UNIFORMLY GÂTEAUX DIFFERENTIABLE NORMS IN SPACES WITH UNCONDITIONAL BASIS

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ABSTRACT. It is shown that a Banach space $X$ admits an equivalent uniformly Gâteaux differentiable norm if it has an unconditional basis and $X^*$ admits an equivalent norm which is uniformly rotund in every direction.

Let $(X, \| \cdot \|)$ be a Banach space. Let $S_X$ and $B_X$ denote the unit sphere and the unit ball respectively, i.e. $S_X = \{ x \in X ; \| x \| = 1 \}$ and $B_X = \{ x \in X ; \| x \| \leq 1 \}$. Let $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ denote the sets of positive integers, rational numbers and real numbers respectively. Let $X^*$ denote the dual to the Banach space $X$ and let $\| \cdot \|^{**}$ denote the norm on $X^*$ that is dual to the norm $\| \cdot \|$ on $X$. A biorthogonal system $\{ x_\gamma, f_\gamma \}_{\gamma \in \Gamma} X \times X^*$ is called an unconditional basis for a Banach space $X$ if for each $x \in X$ $x = \sum_{\gamma \in \Gamma} f_\gamma(x)x_\gamma$ and the sum converges unconditionally. The

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norm \( \| \cdot \| \) on a Banach space \( X \) is said to be **uniformly rotund in every direction** (URED for short), if \( \lim \| x_n - y_n \| = 0 \) whenever \( x_n, y_n \in S_X \) are such that \( \lim \| x_n + y_n \| = 2 \) and \( x_n - y_n = \lambda_n z \) for some \( z \in X \), \( \lambda_n \in \mathbb{R} \). The norm \( \| \cdot \| \) on \( X \) is called **uniformly Gâteaux differentiable** (UG) if

\[
\lim_{t \to 0} \frac{1}{t} \left( \sup_{x \in S_X} \| x + th \| + \| x - th \| - 2 \right) = 0
\]

for every \( h \in S_X \). A compact space \( K \) is called a **uniform Eberlein compact** (UEC) if \( K \) is homeomorphic to a weakly compact subset of a Hilbert space in its weak topology. The space \( (\Sigma(\mathbb{R}^\Gamma), \tau) \) is a subspace of a product space \( \mathbb{R}^\Gamma \) with the product topology, such that \( x \in (\Sigma(\mathbb{R}^\Gamma), \tau) \) iff \( x(\gamma) \neq 0 \) for at most countably many \( \gamma \in \Gamma \).

The main result in this paper is the following theorem.

**Theorem 1.** Let \( X \) be a Banach space with an unconditional basis \( \{x_\gamma, f_\gamma\}_{\gamma \in \Gamma} \). If \( X^* \) admits an equivalent (not necessarily dual) URED norm, then \( B_{X^*} \) in its weak* topology is a UEC.

By putting this implication together with other already known results (we refer to [3], [4], [5, Chap. II], we obtain Theorem 2. Note that, except (i) \( \Rightarrow \) (ii), all the remaining implications hold without the assumption of existence of an unconditional Schauder basis.

**Theorem 2.** Let \( X \) be a Banach space with an unconditional basis \( \{x_\gamma, f_\gamma\}_{\gamma \in \Gamma} \). Then the following are equivalent

- (i) \( X^* \) admits an equivalent (not necessarily dual) URED norm.
- (ii) \( B_{X^*} \) in its weak* topology is a UEC.
- (iii) \( X \) admits an equivalent UG norm.

In the proof of Theorem 1 we shall use the following statements.

**Fact 3.** Let \( (X, \| \cdot \|) \) be a Banach space and let \( \{x_\gamma, f_\gamma\}_{\gamma \in \Gamma} \) be a normalized unconditional basis for \( X \). For \( x \in X \) put \( \|x\|_1 = \sup_{\alpha_\gamma = \pm 1} \left\| \sum_{\gamma \in \Gamma} \alpha_\gamma f_\gamma(x) x_\gamma \right\| \). Then

- (i) \( \| \cdot \|_1 \) is an equivalent norm on \( X \),
(ii) \[ \left\| \sum_{\gamma \in F} a_\gamma x_\gamma \right\|_1 \leq \left\| \sum_{\gamma \in F} b_\gamma x_\gamma \right\|_1, \] and \[ \left\| \sum_{\gamma \in F} a_\gamma f_\gamma \right\|_* \leq \left\| \sum_{\gamma \in F} b_\gamma f_\gamma \right\|_* \] whenever \( F \subset \Gamma \) is finite and \( a_\gamma, b_\gamma \in \mathbb{R} \) satisfy \( |a_\gamma| \leq |b_\gamma| \) for every \( \gamma \in F \).

(iii) \( \|P_A\|_1 = 1 \) for \( A \subset \Gamma \) finite, where \( P_A(x) = \sum_{\gamma \in A} f_\gamma(x)x_\gamma \).

(iv) \( \|x_\gamma\|_1 = \|f_\gamma\|_*^1 = 1 \).

Proof. The proof is based on the similar ideas as [9, p. 499–505], where analogous statements for a countable set \( \Gamma \) are proved. \( \square \)

The following lemma is due to Troyanski [10].

Lemma 4. Let \( X \) be a Banach space and let \( \|\cdot\| \) be an equivalent URED norm on \( X \). Then for any \( \varepsilon > 0 \) there exists a decomposition \( \{S_i^{(\varepsilon)}\}_{i=1}^{\infty} \) of the unit sphere \( S_X \) such that for distinct \( \{x_j\}_{j=1}^{i} \subset S_i^{(\varepsilon)} \) we have \( \max_{a_j = \pm 1} \left| \sum_{j=1}^{i} a_j x_j \right| > \varepsilon^{-1} \).

We shall use the following topological characterization of uniform Eberlein compacts which can be found in [3].

Lemma 5. A compact space \( K \) is UEC iff there exists a family \( \{V_\delta, \delta \in \Delta\} \) of open \( F_\sigma \) subsets of \( K \) such that

(i) The family \( \{V_\delta, \delta \in \Delta\} \) separates points of \( K \), i.e. for \( x \neq y \in K \) there exists \( \delta \in \Delta \) such that \( x \in V_\delta \) and \( y \notin V_\delta \), or \( x \notin V_\delta \) and \( y \in V_\delta \).

(ii) There exists a decomposition of \( \Delta \) into a sequence \( \{\Delta_n\}_{n=1}^{\infty} \) and natural numbers \( \{k(n)\}_{n=1}^{\infty} \) such that \( \{V_\delta, \delta \in \Delta_n\} \) is \( k(n) \)-finite, i.e. any \( x \in K \) belongs to at most \( k(n) \) sets of the family \( \{V_\delta, \delta \in \Delta_n\} \).

The core of our note is the following lemma.

Lemma 6. Let \( (X, \|\cdot\|_1) \) be a Banach space, \( \{x_\gamma, f_\gamma\}_{\gamma \in \Gamma} \) be a normalized unconditional basis, \( \|\cdot\|_1 \) be a norm as in Fact 3, \( \|\cdot\|_2 \) be an equivalent URED norm on \( X^* \). Then there is a bounded linear, weak* to pointwise continuous, and one-to-one operator \( T : (X^*, w^*) \to (\Sigma(\mathbb{R}^\Gamma), \tau) \) such that for any \( \varepsilon > 0 \) there exists a decomposition \( \{\Gamma_i^{(\varepsilon)}\}_{i=1}^{\infty} \) of \( \Gamma \) such that

\[ \text{card} \{\gamma \in \Gamma_i^{(\varepsilon)}, |T(x^*)(\gamma)| > \varepsilon\} < i. \]

for all \( x^* \in S_{(X^*, \|\cdot\|_1^1)} \) and \( i \in \mathbb{N} \).
Proof. Let \( k > 1 \) be such that \( k\|x^*\|_2 \geq \|x^*\|_1^* \geq k^{-1}\|x^*\|_2 \). Let 
\[ f_\gamma = \frac{f_\gamma}{\|f_\gamma\|_2} \]
for all \( \gamma \in \Gamma \). Put \( \Gamma_i^{(e)} = \{ \gamma \in \Gamma; f_\gamma \in S_i^{(e/k^2)} \} \), where according to Lemma 4, \( \{ S_i^{(e/k^2)} \}_{i=1}^\infty \) is the decomposition of \( S(X^*,\|\cdot\|_2) \) such that for all distinct \( \{ x_j^{(i)} \}_{j=1}^i \subset S_i^{(e/k^2)} \) we have 
\[ \max_{a_j=\pm 1} \left| \sum_{j=1}^i a_j x_j^{(i)} \right| > k^2/\varepsilon. \]
Let \( T(X^*,w^*) \rightarrow (\Sigma(\mathbb{R}^k),\tau) \) be defined for \( x^* \in X^* \) by \( (x^*)_{\gamma,\epsilon} = \{ x^*(x_\gamma) \}_{\gamma \in \Gamma} \). Clearly, the operator \( T \) is weak*-pointwise continuous and one-to-one. Put \( U_{x^*,i}^{(e)} = \{ \gamma \in \Gamma_i^{(e)}; |T(x^*)_\gamma| > \varepsilon \} \)
and let us suppose that there is \( \varepsilon > 0, x^* \in B_{X^*} \) and \( i \in \mathbb{N} \) such that \( |T_{x^*,i}^{(e)}| \geq i \). Let 
\( A \subset U_{x^*,i}^{(e)} \) be such that \( \text{card } A = i \). Then

\[
1 \geq \|x^*\|_1^* \geq \|P_A^*(x^*)\|_1^* = \left\| \sum_{\gamma \in A} x^*(x_\gamma)f_\gamma \right\|_1^*
\geq \min_{\gamma \in A} |x^*(x_\gamma)| \left\| \sum_{\gamma \in A} f_\gamma \right\|_1 > \varepsilon \left\| \sum_{\gamma \in A} f_\gamma \right\|_1^* = \varepsilon \max_{a_{\gamma} = \pm 1} \left\| \sum_{\gamma \in A} a_\gamma f_\gamma \right\|_1^*
\geq \varepsilon k^{-1} \max_{a_{\gamma} = \pm 1} \left\| \sum_{\gamma \in A} a_\gamma f_\gamma \right\|_1^* \geq \varepsilon k^{-2} \max_{a_{\gamma} = \pm 1} \left\| \sum_{\gamma \in A} a_\gamma f_\gamma \right\|_2 > 1,
\]
which is a contradiction. Hence, in particular, \( T(X^*,w^*) \rightarrow (\Sigma(\mathbb{R}^k),\tau) \). \( \square \)

Proof of the Theorem 1. We shall use the notation of Lemma 6. According to Lemma 5, in order to prove that \( (B_{X^*},w^*) \) is UEC, we put 
\( \Delta = \{ (\gamma, r); \gamma \in \Gamma, r \in \mathbb{Q} \setminus \{0\} \} \). We put \( V_{(\gamma,r)} = \{ f \in B_{X^*}; T(f)(\gamma) > r \} \)
for \( r > 0 \) and \( V_{(\gamma,r)} = \{ f \in B_{X^*}; T(f)(\gamma) < r \} \) for \( r < 0 \). Clearly, each \( V_{(\gamma,r)} \)
is weak*-open and \( F_\sigma \)-set. If \( f, g \in B_{X^*}, f \neq g \) then there is \( \gamma \in \Gamma \) such that 
\( f(x_\gamma) \neq g(x_\gamma) \). Assume that \( f(x_\gamma) < g(x_\gamma) \). We choose \( 0 \neq r \in \mathbb{Q} \) such that 
\( f(x_\gamma) < r < g(x_\gamma) \). If \( r > 0 \), then \( g \in V_{(\gamma,r)} \) and \( f \notin V_{(\gamma,r)} \). If \( r < 0 \), then 
\( f \in V_{(\gamma,r)} \) and \( g \notin V_{(\gamma,r)} \). The case when \( f(x_\gamma) > g(x_\gamma) \) can be treated similarly.
Hence the family \( \{ V_{(\gamma,r)}; (\gamma, r) \in \Delta \} \) separates points of \( (B_{X^*},w^*) \). Let \( \{ \Gamma_i^{(e)} \}_{i=1}^\infty \)
be the decomposition of \( \Gamma \) by Lemma 6. For \( r \in \mathbb{Q} \setminus \{0\} \) and \( i \in \mathbb{N} \) put 
\[ \Delta_{(r,i)} = \Gamma_i^{(e)} \times \{ r \}. \]

Now fix one such \( (r, i) \) and consider any \( f \in B_{X^*} \). If \( (\gamma, r) \in \Delta_{(r,i)} \) and \( f \in V_{(\gamma,r)} \),
then \( |T(f)(\gamma)| > |r| \). Therefore, by Lemma 6,
\[
\text{card } \{ V_{(\gamma,r)}; (\gamma, r) \in \Delta_{(i,r)}, V_{(\gamma,r)} \ni f \} \leq \text{card } \{ \gamma \in \Gamma_i^{(e)}; |T(f)(\gamma)| > |r| \} < i.
\]
It means that the family \( \{ V(\gamma, r); (\gamma, r) \in \Delta_{(r, i)} \} \) is \((i - 1)\)–finite. Hence \((B_{X^*}, w^*)\) is UEC by Lemma 5. □

Remarks.

1) The condition of the existence of an unconditional basis can not be dropped. Indeed, consider the space \( X = C[0, \omega_1] \) of all continuous functions on the ordinal segment \([0, \omega_1]\). The space \( X^* \) is isometric to \( l_1([0, \omega_1]) \) and hence admits an equivalent URED norm (see e.g. [5], Proposition II.7.7 and II.6.7). However, due to Talagrand’s result (see e.g. [5], Theorem VII.5.2 and Theorem 2.6.7), \( X^* \) admits no dual strictly convex norm, hence \( X \) admits no equivalent UG norm.

2) There is a Banach space \( X \) with an unconditional basis such that the dual space \( X^* \) admits an equivalent (non dual) strictly convex norm and \( X \) admits no equivalent Gâteaux smooth norm. See [2].

3) There is a Banach space \( X \) with no equivalent UG norm such that the dual space \( X^* \) admits an equivalent dual URED norm. See [7].

REFERENCES


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