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EXPONENTS OF SUBVARIETIES OF UPPER TRIANGULAR MATRICES OVER ARBITRARY FIELDS ARE INTEGRAL

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Abstract. Let $\mathbf{U}_c$ be the variety of associative algebras generated by the algebra of all upper triangular matrices, the field being arbitrary. We prove that the upper exponent of any subvariety $\mathbf{V} \subset \mathbf{U}_c$ coincides with the lower exponent and is an integer.

1. Codimension growth and exponents. Let K denote the ground field, we consider it to be arbitrary. Suppose that $\mathbf{V}$ is a variety of (associative) algebras, this is the class of all algebras that satisfy some fixed set of identical relations. Let $F(\mathbf{V}, X)$ be its free algebra generated by a countable set of generators $X = \{x_i \,|\, i \in \mathbb{N}\}$. We denoteby $P_n(\mathbf{V}) \subset F(\mathbf{V}, X)$ the subspace of all

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multilinear elements of degree n in $\{x_1,\dots,x_n\}$. We also consider the dimension of this subspace

$$c_n(\mathbf{V}) = c_n(F(\mathbf{V},X),X) = \dim_K P_n(\mathbf{V}), \quad n=1,2,\dots.$$

The *codimension growth sequence* $c_n(\mathbf{V}),\ n=1,2,\dots$, is an important characteristic of $\mathbf{V}$.

The sequence $c_n(\mathbf{V})$ is bounded by an exponential function, provided that the variety of associative algebras $\mathbf{V}$ is nontrivial. This fact was essentially used by A. Regev to prove that the tensor product of associative PI-algebras is again a PI-algebra [10]. The proof was simplified by V. N. Latyshev [4]. One defines the *upper exponent* and *lower exponent* of a variety

$$\operatorname{Exp}\mathbf{V} = \overline{\lim_{n\to\infty}} \sqrt[n]{c_n(\mathbf{V})}, \qquad \underline{\operatorname{Exp}}\,\mathbf{V} = \underline{\lim_{n\to\infty}} \sqrt[n]{c_n(\mathbf{V})}.$$

Almost all results on the growth of associative algebras are concerned with the case of a field of characteristic zero. For example, there are precise asymptotics for the growth of prime varieties of associative algebras over a field of characteristic zero [11]. The main approach there is the technique of representations of the symmetric group. It has been recently proved that the exponent of any variety of associative algebras over a field of characteristic zero is always an integer [2]. The proof of this theorem uses the classification results of A. R. Kemer [3]. But little is known about the growth in case of fields of positive characteristic.

Let us also recall some facts on the growth of varieties of Lie algebras. In this case the growth might be overexponential [13]. The author suggested a scale for describing a superexponential growth of varieties of Lie algebras [7], it was sharpened in [8]. Recall that $\mathbf{N}_s\mathbf{A}$ denotes the variety of Lie algebras whose commutator subalgebras are nilpotent of class s. In [6] it was established that the exponent of any subvariety $\mathbf{V}\subset\mathbf{N}_s\mathbf{A}$ is an integer, the field being of characteristic zero. The proof used techniques of Young diagrams. Another method allowed us to lift any restrictions on the field [9]. Namely, the method of "necklaces" was developed to study subvarieties of $\mathbf{N}_s\mathbf{A}$. This method was also applied to study the overexponential growth of subvarieties in $\mathbf{A}^3$, the variety of Lie algebras that are soluble of length 3 [9]. Recently it was found that soluble varieties might have nonintegral exponents [14].

Denote by $\mathbf{U}_s$ the variety of associative algebras generated by the algebra of $s\times s$ upper triangular matrices. It is well known [5] that in case of characteristic zero $\mathbf{U}_s$ is defined by the identity

$$[X_1, X_2] \cdot [X_3, X_4] \cdots [X_{2s-1}, X_{2s}] \equiv 0. \tag{1}$$

Let $A = A(X)$ be the free associative algebra in the countable set of variables $X = \{x_1, x_2, \ldots\}$. We denote by $T(\mathbf{V}) \subset A(X)$ the ideal of identities of a variety (or an algebra). It is known that $T(\mathbf{U}_s) = (T(K))^s$, where $T(K)$ is the T-ideal of the field K, for an arbitrary field K. If the field is infinite then the latter ideal is generated by one identity $[X, Y] \equiv 0$. If the field is finite and $|K| = q$, then it is enough to add to the generating set one more identity $X^q - X \equiv 0$ [12]. Properties of finitely generated algebras in $\mathbf{U}_s$ were studied in [1].

Our goal is to prove the following statement using the technique of necklaces from [9].

Theorem. *Let* $\mathbf{V}$ *be a subvariety of* $\mathbf{U}_s$*, the field* K *being arbitrary. Then*

1. $\operatorname{Exp} \mathbf{V} = \underline{\operatorname{Exp}} \mathbf{V}$.
2. *The exponent of* $\mathbf{V}$ *is an integer:* $\operatorname{Exp} \mathbf{V} \in \{1, \ldots, s\}$.
3. *If* $\operatorname{Exp} \mathbf{V} = 1$*, then* $c_n(\mathbf{V})$ *is bounded by a polynomial.*

2. Necklaces. We shall use some combinatorial constructions of [9]. We present them for convenience of the reader.

We shall consider disjoint subsets of $\{1, 2, \ldots, n\}$. Let $A, B \subset \{1, 2, \ldots, n\}$. We write $A < B$ if $a < b$ for any $a \in A$, $b \in B$. Obviously $<$ is a partial order. Let $I_1, I_2, \ldots, I_c \subset \{1, 2, \ldots, n\}$ be disjoint subsets (some of which may be empty). We shall refer to the I_i as *chains*, and $(I_1, I_2, \ldots, I_c)$ is a *chain tuple*.

We consider also another partial ordering on chains. Let $I = \{i_1, \ldots, i_t\}$, $J = \{j_1, \ldots, j_s\}$, where $i_1 > \ldots > i_t$, $j_1 > \ldots > j_s$. Then we set $I \prec J$ if $(i_1, \ldots, i_t) \prec (j_1, \ldots, j_s)$, where the latter ordering is lexicographic from left to right. The ordering of chains is extended lexicographically from left to right to chain tuples of the same length $(I_1, \ldots, I_c)$ using the same sign $\prec$.

Suppose that $(I_1, \ldots, I_c)$ is a fixed chain tuple, then $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_c)$ is called a *necklace* if $\Omega_i \subset I_i$, $i = 1, \ldots, c$, $\Omega_1 < \Omega_2 < \ldots < \Omega_c$ (if a component Ω_i is empty then the corresponding inequalities are regarded as valid). We call a chain tuple (or a necklace) non-empty if at least one component is non-empty.

Let m be a fixed number. Now we describe the *m-algorithm of extracting a necklace* $\Omega=(\Omega_1,\Omega_2,\dots,\Omega_c)$ from a chain tuple $(I_1,I_2,\dots,I_c)$. We shall look at the components $I_1,I_2,\dots$ one after other. Suppose that $\Omega_1,\Omega_2,\dots,\Omega_{i-1}$ as well as other accompanying sets in the components $I_1,I_2,\dots,I_{i-1}$ are already constructed. We partition the chain $I_i=J_i\cup\widetilde{I}_{i-1}$, where J_i consists of all elements greater than all elements of the set $\widetilde{\Omega}_{i-1}=\Omega_1\cup\Omega_2\cup\dots\cup\Omega_{i-1}$, while $\widetilde{I}_{i-1}$ consists of the remaining elements. By definition, we set $\widetilde{\Omega}_0=\emptyset$. If $\widetilde{\Omega}_{i-1}=\emptyset$, then we put $J_i=I_i$, $\widetilde{I}_i=\emptyset$.

1) If $|J_i|\le m$, then we put $\Omega_i=\emptyset$ and form a *false segment* $F_i=J_i$.

2) In the case $|J_i|>m$ we form a *(genuine) segment* S_i, consisting of m maximal elements of the set J_i. The remaining elements are included into the necklace $\Omega_i=J_i\setminus S_i$.

Then we pass to the next chain I_{i+1}.

Lemma 1. *The m-algorithm has the following properties:*

1. *The original chain tuple $(I_1,I_2,\dots,I_c)$ decomposes into the necklace $\Omega=(\Omega_1,\Omega_2,\dots,\Omega_c)$. In each chain I_j, $j=1,\dots,c$, the algorithm cuts off a segment (a genuine one S_j with $|S_j|=m$, or a false one F_j with $|F_j|\le m$). As a result it remains a new chain tuple $(\widetilde{I}_1,\widetilde{I}_2,\dots,\widetilde{I}_c)$.*

2. *An empty necklace is formed if and only if $|I_i|\le m$, $i=1,\dots,c$.*

3. *For $\widetilde{I}_s\ne\emptyset$ there exists a preceding chain I_t, $t<s$, that has a greater segment $S_t>\widetilde{I}_s$.*

P r o o f. The first property follows directly from the description of the m-algorithm. To prove the second property it is sufficient to observe that the first non-zero component of the necklace Ω_i is formed in the first chain whose length is greater than m.

Let us prove the third property. Suppose that $\widetilde{I}_s\ne\emptyset$. We choose the greatest $t<s$ such that $\Omega_t\ne\emptyset$, namely $\widetilde{I}_s$ appeared by comparing with $\widetilde{\Omega}_t$. Let θ be the maximal element of Ω_t. We formed J_s from all elements of I_s that are greater than θ. Therefore $\theta>\widetilde{I}_s$. The algorithm extracted from I_t the genuine segment S_t, hence $S_t>\theta>\widetilde{I}_s$. $\square$

If a non-empty necklace is constructed, then we apply the m-algorithm to the new chain tuple $(\widetilde{I}_1,\widetilde{I}_2,\dots,\widetilde{I}_c)$, and so on. We use additional upper indices to indicate the number of the step at which a given set is formed. The original chain tuple is denoted by the upper index 1.

Lemma 2. *Suppose that k non-empty necklaces $\Omega^1, \dots, \Omega^k$ were constructed as a result of the m-algorithm. Then there exists a decreasing series of segments*

$$(2) \quad S^1_{i_1} > S^2_{i_2} > \dots > S^k_{i_k}, \quad i_1 < i_2 < \dots < i_k, \quad |S^j_{i_j}| = m, \ j = 1, \dots, k.$$

Proof. Indeed, suppose that $\Omega^k_{i_k} \neq \emptyset$. Then a genuine segment $S^k_{i_k} \subset \widetilde{I}^{k-1}_{i_k}$ was constructed in this component at step k. By Property 3 of Lemma 1 there exists $S^{k-1}_{i_{k-1}} > \widetilde{I}^{k-1}_{i_k}$, $i_{k-1} < i_k$, so we obtain $S^{k-1}_{i_{k-1}} > S^k_{i_k}$. Then we again apply Property 3 of Lemma 1 to $S^{k-1}_{i_{k-1}} \subset \widetilde{I}^{k-2}_{i_{k-1}}$, and so on. Finally we arrive at (2). □

3. Proof of the main result. We denote $[y_1, y_2] = y_1y_2 - y_2y_1$, $[y_1, y_2, \dots, y_i] = [[y_1, y_2, \dots, y_{i-1}], y_i]$, $i \geq 3$.

Let $B(X)$ be an associative algebra generated by a countable set $X = \{x_1, x_2, \dots\}$. Let $Y = \{y_{i_1}, \dots, y_{i_n}\} \subset X$. By $P_n(Y) \subset B(X)$ we denote the space of multilinear elements of degree n in Y. Suppose that for a subspace $U \subset B(X)$ the dimension of $U \cap P_n(Y)$ does not depend on Y, but depends on n only. In this case we consider $Y = \{x_1, \dots, x_n\}$ and denote

$$P_n(U) = U \cap P_n(Y), \qquad c_n(U) = \dim_K P_n(U), \qquad n \in \mathbb{N}.$$

Let us fix some subvariety $\mathbf{V} \subset \mathbf{U}_s$. We introduce the following vector spaces

$$W_{c,n} = \begin{cases} P_n(A(X)/T(\mathbf{U}_1 \cap \mathbf{V})), & c = 1; \\ P_n(T(\mathbf{U}_{c-1} \cap \mathbf{V})/T(\mathbf{U}_c \cap \mathbf{V})), & c = 2, \dots, s; \end{cases}$$

$$(3) \qquad c_n(\mathbf{V}) = \sum_{c=1}^{s} \dim W_{c,n}, \quad n \geq 1.$$

We apply identity (1), and observe that these spaces are spanned by the following elements [12]:

$$(4) \quad W_{c,n} = \langle\, x_{11} \cdots x_{1a_1} [x_{21}, \dots, x_{2a_2}] \cdots [x_{c1}, \dots, x_{ca_c}] \mid$$

$$a_2 \geq 2, \dots, a_c \geq 2; \ a_1 + \dots + a_c = n; \ \{x_{ij}\} = \{x_1, \dots, x_n\};$$

$$(5) \qquad x_{11} < \dots < x_{1a_1}; \ x_{i1} > x_{i2} < \dots < x_{ia_i}, \ i = 2, \dots, s\rangle_K.$$

Now we demonstrate why we need the chain tuples. Note that the products (4) do not change under an arbitrary permutation of the elements $x_{11}, \ldots, x_{1a_1}$, as well as under any permutation in the brackets of $x_{i3}, \ldots x_{ia_i}$, $i = 2, \ldots, s$, since interchanging of any two adjacent elements produces an additional element from $T(\mathbf{U}_c)$, which is zero in $W_{c,n}$. We denote this property by $(*)$. Let $I = \{i_1, \ldots, i_t\} \subset \{1, \ldots, n\}$, to simplify the notation we use the same symbol for the set of variables with the corresponding indices $I = \{x_{i_1}, \ldots, x_{i_t}\}$. We denote $(y, I) = y x_{i_1} \cdots x_{i_t}$, where $i_1 \le \ldots \le i_t$, also we denote $(I) = (1, I)$. In case $I = \emptyset$ we set $(y, I) = y$. Analogously $[y, I] = [y, x_{i_1}, \ldots, x_{i_t}]$, where $i_1 \le \ldots \le i_t$, also we set $[y, \emptyset] = y$. We consider the following chains $I_1 = \{x_{11}, \ldots, x_{1a_1}\}$ and $I_i = \{x_{i3}, \ldots, x_{ia_i}\}$, $i = 2, \ldots, c$. This enables us to rewrite (4) in the form

$$(6) \quad W_{c,n} = \langle\, (I_1)[x_{21}, x_{22}, I_2] \cdots [x_{c1}, x_{c2}, I_c] \,| \, \{x_{ij} \,|\, i = 2, \ldots, c,\ j = 1, 2\} \cup I_1 \cup \ldots \cup I_c = \{x_1, \ldots, x_n\} \rangle_K.$$

Let us construct a special identity for interchanging of the elements of the necklaces.

Lemma 3. *Let $c \in \mathbb{N}$ be a fixed number. Suppose that the variety $\mathbf{V}$ is such that $\underline{\mathrm{Exp}}\,\mathbf{V} < k$, where $k \in \mathbb{N}$, $k \le c$. Then there exists a natural number n of the form $n = n' + d$, where $n' = mk$ and $c \le d < c + k$, that satisfy the following property. We fix Lie words $y_1, \ldots, y_k$ in the letters $x_{n'+1}, \ldots, x_n$ as well as letters $z_1, \ldots, z_{k-1}$, so that each letter $x_{n'+1}, \ldots, x_n$ enters some y_i or z_j exactly once. Then $\mathbf{V}$ satisfies*

1. *A nontrivial multilinear identity of type*

$$[y_1, x_{n'}, \ldots, x_{n'-m+1}] z_1 \cdots z_{k-2} [y_{k-1}, x_{2m}, \ldots, x_{m+1}] z_{k-1} [y_k, x_m, \ldots, x_1] \equiv$$
$$\sum_\sigma \lambda_\sigma [y_1, x_{\sigma(1,1)}, \ldots, x_{\sigma(1,m)}] z_1 \cdots z_{k-1} [y_k, x_{\sigma(k,1)}, \ldots, x_{\sigma(k,m)}];\ \lambda_\sigma \in K.$$

   Here the summands on the second line correspond to all possible partitions σ of the set $\{1, \ldots, n'\}$ into k groups of size m:

$$\{\sigma(1,1), \ldots, \sigma(1,m)\} \cup \ldots \cup \{\sigma(k,1), \ldots, \sigma(k,m)\} = \{1, \ldots, n'\};$$
$$\sigma(i,1) > \ldots > \sigma(i,m), \qquad i = 1, \ldots, k;$$

   while we write on the left-hand side the summand that corresponds to the partition $\{n', \ldots, n' - m + 1\} \cup \ldots \cup \{2m, \ldots, m + 1\} \cup \{m, \ldots, 1\}$.

2. *An analogous identity, where instead of the first commutators we have products* $x_{n'} \cdots x_{n'-m+1}$ *and* $x_{\sigma(1,1)} \cdots x_{\sigma(1,m)}$.

P r o o f. By hypothesis there exists a subsequence

$$n_i, \quad i = 1, 2, \ldots, \qquad \lim_{i\to\infty} \sqrt[n_i]{c_{n_i}(\mathbf{V})} < k. \tag{7}$$

For each n_i we choose the unique m_i, such that $n_i = km_i + d_i$, where $c \le d_i < c + k$. We fix $n = n_i$, $m = m_i$, $d = d_i$, $n' = km_i$.

Let α_n be the number of monomials of the form indicated in the lemma, with the elements $z_1, \ldots, z_{k-1}, y_1, \ldots, y_k$ being fixed. We order the letters $x_n > \ldots > x_1$. Without loss of generality we assume that the letters $x_n, \ldots, x_{n'+1}$ stand in all monomials in fixed positions in decreasing order. Then the leading terms of the monomials of the identity are obtained simply by erasing of the brackets. One easily observes that these elements are linearly independent in the free associative algebra. By Stirling's formula we have

$$\alpha_n = \frac{n'!}{((n'/k)!)^k} \approx \frac{k^{n'} k^{k/2}}{(2\pi n')^{(k-1)/2}} = \frac{k^{n-d} k^{k/2}}{(2\pi(n-d))^{(k-1)/2}}; \quad \lim_{i\to\infty} \sqrt[n_i]{\alpha_{n_i}} = k.$$

Using (7) we obtain that for a sufficiently large $n = n_i$ our monomials are linearly dependent modulo $P_n(\mathbf{V}, \{x_1, \ldots, x_n\})$. This yields a non-trivial identity of the required form. The monomial indicated on the left-hand side is isolated by relabelling the variables. $\square$

P r o o f o f T h e o r e m. Let us evaluate the growth of $c_n(\mathbf{U}_s)$. First, we find an upper bound for $\dim W_{c,n}$ in (4). We can decompose letters $\{x_1, \ldots, x_n\}$ into c sets: $\{x_{11}, \ldots, x_{1a_1}\}$, $\{x_{21}, \ldots, x_{2a_2}\}, \ldots$, $\{x_{c1}, \ldots, x_{ca_c}\}$, in at most c^n ways. Next, if we choose elements for the first two places in each bracket (4), then we uniquely determine the element (4) by (5). Therefore, $\dim W_{c,n} \le c^n n^{2c-2}$, $c = 1, \ldots, s$. By (3), we derive $\operatorname{Exp} \mathbf{U}_s \le s$.

Now we suppose that $\underline{\operatorname{Exp}} \mathbf{V} < k$ for a subvariety $\mathbf{V} \subset \mathbf{U}_s$, where $k \in \{2, \ldots, s\}$. We shall prove that these conditions imply that $\operatorname{Exp} \mathbf{V} \le k - 1$, thus yielding claims 1), 2).

We fix $c \in \{1, \ldots, s\}$ and choose n according to Lemma 3. We prove that $W_{c,n}$ is a linear span of elements of type (6) and such that, an iteration of the m-algorithm to the chain tuple $(I_1, \ldots, I_c)$ gives at most $k-1$ non-empty necklaces. Indeed, we consider an element (6), and suppose that the algorithm gives k non-empty necklaces $\Omega^1, \ldots, \Omega^k$. Then by Lemma 2 we obtain a decreasing sequence of segments

$$(8)\qquad S^1_{i_1} > S^2_{i_2} > \ldots > S^k_{i_k},\quad i_1 < i_2 < \ldots < i_k,\quad |S^j_{i_j}| = m,\ j = 1,\ldots,k.$$

We remark that this is only possible in the case $c \geq k$. We denote $D_{i_j} = I_{i_j} \setminus S^j_{i_j}$, $j = 1,\ldots,k$. We apply $(*)$ to the element (6) and shift segments to the ends of brackets in the brackets with numbers $i_1, i_2, \ldots, i_k$. This presents our element as follows:

$$(9)\qquad \begin{cases} (I_1)\cdots[z_{i_1}, D_{i_1}, S^1_{i_1}]\cdots[z_{i_k}, D_{i_k}, S^k_{i_k}]\cdots[z_c, I_c], & i_1 > 1;\\ (D_1, S^1_1)\cdots[z_{i_2}, D_{i_2}, S^2_{i_2}]\cdots[z_{i_k}, D_{i_k}, S^k_{i_k}]\cdots[z_c, I_c], & i_1 = 1; \end{cases}$$
$$z_i = [x_{i1}, x_{i2}],\quad i = 2,\ldots,c.$$

In order to apply Lemma 3 it is sufficient that the total degree of words apart from the chosen segments exceeds $c+k$. This is the case when n is large enough. In virtue of (8) we apply the identity of the lemma to the element (9) and obtain another element with a new chain tuple $(\widetilde{I}_1,\ldots,\widetilde{I}_c) \prec (I_1,\ldots,I_c)$. (If $i_1 > 1$ then we apply the identity of the first type, if $i_1 = 1$ then we use the identity of the second type). Since the number of monomials in the given letters is finite all elements of $W_{c,n}$ will be expressed via monomials from which the m-algorithm extracts at most $k-1$ non-empty necklaces.

We give an upper bound of the number of such monomials. An empty necklace is formed not late than at Step k. Hence at most $q = kmc + 2(c-1)$ elements are involved in segments of both types and first two terms of the brackets. We choose at most q elements from $\{1,\ldots,n\}$ and then choose positions for them in (4), this can be done in at most n^{2q} ways. The remaining elements in $\{1,\ldots,n\}$ are distributed among at most $k-1$ necklaces, which can be done in at most $(k-1)^n$ ways. Further, each necklace Ω can be split up into its components $\Omega = (\Omega_1,\ldots,\Omega_c)$, $\Omega_1 < \ldots < \Omega_c$ in at most $\binom{n+c-1}{c-1} \approx \frac{n^{c-1}}{(c-1)!}$ ways. As a result we obtain the bound

$$(10)\qquad \dim W_{c,n} \leq (k-1)^n\,\frac{n^{2q+c-1}}{(c-1)!}.$$

Taking into account (3) it follows that $\operatorname{Exp}\mathbf{V} \leq k-1$. This proves the first two statements.

In order to prove the third statement we suppose that $\operatorname{Exp}\mathbf{V} = 1 < k = 2$, then (10) yields a polynomial bound. □

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