Serdica

Mathematical Journal

Сердика

Математическо списание

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **26** (2000), 221-228

Serdica
Mathematical Journal

Institute of Mathematics
Bulgarian Academy of Sciences

COMPLETE SYSTEMS OF HERMITE ASSOCIATED FUNCTIONS

Peter Rusev

Communicated by S. L. Troyanski

Abstract. It is proved that if the increasing sequence $\{k_n\}_{n=0}^{\infty}$ of non-negative integers has density greater than 1/2 and D is an arbitrary simply connected subregion of $\mathbb{C}\backslash\mathbb{R}$ then the system of Hermite associated functions $\left\{G_{k_n}(z)\right\}_{n=0}^{\infty}$ is complete in the space $H(D)$ of complex functions holomorphic in D.

1. Completeness in spaces of holomorphic functions.

1.1. A subset X of a topological vector space V is called *complete* in V if its linear span is everywhere dense in the space V.

Suppose that X is at most denumerable, i.e $X = \{x_n\}_{n=0}^{\omega}, 0 \leq \omega \leq \infty$. Then X is complete in V iff for every $v \in V$ and every neighbourhood U of the origin of V there exists a linear combination $x = \sum_{n=0}^{N} a_n x_n (0 \leq N \leq \omega$ if $\omega < \infty$ and $0 \leq N < \infty$ if $\omega = \infty; a_1, a_2, a_3, \ldots, a_N$ are scalars) such that $v - x \in U$.

Remark. If $\{\lambda_n\}_{n=0}^{\infty}$ are nonzero scalars then the systems $\{x_n\}_{n=0}^{\infty}$ and $\{\lambda_n x_n\}_{n=0}^{\infty}$ are simultaneously complete or incomplete.

2000 *Mathematics Subject Classification:* 30B60, 33C45.
Key words: Hermite polynomials, Hermite associated functions, completeness.

1.2. Let B be a nonempty open subset of the complex plane $\mathbb{C}$ and $H(B)$ be the $\mathbb{C}$-vector space of complex functions holomorphic in B. As usual, we consider $H(B)$ with the topology of uniform convergence on compact subsets of B.

A sequence $\{f_n(z)\}_{n=0}^{\infty} \subset H(B)$ is complete in $H(B)$ iff vor every $f \in H(B)$, every compact set $K \subset B$ and every $\varepsilon > 0$ there exists a "polynomial" $p(z) = \sum_{n=0}^{N} a_n f_n(z)(0 \le N < \infty; a_n \in \mathbb{C}, n = 0, 1, 2, \ldots, N)$ such that $|f(z) - p(z)| < \varepsilon$ whenever $z \in K$.

Let $\gamma \subset \mathbb{C}$ be a Jordan curve and H_γ be the $\mathbb{C}$-vector space of complex functions holomorphic on the closed set $C_\gamma = \overline{\mathbb{C}} \setminus int\gamma$ and vanishing at the infinite point. That is, H_γ consists of functions F everyone of which is holomorphic in an open set containing C_γ and, in addition, $F(\infty) = 0$.

The following statement is a criterion for completeness in spaces of the kind $H(B)$ [4, p. 211, Theorem 17]:

(CC) *Let $D \subset \mathbb{C}$ be a simply connected region. A sequence $\{f_n(z)\}_{n=0}^{\infty} \subset H(D)$ is complete in the space $H(D)$ iff for every rectifiable Jordan curve $\gamma \subset D$ the only function $F \in H_\gamma$, which is orthogonal on γ to each of the functions $\{f_n(z)\}_{n=0}^{\infty}$, is identically zero, i.e. the equalities $\int_\gamma f_n(z)F(z)\, dz = 0, \quad n = 0, 1, 2, \ldots$ imply $F \equiv 0$.*

2. Series in Hermite polynomials.

2.1. It is known that the region of convergence of a series of the kind

$$\sum_{n=0}^{\infty} a_n H_n(z), \quad a_n \in \mathbb{C}, \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

where $\{H_n(z)\}_{n=0}^{\infty}$ are the Hermite polynomials, is either the whole complex plane or a horizontal strip symmetrically situated with respect to the real axis. More precisely, if

$$\tau_0 := -\limsup_{n\to\infty} (2n+1)^{-1/2} \log |(2n/e)^{n/2} a_n| > 0, \tag{2.2}$$

then the series (2.1) is absolutely uniformly convergent on every compact subset of the region $S(\tau_0) := \{z : |\Im z| < \tau_0\}$ and diverges at every point of the open set $\mathbb{C} \setminus \overline{S(\tau_0)}$ [8, 9.2, (5)]. This statement is a corollary of a particular case of G. Szegö's asymptotic formula for the Hermite polynomials [8, Theorem 8.22.7], namely,

$$H_n(z) = \sqrt{2} \exp(z^2/2)(2n/e)^{n/2}\{\cos((2n+1)^{1/2} z - n\pi/2) + h_n(z)\}. \tag{2.3}$$

In the above representation, $\{h_n(z)\}_{n=1}^{\infty}$ are entire functions and, moreover, the sequence $\{n^{1/2}\exp(-|\Im z|\sqrt{2n+1})h_n(z)\}_{n=1}^{\infty}$ is uniformly bounded on every compact subset of the complex plane.

2.2. Let the complex function f have a representation by a series in Hermite polynomials in the region $S(\tau_0)(0<\tau_0\le\infty)$, i.e. $f(z)=\sum_{n=0}^{\infty}a_nH_n(z)$ for every $z\in S(\tau_0)$. Then it is holomorphic in $S(\tau_0)$ and, moreover,

$$a_n=(\sqrt{\pi}n!2^n)^{-1}\int_{-\infty}^{\infty}\exp(-t^2)H_n(t)f(t)\,dt,\quad n=0,1,2,\dots \tag{2.4}$$

This statement is a corollary of (2.2) as well as of the following inequality for the Hermite polynomials

$$|H_n(x)|\le(\sqrt{\pi}n!2^n)^{1/2}\exp(x^2/2),\quad x\in\mathbb{R},\quad n=0,1,2,\dots. \tag{2.5}$$

Remark. The above inequality is due to O. Szász [7].

2.3. Let $E(\tau_0)(0<\tau_0\le\infty)$ be the $\mathbb{C}$-vector space of complex functions f holomorphic in the region $S(\tau_0)$ and having the property that for every $\tau\in[0,\tau_0)$ there exists a constant $B=B(f,\tau)\ge 0$ such that if $z=x+iy\in\overline{S}(\tau):=\{z:|\Im z|\le\tau\}$, then $|f(z)|=|f(x+iy)|\le B\exp\{x^2/2-|x|(\tau^2-y^2)^{1/2}\}$.

A remarkable result due to E. Hille [3] is that a function $f\in H(S(\tau_0))$ can be represented in $S(\tau_0)$ by a series in Hermite polynomials iff $f\in E(\tau_0)$.

3. Hermite associated functions.

3.1. A well-known fact is that the system of Hermite polynomials $\{H_n(z)\}_{n=0}^{\infty}$ is a solution of the difference equation $y_{n+1}-2zy_n+2ny_{n-1}=0(n\ge1)$.

The system of complex functions $\{G_n(z)\}_{n=0}^{\infty}$, defined for $z\in\mathbb{C}\setminus\mathbb{R}$ by the equalities

$$G_n(z)=-\int_{-\infty}^{\infty}\frac{\exp(-t^2)H_n(t)}{t-z}\,dt,\quad n=0,1,2,\dots, \tag{3.1}$$

is another solution of the same equation. Indeed, for every $n\ge1$,

$$\begin{aligned}&G_{n+1}(z)-2zG_n(z)+2nG_{n-1}(z)\\ &=-\int_{-\infty}^{\infty}\frac{\exp(-t^2)}{t-z}\{H_{n+1}(t)-2zH_n(t)+2nH_{n-1}(t)\}\,dt\\ &=-\int_{-\infty}^{\infty}\frac{\exp(-t^2)}{t-z}\{H_{n+1}(t)-2tH_n(t)+2nH_{n-1}(t)\}\,dt\end{aligned}$$

$$-2\int_{-\infty}^{\infty}\exp(-t^2)H_n(t)\,dt=0.$$

We call the functions (3.1) *Hermite associated functions.* Since they are defined by means of Cauchy-type integrals it is clear that all they are holomorphic in the open set $\mathbb{C}\setminus\mathbb{R}$.

3.2. Let us define the system of complex functions

$$\tilde{G}_n(z,\zeta):=(\sqrt{\pi}n!2^n)^{-1}H_n(\zeta)G_n(z),\quad n=0,1,2,\dots,\tag{3.2}$$

where ζ is an arbitrary complex number. Then Mehler's generating function [1, II, 10.13,(22)] implies that if $|w|<1$ then

$$\sum_{n=0}^{\infty}\tilde{G}_n(z,\zeta)w^n=-\sum_{n=0}^{\infty}\Big\{(\sqrt{\pi}n!2^n)^{-1}H_n(\zeta)\int_{-\infty}^{\infty}\frac{\exp(-t^2)H_n(t)}{t-z}\,dt\Big\}w^n$$

$$=-\pi^{-1/2}\int_{-\infty}^{\infty}\frac{\exp(-t^2)}{t-z}\Big\{\sum_{n=0}^{\infty}(n!2^n)^{-1}H_n(\zeta)H_n(t)w^n\Big\}\,dt$$

$$=-\frac{\pi^{-1/2}\exp\Big\{-\frac{\zeta^2w^2}{1-w^2}\Big\}}{\sqrt{1-w^2}}\int_{-\infty}^{\infty}(t-z)^{-1}\exp\Big\{-\frac{t^2}{1-t^2}+\frac{2wt\zeta}{1-w^2}\Big\}\,dt.\tag{3.3}$$

Let us note that if $|w|<1$ then $\Re\{(1-w^2)^{-1}\}>0$ and, therefore, the last integral in (3.3) is (absolutely) convergent. Moreover, the change of summation and integration we have just performed is allowed. Indeed, as a corollary of the asymptotic formula (2.3), Szász's inequality (2.5) as well as Stirling's formula, we can conclude that there exists a positive constant M independent of n, t, ζ and w and such that the inequality

$$(n!2^n)^{-1}|H_n(\zeta)H_n(t)w^n|\le M\exp(t^2/2+|\Im\zeta|\sqrt{2n+1})|w|^n$$

holds for each $t\in\mathbb{R}$ and each $n=0,1,2,\dots$.

Further, we define

$$\varphi(t,\zeta,w)=\exp\{-(1-w^2)^{-1}t^2+2\zeta wt(1-w^2)^{-1}\}\tag{3.4}$$

provided that $t\in\mathbb{R}$, $\zeta\in\mathbb{C}$ and $w\in\mathbb{C}\setminus\{-1,1\}$.

Let W be a region in $\mathbb{C}$ determined by the inequality $\Re\{(1-w^2)^{-1}\}>0$. It is easy to see that W is a subregion of $\mathbb{C}\setminus\{(-\infty,-1]\bigcup[1,\infty)\}$. Indeed, the inequality $\Re\{(1-w^2)^{-1}\}>0$ is equivalent to $\Re\overline{w}^2<1$, i.e. W is nothing but the exterior of the hyperbola with Cartesian equation $u^2-v^2=1 (w=u+iv)$.

Let us define

$$\Phi(z,\zeta,w) = \int_{-\infty}^{\infty} \frac{\varphi(t,\zeta,w)}{t-z}\,dt \tag{3.5}$$

and

$$\tilde{G}(z,\zeta,w) = \pi^{-1/2}(1-w^2)^{-1/2}\exp\{-\zeta^2 w^2(1-w^2)^{-1}\}\Phi(z,\zeta,w) \tag{3.6}$$

where $z \in \mathbb{C}\setminus\mathbb{R}$, $\zeta \in \mathbb{C}$ and $w \in W$.

It is clear that $\tilde{G}(z,\zeta,w)$, as a function of w, is holomorphic in the region W. Moreover, as a corollary of (3.2), (3.3), (3.4), (3.5) and (3.6) we obtain that its Taylor expansion centered at the origin is

$$\tilde{G}(z,\zeta,w) = \sum_{n=0}^{\infty} \tilde{G}_n(z,\zeta)w^n. \tag{3.7}$$

Since the unit disk is contained in the region W, the radius of convergence of the above power series is at least equal to one.

4. The main result. The main result in this paper is the following statement:

Let $k = \{k_n\}_{n=0}^{\infty}$ be an increasing sequence of positive integers with density greater than 1/2, i.e. there exists

$$\lim_{n\to\infty} \frac{n}{k_n} = \delta(k) > 1/2. \tag{4.1}$$

Then the system $\{G_{k_n}(z)\}_{n=0}^{\infty}$ is complete in the space $H(D)$ provided that D is any simply connected subregion of $\mathbb{C}\setminus\mathbb{R}$.

Proof. Suppose the statement we wish to prove is not true. Then, in view of the completeness criterion (CC), there exists a simply connected region $\tilde{D} \subset \mathbb{C}\setminus\mathbb{R}$ such that the system $\{G_{k_n}(z)\}_{n=0}^{\infty}$ is not complete in the space $H(\tilde{D})$. Let $\zeta \in \mathbb{C}\setminus\mathbb{R}$ be fixed. Since $H_n(\zeta) \neq 0$ for every $n = 0,1,2,\ldots$, the system $\{\tilde{G}_{k_n}(z,\zeta)\}_{n=0}^{\infty}$ is also not complete in the space $H(\tilde{D})$. That means (because of (CC)) that there exists a rectifiable Jordan curve $\tilde{\gamma} \subset \tilde{D}$ and a function $\tilde{F} \in H_{\tilde{\gamma}}$ which is not identically zero but $\int_{\tilde{\gamma}} \tilde{G}_{k_n}(z,\zeta)\tilde{F}(z)\,dz = 0, \quad n = 0,1,2,\ldots$. We can assume that the curve $\tilde{\gamma}$ is negatively oriented.

We define the function $A(\tilde{F};\zeta,w)$ in the region W by the equality

$$A(\tilde{F};\zeta,w) = \int_{\tilde{\gamma}} \tilde{G}(z,\zeta,w)\tilde{F}(z)\,dz \tag{4.2}$$

and, as a corollary of (3.7), we obtain that if $|w| < 1$ then

$$A(\tilde{F};\zeta,w) = \sum_{n=0}^{\infty} A_n(\tilde{F};\zeta)w^n, \tag{4.3}$$

where $A_n(\tilde{F};\zeta) = \int_{\tilde{\gamma}} \tilde{G}_n(z,\zeta)\tilde{F}(z)\,dz$, $\quad n = 0,1,2,\ldots$. Then (3.1) and (3.2) imply that $(n = 0,1,2,\ldots)$

$$A_n(\tilde{F};\zeta) = -(\sqrt{\pi}n!2^n)^{-1}H_n(\zeta)\int_{\tilde{\gamma}}\left\{\int_{-\infty}^{\infty}\frac{\exp(-t^2)H_n(t)}{t-z}\,dt\right\}\tilde{F}(z)\,dz. \tag{4.4}$$

Since $\tilde{\gamma}$ is a compact subset of $\mathbb{C}\backslash\mathbb{R}$, $\mu := \min_{z\in\tilde{\gamma}}|\Im z|$ is positive. Moreover the inequality $|t-z| \geq \mu$ holds for every $t \in \mathbb{R}$ and every $z \in \tilde{\gamma}$. Then from (2.5) it follows that $|(t-z)^{-1}\exp(-t^2)H_n(t)| \leq \mu^{-1}(\sqrt{\pi}n!2^n)^{1/2}\exp(-t^2/2)$, $\quad n = 0,1,2,\ldots; t \in \mathbf{R}, z \in \tilde{\gamma}$.

Therefore, the improper integral in (4.4) is absolutely uniformly convergent with respect to z on the curve $\tilde{\gamma}$. After changing the order of integration in (4.4) we obtain that $(n = 0,1,2,\ldots)$

$$A_n(\tilde{F};\zeta) = (\sqrt{\pi}n!2^n)^{-1}H_n(\zeta)\int_{-\infty}^{\infty}\exp(-t^2)H_n(t)\,dt\int_{\tilde{\gamma}}\frac{\tilde{F}(z)}{(z-t)}\,dz.$$

As a connected set D is contained either in the upper or in the lower half-plane. Since $F(\infty) = 0$ in both cases the Cauchy integral formula gives that for every $t \in \mathbf{R}$ we have

$$\int_{\tilde{\gamma}}\frac{\tilde{F}(z)}{z-t}\,dz = 2\pi i\tilde{F}(t).$$

In this way we obtain that $A_n(\tilde{F};\zeta) = 2\pi iH_n(\zeta)a_n(\tilde{F})(n = 0,1,2,\ldots)$, where

$$a_n(\tilde{F}) = (\sqrt{\pi}n!2^n)^{-1}\int_{-\infty}^{\infty}\exp(-t^2)H_n(t)\tilde{F}(t)\,dt, \quad n = 0,1,2,\ldots. \tag{4.5}$$

Let T be the set of all positive τ such that the function $\tilde{F}$ is holomorphic in the strip $S(\tau)$. Since $\tilde{F} \in H(S(\mu))$, T is not empty. It is clear that $\tau_0(\tilde{F}) := \sup T < \infty$. Otherwise $\tilde{F}$ would be holomorphic in the extended complex plane $\overline{\mathbf{C}}$ and, since $\tilde{F}(\infty) = 0$, it would be identically zero.

For every $0 < \tau < \tau_0(\tilde{F})$ the function $\tilde{F}$ is continuous on the closed stripe $\overline{S(\tau)}$ and since $\tilde{F}(\infty) = 0$, it is bounded on $\tilde{S}(\tau)$. Therefore, $\tilde{F}$ is in the space $E(\tau_0(\tilde{F}))$, i.e. it has an expansion in Hermite polynomials in the strip $S(\tau_0(\tilde{F}))$. In view of (2.4) the coefficients of this expansion are given by the equalities (4.5).

Now we claim that the radius of convergence $R(\tilde{F};\zeta)$ of the power series in (4.3) is exactly equal to one. At first we note that, since $0 < \tau_0(\tilde{F}) < \infty$, (4.5) and (2.2) yield

$$-\infty < \limsup_{n\to\infty}(2n+1)^{-1/2}\log|(2n/e)^{n/2}a_n(\tilde{F})| < 0 \tag{4.6}$$

Further, the asymptotic formula (2.3) implies that

$$\log|H_n(\zeta)a_n(\tilde{F})| = \log|(2n/e)^{n/2}a_n(\tilde{F})| + \eta_n(\zeta), \tag{4.7}$$

where $\eta_n(\zeta) = O(n^{1/2})$ when $n \to \infty$. Then, as a corollary of (4.5), (4.6) and (4.7) we have that

$$\limsup_{n\to\infty} n^{-1}\log|A_n(\tilde{F})|$$
$$= \limsup_{n\to\infty} n^{-1}(2n+1)^{1/2}(2n+1)^{-1/2}\log|(2n/e)^{n/2}a_n(\tilde{F})| = 0$$

and, therefore,

$$\{R(\tilde{F};\zeta)\}^{-1}$$
$$= \limsup_{n\to\infty}\exp\{n^{-1}\log|A_n(\tilde{F};\zeta)\} = \exp\{\limsup_{n\to\infty} n^{-1}\log|A_n(\tilde{F};\zeta)|\} = 1.$$

It is clear that the function (4.2) is holomorphic in the region W. Since W contains the closed unit disk except the points 1 and -1, at least one of these points is a singular point for the function (4.2).

Since the power series in (4.3) cannot be reduced to a polynomial, the complementary sequence $\tilde{k}$ of the sequence k with respect to the set of nonnegative integers is not finite. Moreover, as it is easy to prove, $\tilde{k}$ also has density and $\delta(\tilde{k}) = 1 - \delta(k)$.

The sequence k^* of the indices of the nonzero coefficients of the power series in (4.3) is also ifinite. If $\Delta(k^*)$ is its maximal density, then $\Delta(k^*) \le \delta(\tilde{k})$ [4, Note I, 2] and, in view of (4.1), we have that $\Delta(k^*) < 1/2$.

By a Theorem of G. Pólya [5, p. 625, Satz IV, a] every closed arc of the unit circle with angular measure $2\pi\Delta(k^*) < \pi$ contains at least one singular point of the function $A(\tilde{F};\zeta,w)$. But that is a contradiction since each point $\exp i\theta$ is a regular point for the power series in(4.3) provided that $0 < |\theta| < \pi$.

It is easy to see that the statement just proved is not true when $\delta(k) = 1/2$. Indeed, let a be a nonzero real number, $D \subset \mathbf{C} \setminus \{(-\infty,\infty) \cup \{-ia, ia\}\}$ be a simply connected region and $\gamma \subset D$ be an arbitrary rectifiable negatively oriented Jordan curve. Then, as a corollary of the equalities (3.1) as well as of the Cauchy integral formula, we obtain that

$$(2\pi i)^{-1}\int_{\gamma} G_{2n+1}(z)(z^2+a^2)^{-1}\,dz$$

$$=\int_{-\infty}^{\infty}\exp(-t^2)H_{2n+1}(t)(t^2+a^2)^{-1}\,dt=0,\quad n=0,1,2,\dots$$

and, therefore, the system $\{G_{2n+1}(z)\}_{n=0}^{\infty}$ is not complete in the space $H(D)$.

REFERENCES

[1] H. Bateman, A. Erdélyi. Higher Transcendental Functions, I, II, III. New York, Mc Graw-Hill BC, 1953.

[2] V. Bernstein. Leçons sur les progres récent de la théorie des séries de Dirichlet. Paris, Gauthier-Villars, 1933.

[3] PE. Hille. Contribution to the theory of Hermitian series, II: The representation problem. *Trans. Amer. Math. Soc.* **47** (1940), 80–94.

[4] B. Ja. Levin. Zeros of Entire Functions. Transl. Math. Monogr. vol. **5**, Providence RI, 1964.

[5] G. Pólya. Untersuchungen über Lücken und Singularitäten von Potenzreihen. *Math. Z.* **29** (1929), 549–640.

[6] S. Saks, A. Zygmund. Analytic Functions. Monografie Matematiczne vol. **28**, Warszawa, 1952.

[7] O. Szász. On the relative extrema of the Hermitian orthogonal functions. *J. Indiana Math. Soc.* **25** (1951), 129–134.

[8] G. Szegö. Orthogonal polynomials. Colloquium publications vol. **23**, Providence RI, 1939.

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics
Acad. G. Bontchev Str., Block 8
1113 Sofia, Bulgaria
e-mail: pkrusev@math.bas.bg

Received April 7, 2000
Revised August 7, 2000