Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **26** (2000), 353-358

Serdica
Mathematical Journal

Institute of Mathematics
Bulgarian Academy of Sciences

UNIFORMLY GÂTEAUX DIFFERENTIABLE NORMS IN SPACES WITH UNCONDITIONAL BASIS

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Communicated by S. L. Troyanski

Abstract. It is shown that a Banach space X admits an equivalent uniformly Gâteaux differentiable norm if it has an unconditional basis and X^* admits an equivalent norm which is uniformly rotund in every direction.

Let $(X, \|.\|)$ be a Banach space. Let S_X and B_X denote the unit sphere and the unit ball respectively, i. e. $S_X = \{x \in X; \|x\| = 1\}$ and $B_X = \{x \in X; \|x\| \leq 1\}$. Let $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ denote the sets of positive integers, rational numbers and real numbers respectively. Let X^* denote the dual to the Banach space X and let $\|.\|^*$ denote the norm on X^* that is dual to the norm $\|.\|$ on X. A biorthogonal system $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}1X \times X^*$ is called an *unconditional basis* for a Banach space X if for each $x \in X$ $x = \sum\limits_{\gamma\in\Gamma} f_\gamma(x)x_\gamma$ and the sum converges unconditionally. The

2000 *Mathematics Subject Classification:* 46B03, 46B15, 46B20.

Key words: Unconditional basis, uniformly Gâteaux smooth norms, uniform Eberlein compacts, uniform rotundity in every direction.

*Supported in part by GAČR 201-98-1449 and AV 101 9003. This paper is based on a part of the author's MSc thesis written under the supervison of Professor V. Zizler.

norm $\|.\|$ on a Banach space X is said to be *uniformly rotund in every direction* (*URED* for short), if $\lim \|x_n - y_n\| = 0$ whenever $x_n, y_n \in S_X$ are such that $\lim \|x_n + y_n\| = 2$ and $x_n - y_n = \lambda_n z$ for some $z \in X$, $\lambda_n \in \mathbb{R}$. The norm $\|.\|$ on X is called *uniformly Gâteaux differentiable* (UG) if

$$\lim_{t\to 0} \frac{1}{t}\left(\sup_{x\in S_X} \|x+th\| + \|x-th\| - 2\right) = 0$$

for every $h \in S_X$. A compact space K is called a *uniform Eberlein compact* (UEC) if K is homeomorphic to a weakly compact subset of a Hilbert space in its weak topology. The space $(\Sigma(\mathbb{R}^\Gamma), \tau)$ is a subspace of a product space $\mathbb{R}^\Gamma$ with the product topology, such that $x \in (\Sigma(\mathbb{R}^\Gamma), \tau)$ iff $x(\gamma) \neq 0$ for at most countably many $\gamma \in \Gamma$.

The main result in this paper is the following theorem.

Theorem 1. *Let X be a Banach space with an unconditional basis $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$. If X^* admits an equivalent (not necessarily dual) URED norm, then B_{X^*} in its weak* topology is a UEC.*

By putting this implication together with other already known results (we refer to [3], [4], [5, Chap. II], we obtain Theorem 2. Note that, except (i) $\Rightarrow$ (ii), all the remaining implications hold without the assumption of existence of an unconditional Schauder basis.

Theorem 2. *Let X be a Banach space with an unconditional basis $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$. Then the following are equivalent*

(i) *X^* admits an equivalent (not necessarily dual) URED norm.*

(ii) *B_{X^*} in its weak* topology is a UEC.*

(iii) *X admits an equivalent UG norm.*

In the proof of Theorem 1 we shall use the following statements.

Fact 3. *Let $(X, \|.\|)$ be a Banach space and let $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ be a normalized unconditional basis for X. For $x \in X$ put $\|x\|_1 = \sup_{\alpha_\gamma=\pm 1}\left\|\sum_{\gamma\in\Gamma} \alpha_\gamma f_\gamma(x) x_\gamma\right\|$. Then*

(i) *$\|.\|_1$ is an equivalent norm on X,*

(ii) $\left\|\sum_{\gamma\in F} a_\gamma x_\gamma\right\|_1 \leq \left\|\sum_{\gamma\in F} b_\gamma x_\gamma\right\|_1$, *and* $\left\|\sum_{\gamma\in F} a_\gamma f_\gamma\right\|_1^* \leq \left\|\sum_{\gamma\in F} b_\gamma f_\gamma\right\|_1^*$, *whenever* $F \subset \Gamma$ *is finite and* $a_\gamma, b_\gamma \in \mathbb{R}$ *satisfy* $|a_\gamma| \leq |b_\gamma|$ *for every* $\gamma \in F$,

(iii) $\|P_A\|_1 = 1$ *for* $A \subset \Gamma$ *finite, where* $P_A(x) = \sum_{\gamma\in A} f_\gamma(x)x_\gamma$,

(iv) $\|x_\gamma\|_1 = \|f_\gamma\|_1^* = 1$.

Proof. The proof is based on the similar ideas as [9, p. 499–505], where analogous statements for a countable set Γ are proved. □

The following lemma is due to Troyanski [10].

Lemma 4. *Let X be a Banach space and let* $\|.\|$ *be an equivalent URED norm on X. Then for any* $\varepsilon > 0$ *there exists a decomposition* $\{S_i^{(\varepsilon)}\}_{i=1}^\infty$ *of the unit sphere* S_X *such that for distinct* $\{x_j\}_{j=1}^i \subset S_i^{(\varepsilon)}$ *we have* $\max_{a_j=\pm1}\left\|\sum_{j=1}^i a_j x_j\right\| > \varepsilon^{-1}$.

We shall use the following topological characterization of uniform Eberlein compacts which can be found in [3].

Lemma 5. *A compact space K is UEC iff there exists a family* $\{V_\delta, \delta \in \Delta\}$ *of open* F_σ *subsets of K such that*

(i) *The family* $\{V_\delta, \delta \in \Delta\}$ *separates points of K, i. e. for* $x \neq y \in K$ *there exists* $\delta \in \Delta$ *such that* $x \in V_\delta$ *and* $y \notin V_\delta$, *or* $x \notin V_\delta$ *and* $y \in V_\delta$.

(ii) *There exists a decomposition of* Δ *into a sequence* $\{\Delta_n\}_{n=1}^\infty$ *and natural numbers* $\{k(n)\}_{n=1}^\infty$ *such that* $\{V_\delta, \delta \in \Delta_n\}$ *is* $k(n)$*-finite, i. e. any* $x \in K$ *belongs to at most* $k(n)$ *sets of the family* $\{V_\delta, \delta \in \Delta_n\}$.

The core of our note is the following lemma.

Lemma 6. *Let* $(X, \|.\|_1)$ *be a Banach space,* $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ *be a normalized unconditional basis,* $\|.\|_1$ *be a norm as in Fact 3,* $\|.\|_2$ *be an equivalent URED norm on* X^*. *Then there is a bounded linear, weak* to pointwise continuous, and one-to-one operator* $T : (X^*, w^*) \to (\Sigma(\mathbb{R}^\Gamma), \tau)$ *such that for any* $\varepsilon > 0$ *there exists a decomposition* $\{\Gamma_i^{(\varepsilon)}\}_{i=1}^\infty$ *of* Γ *such that*

$$card\,\{\gamma \in \Gamma_i^{(\varepsilon)}, |T(x^*)(\gamma)| > \varepsilon\} < i.$$

for all $x^* \in S_{(X^*,\|.\|_1^*)}$ *and* $i \in \mathbb{N}$.

Proof. Let $k > 1$ be such that $k\|x^*\|_2 \geq \|x^*\|_1^* \geq k^{-1}\|x^*\|_2$. Let $\tilde{f}_\gamma = \dfrac{f_\gamma}{\|f_\gamma\|_2}$ for all $\gamma \in \Gamma$. Put $\Gamma_i^{(\varepsilon)} = \{\gamma \in \Gamma, \tilde{f}_\gamma \in S_i^{(\varepsilon/k^2)}\}$, where according to Lemma 4, $\{S_i^{(\varepsilon/k^2)}\}_{i=1}^\infty$ is the decomposition of $S_{(X^*,\|.\|_2)}$ such that for all distinct $\{x_j^*\}_{j=1}^i \subset S_i^{(\varepsilon/k^2)}$ we have $\max_{a_j=\pm 1}\Big\|\sum_{j=1}^i a_j x_j^*\Big\| > k^2/\varepsilon$. Let $T\ (X^*, w^*) \to (\mathbb{R}^\Gamma, \tau)$ be defined for $x^* \in X^*$ by $T(x^*) = \{x^*(x_\gamma)\}_{\gamma\in\Gamma}$. Clearly, the operator T is weak* to pointwise continuous and one-to-one. Put $U_{x^*,i}^{(\varepsilon)} = \{\gamma \in \Gamma_i^{(\varepsilon)}; |T(x^*)(\gamma)| > \varepsilon\}$ and let us suppose that there is $\varepsilon > 0$, $x^* \in B_{X^*}$ and $i \in \mathbb{N}$ such that $|U_{x^*,i}^{(\varepsilon)}| \geq i$. Let $A \subset U_{x^*,i}^{(\varepsilon)}$ be such that $\operatorname{card} A = i$. Then

$$\begin{aligned}
1 &\geq \|x^*\|_1^* \geq \|P_A^*(x^*)\|_1^* = \Big\|\sum_{\gamma\in A} x^*(x_\gamma) f_\gamma\Big\|_1^* \\
&\geq \min_{\gamma\in A}|x^*(x_\gamma)|\ \Big\|\sum_{\gamma\in A} f_\gamma\Big\|_1^* > \varepsilon\Big\|\sum_{\gamma\in A} f_\gamma\Big\|_1^* = \varepsilon \max_{a_\gamma=\pm 1}\Big\|\sum_{\gamma\in A} a_\gamma f_\gamma\Big\|_1^* \\
&\geq \varepsilon k^{-1}\max_{a_\gamma=\pm 1}\Big\|\sum_{\gamma\in A} a_\gamma \tilde{f}_\gamma\Big\|_1^* \geq \varepsilon k^{-2}\max_{a_\gamma=\pm 1}\Big\|\sum_{\gamma\in A} a_\gamma \tilde{f}_\gamma\Big\|_2 > 1,
\end{aligned}$$

which is a contradiction. Hence, in particular, $T\ (X^*, w^*) \to (\Sigma(\mathbb{R}^\Gamma), \tau)$. □

Proof of the Theorem 1. We shall use the notation of Lemma 6. According to Lemma 5, in order to prove that (B_{X^*}, w^*) is UEC, we put $\Delta = \{(\gamma, r); \gamma \in \Gamma, r \in \mathbb{Q} \setminus \{0\}\}$. We put $V_{(\gamma,r)} = \{f \in B_{X^*}; T(f)(\gamma) > r\}$ for $r > 0$ and $V_{(\gamma,r)} = \{f \in B_{X^*}; T(f)(\gamma) < r\}$ for $r < 0$. Clearly, each $V_{(\gamma,r)}$ is weak*-open and F_σ-set. If $f, g \in B_{X^*}, f \neq g$ then there is $\gamma \in \Gamma$ such that $f(x_\gamma) \neq g(x_\gamma)$. Assume that $f(x_\gamma) < g(x_\gamma)$. We choose $0 \neq r \in \mathbb{Q}$ such that $f(x_\gamma) < r < g(x_\gamma)$. If $r > 0$, then $g \in V_{(\gamma,r)}$ and $f \notin V_{(\gamma,r)}$. If $r < 0$, then $f \in V_{(\gamma,r)}$ and $g \notin V_{(\gamma,r)}$. The case when $f(x_\gamma) > g(x_\gamma)$ can be treated similarly. Hence the family $\{V_{(\gamma,r)}; (\gamma, r) \in \Delta\}$ separates points of (B_{X^*}, w^*). Let $\{\Gamma_i^{(\varepsilon)}\}_{i=1}^\infty$ be the decomposition of Γ by Lemma 6. For $r \in \mathbb{Q} \setminus \{0\}$ and $i \in \mathbb{N}$ put

$$\Delta_{(r,i)} = \Gamma_i^{|r|} \times \{r\}.$$

Now fix one such (r, i) and consider any $f \in B_{X^*}$. If $(\gamma, r) \in \Delta_{(r,i)}$ and $f \in V_{(\gamma,r)}$, then $|T(f)(\gamma)| > |r|$. Therefore, by Lemma 6,

$$card\,\{V_{(\gamma,r)};\ (\gamma, r) \in \Delta_{(i,r)}, V_{(\gamma,r)} \ni f\} \leq card\,\{\gamma \in \Gamma_i^{|r|};\ |T(f)(\gamma)| > |r|\} < i.$$

It means that the family $\{V_{(\gamma,r)}; (\gamma,r) \in \Delta_{(r,i)}\}$ is $(i-1)$–finite. Hence (B_{X^*}, w^*) is UEC by Lemma 5. □

Remarks.

1) The condition of the existence of an unconditional basis can not be dropped. Indeed, consider the space $X = C[0,\omega_1]$ of all continuous functions on the ordinal segment $[0,w_1]$. The space X^* is isometric to $l_1([0,\omega_1])$ and hence admits an equivalent URED norm (see e.g. [5], Proposition II.7.7 and II.6.7). However, due to Talagrand's result (see e.g. [5], Theorem VII.5.2 and Theorem 2.6.7), X^* admits no dual strictly convex norm, hence X admits no equivalent UG norm.

2) There is a Banach space X with an unconditional basis such that the dual space X^* admits an equivalent (non dual) strictly convex norm and X admits no equivalent Gâteaux smooth norm. See [2].

3) There is a Banach space X with no equivalent UG norm such that the dual space X^* admits an equivalent dual URED norm. See [7].

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Received November 17, 2000