^m_k}), \frac{1}{m}) \leq \omega^\xi$. Then for the sequence $f'_k = f_{n^k_k}$, $k \in \mathsf{N}$ we have $\gamma((f'_k)) \leq \omega^\xi$, a contradiction. The proof of Claim is complete. $\square$

For every $\alpha < \omega_1$ we set $\mathcal{S}_\alpha = \{F \in [\mathsf{N}]^{<\omega} : F \subseteq H \text{ for some } H \in \mathcal{M}_\alpha[\mathsf{N}]\}$. The families $\{\mathcal{S}_\alpha\}_{\alpha<\omega_1}$ have the generalized Schreier property. By Claim and Proposition 1.16, there exists a subsequence (f'_k) of (f_k) such that for every $F = \{k_1 < \ldots < k_\lambda\} \in \mathcal{S}_\xi$ there exists $x_F \in K$ with

$$|f'_{2k+1}(x_F) - f'_{2k}(x_F)| > \frac{\epsilon}{4} \quad \text{for all} \quad k \in F.$$

For every $x \in K$ we set

$$F_x = \left\{k \in \mathsf{N} : |f'_{2k+1}(x) - f'_{2k}(x)| > \frac{\epsilon}{4}\right\},$$

$$F_x^+ = \left\{k \in \mathsf{N} : f'_{2k+1}(x) - f'_{2k}(x) > \frac{\epsilon}{4}\right\}$$

and

$$F_x^- = \left\{k \in \mathsf{N} : f'_{2k+1}(x) - f'_{2k}(x) < -\frac{\epsilon}{4}\right\}.$$

So, for every $H \in \mathcal{M}_\xi[\mathsf{N}]$ there exists $x \in K$ such that $a^\xi_{F_x}(H) = 1$. Also

$$a^\xi_{F_x}(H) = a^\xi_{F_x^+}(H) + a^\xi_{F_x^-}(H)$$

for every $x \in K$ and $H \in \mathcal{M}_\xi[\mathsf{N}]$. Therefore for every $N \in [\mathsf{N}]$ and $H \in \mathcal{M}_\xi[N]$ there exists $x \in K$ such that $a^\xi_{F_x^+}(H) \geq \frac{1}{2}$ or $a^\xi_{F_x^-}(H) \geq \frac{1}{2}$. By using Theorem 2.1, there exists $N \in [\mathsf{N}]$ such that

either for every $H \in \mathcal{M}_\xi[N]$ there exists $x \in K$ with $a^\xi_{F_x^+}(H) \geq \frac{1}{2}$;

or for every $H \in \mathcal{M}_\xi[N]$ there exists $x \in K$ with $a^\xi_{F_x^-}(H) \geq \frac{1}{2}$.

By Theorem 2.3, there exists a strictly increasing sequence (m_k) of elements of N such that

either for every $F = \{k_1 < \ldots < k_\lambda\} \in \mathcal{S}_\xi$ there exists $x \in K$ such that $f'_{2m_{k_i}+1}(x) - f'_{2m_{k_i}}(x) > \frac{\epsilon}{4}$ for every $1 \leq i \leq \lambda$;

or for every $F = \{k_1 < \ldots < k_\lambda\} \in \mathcal{S}_\xi$ there exists $x \in K$ with $f'_{2m_{k_i}+1}(x) - f'_{2m_{k_i}}(x) < -\frac{\epsilon}{4}$ for every $1 \leq i \leq \lambda$.

So, $\|a^\xi((f'_{2m_k+1} - f'_{2m_k}); H)\|_\infty \geq \frac{\epsilon}{4}$ for every $H \in \mathcal{M}_\xi[\mathsf{N}]$. We set $f''_1 = f'_1$, $f''_{2k+1} = f'_{2m_k+1}$ and $f''_{2k} = f'_{2m_k}$ for every $k \in \mathsf{N}$. Then $\|a^\xi((f''_{2k+1} - f''_{2k}); H)\|_\infty \geq \frac{\epsilon}{4}$ for every $H \in \mathcal{M}_\xi[\mathsf{N}]$, a contradiction by the hypothesis. This finishes the proof of Proposition. □

Corollary 2.8. *Let K be a compact metric space, $1 \leq \xi < \omega_1$ and $(f_k) \subseteq C(K)$ pointwise converging to f.*

(i) *If for every subsequence (f'_k) of (f_k) there exists a strictly increasing sequence (H_n) of members of $\mathcal{M}_\xi[\mathsf{N}]$ such that the sequence $g_n = a^\xi((f'_{2k+1} - f'_{2k}); H_n)$, $n \in \mathsf{N}$ converges uniformly to zero then $\beta(f) \leq \omega^\xi$.*

(ii) *If f is bounded and $\beta(f) \leq \omega^\xi$ then there exists a sequence (h_k) of convex blocks of (f_k) (i.e., $h_k \in conv((f_p)_{p\geq k})$ for all k) such that for every $M \in [\mathsf{N}]$ there exists $N \in [M]$ such that for every strictly increasing sequence (H_n) of members of $\mathcal{M}_\xi[N]$ the sequence $g_n = a^\xi((h_{2k+1} - h_{2k}); H_n)$, $n \in \mathsf{N}$ converges uniformly to zero. (Here $conv((\phi_k))$ denotes the set of convex combinations of the ϕ_k ' s.)*

P r o o f. (i) By Proposition 2.7 there exists a subsequence (f'_k) of (f_k) with $\gamma((f'_k)) \leq \omega^\xi$. Also $\beta(f) \leq \gamma((f'_k))$ by Proposition 1.1 of [9]. Hence $\beta(f) \leq \omega^\xi$.

(ii) By using [9; Theorem 1.3] or the proof of [11; Theorem 17] we prove that there exists a sequence (h_k) of convex blocks of (f_k) with $\gamma((h_k)) \leq \omega^\xi$. Therefore, by Proposition 2.6, the conclusion is immediate. □

Corollary 2.9. *Let K be a compact metric space and $1 \leq \xi < \omega_1$ such that for every sequence $(f_k) \subseteq C(K)$ pointwise converging to zero there exists a strictly increasing sequence (H_n) of members of $\mathcal{M}_\xi[\mathsf{N}]$ such that the sequence $g_n = a^\xi((f_k); H_n)$, $n \in \mathsf{N}$ converges uniformly to zero. Then $\beta(f) \leq \omega^\xi$ for every Baire-1 function f on K.*

P r o o f. Let $(f_k) \subseteq C(K)$ pointwise converging to f. Then the sequence $(f_{2k+1} - f_{2k})$ converges pointwise to zero. So, by the hypothesis, for every subsequence (f'_k) of (f_k) there exists a strictly increasing sequence (H_n) of members of $\mathcal{M}_\xi[\mathsf{N}]$ such that the sequence $g_n = a^\xi((f'_{2k+1} - f'_{2k}); H_n)$, $n \in \mathsf{N}$ converges uniformly to zero. Therefore, by Corollary 2.8 (i), $\beta(f) \leq \omega^\xi$. □

Corollary 2.10. *Let E be a Banach space, $A = (x_n) \subseteq E$, $M \in [\mathsf{N}]$ and $\xi < \omega_1$. Then there exists $N \in [M]$ such that*

either $\lim_{n\to\infty} \|\xi^L_n \cdot A\| = 0$ *for every* $L \in [N]$;

or does not exist $L \in [N]$ *such that* $\lim_{n\to\infty} \|\xi^L_n \cdot A\| = 0$.

P r o o f. We consider the elements x_n, $n \in \mathsf{N}$, as functions on the dual unit ball. So, by using Theorem 2.5, Proposition 1.5 and Remarks 1.8(ii), we get the conclusion. □

The next Proposition is a result stronger than a theorem of Pták in [17].

Proposition 2.11. *Let (f_k) be a uniformly bounded sequence of real-valued functions defined on a set X, $\delta > 0$ such that for every $\xi < \omega_1$ there exists $M \in [\mathsf{N}]$ such that for every $N \in [M]$ there exists $H \in \mathcal{M}_\xi[N]$ with*

$\|a^\xi((f_k);H)\|_\infty \geq \delta$. *Then there exists a strictly increasing sequence* (m_k) *of positive integers such that for every* $k \in \mathsf{N}$ *there exists* $x \in X$ *with* $|f_{m_j}(x)| > \dfrac{\delta}{2}$ *for every* $j = 1, \ldots, k$.

Proof. For every $\xi < \omega_1$ there exists $M \in [\mathsf{N}]$ such that for every $N \in [M]$ there exist $H \in \mathcal{M}_\xi[N]$ and $x \in X$ such that $a^\xi_{F_x}(H) \geq \dfrac{\delta}{2\sup_k \|f_k\|_\infty}$, where $F_x = \left\{ k \in \mathsf{N} : |f_k(x)| > \dfrac{\delta}{2} \right\}$.

Therefore, by Theorem 2.3, for every $\xi < \omega_1$ there exists a strictly increasing sequence (m^ξ_k) of positive integers such that for every $F \in \mathcal{S}_\xi$ there exists $x \in X$ with $|f_{m^\xi_k}(x)| > \dfrac{\delta}{2}$ for every $k \in F$.

Consider the tree

$T\left((f_k), \dfrac{\delta}{2}\right) = \{(1)\} \cup \{(1, k_1, \ldots, k_n) \in \mathsf{N}^{n+1} : 1 < k_1 < \ldots < k_n$ and there exists $x \in X$ so that $|f_{k_i}(x)| > \dfrac{\delta}{2}$ for all $i = 1, \ldots, n\}$.

Therefore, by Lemma 2.6 of [10], we have $\left(T((f_k), \dfrac{\delta}{2})\right)^{\omega^\xi} \neq \emptyset$ for every $\xi < \omega_1$. So, by Proposition 1.12, the tree $T\left((f_k), \dfrac{\delta}{2}\right)$ is not well-founded, i.e., there exists a strictly increasing sequence $m_1 < \ldots < m_k < \ldots$ of positive integers such that for every $k \in \mathsf{N}$ there exists $x \in X$ with $|f_{m_j}(x)| > \dfrac{\delta}{2}$ for every $j = 1, \ldots, k$. This finishes the proof of Proposition. $\square$

The next Proposition is a result stronger than Mazur's theorem in [13].

Proposition 2.12. *If* X *is a pseudocompact topological space (i.e., if* (U_n) *is a decreasing sequence of non-empty open subsets of* X *then* $\bigcap_{n=1}^\infty \operatorname{cl} U_n \neq \emptyset$*) then for every uniformly bounded sequence* $(f_k) \subseteq C(X)$ *pointwise converging to zero with* $\inf_k \|f_k\|_\infty > 0$ *there exists* $1 \leq \xi < \omega_1$ *such that for every* $M \in [\mathsf{N}]$ *there exists* $N \in [M]$ *such that for every strictly increasing sequence* (H_n) *of members of* $\mathcal{M}_\xi[N]$ *the sequence* $g_n = a^\xi((f_k); H_n)$, $n \in \mathsf{N}$ *converges uniformly to zero.*

Proof. Let $(f_k) \subseteq C(X)$ uniformly bounded and pointwise converging to zero.

Claim 1. *For every* $\delta > 0$ *there exists* $\xi < \omega_1$ *such that for every* $M \in [\mathsf{N}]$ *there exists* $N \in [M]$ *such that* $\|a^\xi((f_k); H)\|_\infty < \delta$ *for every* $H \in \mathcal{M}_\xi[N]$.

Proof of Claim 1. Assume the contrary. Then there exists $\delta > 0$ such that for every $\xi < \omega_1$ there exists $M \in [\mathsf{N}]$ such that for every $N \in [M]$ there exists $H \in \mathcal{M}_\xi[N]$ with $\|a^\xi((f_k);H)\|_\infty \geq \delta$. So, by Proposition 2.11, there exists a strictly increasing sequence (m_k) of positive integers such that for every $k \in \mathsf{N}$ there exists $x_k \in X$ with $|f_{m_j}(x_k)| > \dfrac{\delta}{2}$ for every $j = 1, \ldots, k$. For every $k \in \mathsf{N}$ we set

$$U_k = \left\{ x \in X : |f_{m_j}(x)| > \frac{\delta}{2} \ \text{ for } \ j = 1, \ldots, k \right\}.$$

For every $k \in \mathsf{N}$ the set U_k is open and non-empty because f_{m_j} is continuous for $1 \leq j \leq k$ and $x_k \in U_k$. Also $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_k \supseteq \ldots$. Therefore, by the hypothesis, there exists $x_0 \in \bigcap_{k=1}^\infty \operatorname{cl} U_k$. By the continuity of f_{m_k}'s we have $|f_{m_k}(x_0)| \geq \dfrac{\delta}{2}$ for every $k \in \mathsf{N}$, a contradiction because (f_k) converges pointwise to zero. This finishes the proof of Claim 1. □

For every $n \in \mathsf{N}$ we choose $l_n, m_n \in \mathsf{N}$ with $n < m_n < l_n$ such that

$$\frac{1}{n} \inf_k \|f_k\|_\infty - \frac{1}{m_n} \sup_k \|f_k\|_\infty > \frac{1}{l_n}.$$

Applying Claim 1 for $\delta = \dfrac{1}{l_n}$, $n \in \mathsf{N}$ we find a sequence (ξ_n) of countable ordinals such that for every $n \in \mathsf{N}$ and $M \in [\mathsf{N}]$ there exists $N_n \in [M]$ such that $\|a^{\xi_n}((f_k);H)\|_\infty < \dfrac{1}{l_n}$ for every $H \in \mathcal{M}_{\xi_n}[N_n]$. We set $\xi = \sup_n \xi_n$. Clearly $\xi < \omega_1$.

If $\xi = 0$ then $\xi_n = 0$ for every $n \in \mathsf{N}$ and so, by using Lemma 2.5.1, for every $M \in [\mathsf{N}]$ there exists $N \in [M]$, $N = (n_k)$ such that the sequence (f_{n_k}) converges uniformly to zero, a contradiction because $\inf_k \|f_k\|_\infty > 0$. Hence $\xi \geq 1$.

Claim 2. *For every $n \in \mathsf{N}$ and $M \in [\mathsf{N}]$ there exists $L_n \in [M]$ such that*

$$\|a^\xi((f_k);H)\|_\infty < \frac{3}{n} \sup_k \|f_k\|_\infty$$

for every $H \in \mathcal{M}_\xi[L_n]$.

Proof of Cl,aim 2. Assume the contrary. Then there exists $n \in \mathsf{N}$ and $M \in [\mathsf{N}]$ such that for every $L \in [M]$ there exists $H \in \mathcal{M}_\xi[L]$ such that $\|a^\xi((f_k);H)\|_\infty \geq \dfrac{3}{n} \sup_k \|f_k\|_\infty$. By Proposition 2.4, there exists $L \in [M]$ such that $\|a^\xi((f_k);H)\|_\infty \geq \dfrac{3}{n} \sup_k \|f_k\|_\infty$ for every $H \in \mathcal{M}_\xi[L]$. Then for every $H \in \mathcal{M}_\xi[L]$ there exists $x \in X$ such that $a^\xi_{F_x}(H) \geq \dfrac{2}{n}$, where $F_x =$

$\left\{k \in \mathsf{N} : |f_k(x)| \geq \frac{1}{m_n} \sup_\lambda \|f_\lambda\|_\infty\right\}$. By using Lemma 2.3.1 and Theorem 2.1, there exists $N \in [L]$ such that for every $H \in \mathcal{M}_{\xi_n}[N]$ there exists $x \in X$ with $a^{\xi_n}_{F_x}(H) \geq \frac{1}{n}$.

Since for every $x \in X$ and $H \in \mathcal{M}_{\xi_n}[\mathsf{N}]$ we have

$$|a^{\xi_n}((f_k); H)(x)| \geq a^{\xi_n}_{F_x}(H) \inf_k \|f_k\|_\infty - \frac{1}{m_n} \sup_k \|f_k\|_\infty$$

it follows that for every $H \in \mathcal{M}_{\xi_n}[N]$ there exists $x \in X$ such that

$$|a^{\xi_n}((f_k); H)(x)| \geq \frac{1}{n} \inf_k \|f_k\|_\infty - \frac{1}{m_n} \sup_k \|f_k\|_\infty > \frac{1}{l_n}.$$

Hence $\|a^{\xi_n}((f_k); H)\| > \frac{1}{l_n}$ for every $H \in \mathcal{M}_{\xi_n}[N]$, a contradiction. This finishes the proof of Claim 2. $\square$

By using Claim 2 and Lemma 2.5.1 we get the desired conclusion. $\square$

Remark 2.13. The conclusion of Proposition 2.12 fails if X is an arbitrary topological space. For example, let $X = \mathsf{N}$ with the discrete topology. For every $k \in \mathsf{N}$ we consider the function $f_k : \mathsf{N} \to \{0, 1\}$ where $f_k(n) = 1$ if $n \geq k$ and $f_k(n) = 0$ if $n < k$. Then (f_k) converges pointwise to zero, but $\|a^\xi((f_k); H)\|_\infty = 1$ for every $\xi < \omega_1$ and $H \in \mathcal{M}_\xi[\mathsf{N}]$.

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