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DESCRIPTIVE SETS AND THE TOPOLOGY OF NONSEPARABLE BANACH SPACES

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Abstract. This paper was extensively circulated in manuscript form beginning in the Summer of 1989. It is being published here for the first time in its original form except for minor corrections, updated references and some concluding comments.

We call a Banach space descriptive (almost descriptive) if its weak topology has a σ-relatively discrete (σ-scattered) network. A key property in deriving many of our results is the fact that whenever the weak topology has a network of either type, then the norm topology will have a network of the same type where the discreteness property is again with respect to the weak topology. Properties known to hold for Banach spaces with an equivalent Kadec norm are shown to hold for the more general class of descriptive Banach spaces. And almost descriptive Banach spaces are shown to coincide with σ-fragmented Banach spaces introduced by Jayne, Namioka and Rogers.

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1. Introduction. In order to study the geometric and topological properties of non-separable Banach spaces it is usually necessary to impose some condition to limit the size of this class. Usually this is accomplished by requiring that the weak topology satisfy some compactness condition. This has led to the study of progressively more general classes such as reflexive Banach spaces, weakly compactly generated Banach spaces (see [1], [49], [65] and [8]) and, more recently, to the classes of (weakly) K-analytic and countably determined Banach spaces [71], [78]. All of these conditions, however, require that the underlying Banach space be a Lindelöf space in the weak topology.

The object of this paper is to take this investigation further by considering classes of non-separable Banach spaces whose weak topologies possess a certain type of network, similar to the types of networks investigated in the study of generalized metric spaces [23]. Recall that a collection $\mathfrak{N}$ of subsets of a space X is said to be a *network* for X if, whenever $x \in U$ and U is open in X, then for some $N \in \mathfrak{N}$ we have $x \in \mathfrak{N} \subset U$; thus, $\mathfrak{N}$ is like a base except the sets need not be open. The results we obtain seem to underscore the fact that this is a potentially useful way to approach the study of certain classes of non-separable Banach spaces. The classes study here provides a natural extension of the class of K-analytic and countably determined Banach spaces, but need not be Lindelöf in the weak topology.

It is often the case that results about separable metric spaces are of such a special character that they provide little if any insight as to how such a result might be generalized to the non-separable case. As an example, consider the following standard result for separable Banach spaces, where we use (X, weak) and (X, norm) to designate which topology is being associated with the Banach space X.

Theorem 1.1. *For a Banach space X the following are equivalent.*

(a) *X is separable.*

(b) (X, weak) *has a countable network.*

(c) (X, norm) *has a countable network consisting of weakly closed sets.*

Note that property (b) follows from the fact that any continuous image of a separable metric space has a countable network. (In fact, this property is characteristic of all Hausdorff spaces that are continuous images of separable metric spaces [51, Proposition 10.2]). Property (c) is especially useful and follows from (a) by taking any countable collection of closed balls whose interiors form a base (or open network) for the norm topology. Some immediate consequences of (c) are that each norm open set is a weakly $\mathcal{F}_\sigma$ set (hence the norm and weak topologies have the same Borel sets), and this implies that the points of *norm* discontinuity of any weakly continuous map into X is a set of the first category (cf. [2, §1]).

It is natural to ask if something similar to the equivalence (b) $\Longleftrightarrow$ (c) in Theorem 1.1 holds for suitable classes of non-separable Banach spaces. For example, suppose X is a Banach space such that (X, weak) has a network of the form $\mathfrak{N} = \bigcup_{n\in\mathbb{N}} \mathfrak{N}_n$ where each collection $\mathfrak{N}_n$ is disjoint and satisfies some "discreteness" property relative to the weak topology. Does it then follow that (X, norm) has a network of the form $\mathfrak{M} = \bigcup_{n\in\mathbb{N}} \mathfrak{M}_n$ where each collection $\mathfrak{M}_n$ satisfies the same discreteness property *relative to the weak topology*? In addition, can we choose the members of $\mathfrak{M}_n$ to be (say) Borel sets relative to the weak topology? We show that this is indeed the case, and that such networks will in fact exist for a wide class of non-separable Banach spaces, including all countably determined Banach spaces. A number of results are deduced from this similar to the above consequences of property (c) of Theorem 1.1.

Standard results of general topology show that separable metric spaces are essentially characterized by the existence of a countable open base. In analogy with this is the fact that non-separable metric spaces have a σ-discrete open base. In view of this and Theorem 1.1 it is tempting to ask if the weak topology of certain non-separable Banach spaces have a σ-discrete network. This turns out to be too much to expect, since discrete collections in Lindelöf spaces must be countable. Thus, for example, a weakly compactly generated Banach space would have this property only if it has a countable network; that is, only if it is separable.

No such problem is encountered however concerning the existence of σ-*relatively* discrete network for the weak topology of a Banach space. Recall that a collection $\mathcal{E}$ is *relatively discrete* if it is discrete relative to the subspace $\cup\mathcal{E}$ (i.e., each point in $\cup\mathcal{E}$ has a neighborhood meeting exactly one member of $\mathcal{E}$). Not only is this a useful property for the weak topology to have, but one which is satisfied (as we will show) by a significantly wide class of Banach spaces. Although the notion of a relatively discrete collections appears to have received little attention in the past, we will see that it is the natural property to consider in this context. We will see that relatively discrete collections have many of the usual preservation properties of discrete collections, except one has to replace references to closed sets by $\mathcal{F}\cap\mathcal{G}$ sets (i.e., sets of the form $F\cap G$ where F is closed and G is open).

Most of our results dealing with the topological properties of a Banach space hold more generally for the case of a function space of the type $C(K)$ with the weak topology replaced by the topology of pointwise convergence. If K is a compact Hausdorff space, then $C(K)$ denotes the Banach space of all real-valued continuous functions on K with the usual supremum norm, and we let $C_p(K)$ denote the same space with the topology of pointwise convergence. More generally, if $X \subset C(K)$, then (X, τ_p) will denote the space X with the topology of pointwise convergence. We can now state our first result which provides a natural extension of Theorem 1.1 to non-separable Banach spaces.

Theorem 1.2. *For any compact Hausdorff space K and for any $X \subset C(K)$ the following are equivalent.*

(a) (X, τ_p) *has a σ-relatively discrete network.*

(b) (X, norm) *has a network which, relative to the topology of point-wise convergence, is σ-relatively discrete and consists of $\mathcal{F} \cap \mathcal{G}$ sets.*

(c) *Each norm discrete collection of subsets of X has a τ_p σ-relatively discrete refinement.*

In particular, the above equivalences hold whenever X is a subset of a Banach space and τ_p is replaced by the weak topology of X.

As we show below, property (b) of Theorem 1.2 implies that the Borel sets for both the norm and τ_p topologies coincide. In order for this property to hold in a dual Banach space X^*, relative to the norm and weak* topologies, it is known that X^* must have the Radon-Nikodým property (RNP) (see, for example, [5, Theorems 4.4.2 and 4.4.3]). With this in mind we have the following analog of Theorem 1.2 for dual Banach spaces.

Theorem 1.3. *For any dual Banach space Z^* with RNP and for any $X \subset Z^*$ the following are equivalent.*

(a) (X, weak^*) *has a σ-relatively discrete network.*

(b) (X, norm) *has a network which, relative to the weak* topology, is σ-relatively discrete and consists of $\mathcal{F} \cap \mathcal{G}$ sets.*

(c) *Each norm discrete collection of subsets of X has a weak* σ-relatively discrete refinement.*

(d) *Any collection of weak* open sets has a σ-relatively discrete refinement.*

Conversely, if the norm topology of a dual Banach space Z^* has a network that is σ-relatively discrete for the weak* topology, then Z^* has RNP.

For general topological spaces property (d) of Theorem 1.3 is substantially weaker than property (a). The topological covering property known as *weak θ-refinability* can be characterized by the condition that every open cover of a space has a σ-relatively discrete refinement (see [4] or [7, Theorem 3.7]). It follows that any space with a σ-relatively discrete network is hereditarily weakly θ-refinable (equivalently, every collection of open sets, not necessarily a cover, has a σ-relatively discrete refinement). Several classes of Banach spaces described in detail below turn out to be equivalent when the weak topology is hereditarily weakly θ-refinable. Although it is shown that a large number of Banach spaces have this property, it remains an open question as to whether the weak topology of every Banach space has this property. (Professor Robert Wheeler has communicated to the author that he had raised this very question some years ago.)

The following theorem gives a list of some properties enjoyed by the spaces described in Theorems 1.2 and 1.3 above. The terms are explained following the theorem. Other properties, satisfied by a more general class of Banach spaces discussed below, are described later in this section.

Theorem 1.4. *Let X be a subset of a Banach space Z and let τ denote either the weak topology of Z, or the topology of pointwise convergence when $Z = C(K)$, or the weak* topology if Z is a dual Banach space with RNP. If (X, τ) has a σ-relatively discrete network, then the following hold.*

(a) *Each norm open set in X is a $(\mathcal{F} \cap \mathcal{G})_\sigma$ set relative to (X, τ).*

(b) *In the case when $X = Z$ and $\tau =$ weak topology, (Z, weak) is a $(\mathcal{F} \cap \mathcal{G})_{\sigma\delta}$ set in $(Z^{**}, \mathrm{weak}^*)$.*

(c) *If X is a norm Souslin subset of Z, then (X, τ) is a Čech analytic space, and every point-finite Souslin-additive family in (X, τ) is σ-relatively discretely decomposable.*

(d) *If X is a norm Souslin subset of Z, then (X, τ) is either a countable union of relatively discrete subspaces or else it contains a non-empty compact perfect subset.*

(e) *(X, τ) is a Radon space, provided it contains no discrete subsets of measurable cardinality.*

If $\mathfrak{A}$ is any collection of sets, then $\mathfrak{A}_\sigma$ and $\mathfrak{A}_\delta$ denote the collections of all countable unions and countable intersections respectively of sets in $\mathfrak{A}$, and $\mathfrak{A}_{\sigma\delta} = (\mathfrak{A}_\sigma)_\delta$. A set A is a *Souslin*($\mathfrak{A}$) set if, for some collection $\{A_{\tau|n} : \tau|n \in \mathbb{N}^n, n \in \mathbb{N}\} \subset \mathfrak{A}$, we have

$$A = \cup \left\{ \bigcap_{n=1}^{\infty} A_{\tau|n} : \tau \in \mathbb{N}^{\mathbb{N}} \right\}$$

where $\tau|n = (\tau_1, \ldots, \tau_n)$ for $\tau = (\tau_1, \tau_2, \ldots) \in \mathbb{N}^{\mathbb{N}}$. Standard results on the properties of the Souslin operation imply that, if $\mathfrak{B}$ denotes the family of Borel sets of a space X, then

$$\text{Souslin}(\mathfrak{B}) = \text{Souslin}(\mathcal{F} \cup \mathcal{G}),$$

where $\mathcal{F} \cup \mathcal{G}$ is the ordinary union of the closed and open sets of X. These sets will simply be called *Souslin sets* in X. For metrizable spaces they coincide with the more familiar Souslin($\mathcal{F}$) sets.

If X is a completely regular Hausdorff space, then X is said to be *Čech complete* if X is a $\mathcal{G}_\delta$ set in some (equivalently, every) compactification of X (see [14, p. 142]). Following Fremlin [17], we say that X is *Čech analytic* if it is a Souslin set in some (equivalently, every) compactification of X. The Čech analytic subsets of a compact Hausdorff space X are precisely the projections to X of Čech complete subsets of $X \times \mathbb{N}^{\mathbb{N}}$. This and other results of Fremlin on Čech analytic sets can be found in [34, Section 5] and in Fremlin unpublished note [17].

Properties of families of sets are always understood to be in the sense of an *indexed* family. In particular, if $\{E_\lambda : \lambda \in \Lambda\}$ is disjoint, then $\lambda \neq \gamma$ implies $E_\lambda \cap E_\gamma = \emptyset$. Similarly, a family of sets $\{E_\lambda : \lambda \in \Lambda\}$ is said to be *point-finite* or *point-countable* if, for any point x, $\{\lambda \in \Lambda : x \in E_\lambda\}$ is finite or countable, respectively. Suppose P denotes any property of sets or families of sets. Then we say that a family is *σ-P* if it is a countable union of subfamilies each of which has property P. We say that $\{E_\lambda : \lambda \in \Lambda\}$ is *σ-P-decomposable* if, for each $\lambda \in \Lambda$, $E_\lambda = \cup\{E_{\lambda n} : n \in \mathbb{N}\}$ where $\{E_{\lambda n} : \lambda \in \Lambda\}$ is disjoint and has property P for each $n \in \mathbb{N}$. A family is said to be *P-additive* if the union of each subset has property P. An important result in non-separable descriptive topology states that a point-finite Souslin-additive family in an analytic metric space is σ-discretely decomposable [44] (see also [26] and [19]). Property (c) of Theorem 1.4 is a generalization of this, since in a metric space any relatively discrete collection is σ–discretely decomposable [26, §1.2].

A Hausdorff space X is said to be a *Radon space* if each Borel measure μ on X is Radon; i.e., for each Borel set B of X, $\mu(B) = \sup\{\mu(K) : K \subset B \text{ and } K \text{ compact}\}$. A classical result of Marczewski and Sikorski [47] shows that any complete (hence any absolutely analytic) metric space is a Radon space if, and only if, it contains no discrete subset of measurable cardinality. (Recall that measurable cardinals are also said not to be real-valued measurable; the precise definitions are recalled in §5 below). Schachermayer [66] has shown that a similar result holds for any weakly compact subset of a Banach space with the weak topology (cf. also [11, p. 676]).

We say that a Hausdorff space (X, τ) is *descriptive* if there is a complete metric space T and a continuous surjection $f : T \to X$ such that, whenever $\{E_\lambda : \lambda \in \Lambda\}$ is a relatively discrete family in T, then $\{f(E_\lambda) : \lambda \in \Lambda\}$ is σ-relatively discretely decomposable in X. If we omit the word "relatively" in this statement, then we obtain a characterization of an analytic metric space (not necessarily separable), that is, a metric space which is a Souslin($\mathcal{F}$) set in every metric embedding [27]. When T is separable the above mapping property is automatically satisfied, and we get the definition of a *Souslin space* [67, p. 96] (also called an *analytic space* [64, §5.5]). More generally, we say that (X, τ) is *K–descriptive* if there is a complete metric space T and an upper semi-continuous compact-valued map $F : T \to X$ such that $X = \cup\{F(t) : t \in T\}$ and, whenever $\{E_\lambda : \lambda \in \Lambda\}$ is a discrete family in T, then $\{F(E_\lambda) : \lambda \in \Lambda\}$ is σ-relatively discretely decomposable in X. Recall that F is upper semi-continuous if $\{t \in T : F(t) \subset U\}$ is open in T whenever $U \subset X$ is open. When T is separable the above mapping property is again automatically satisfied, and we get the usual definition of a K-analytic space, or a (K-) countably determined space when T is not assumed to be complete [64].

We say that a Banach space Z is *descriptive* or *K-descriptive* whenever (Z, weak) is descriptive or K-descriptive, respectively. It is easy to see that any descriptive space has a σ-relatively discrete network (see Theorem 5.1), but that the converse is not true in general, since any metric space has such a network (and, as noted above, a metric space will be descriptive if, and only if, it is analytic). Notwithstanding this, Theorem 1.2 shows that a Banach space Z will be descriptive precisely when (Z, weak) has a σ-relatively discrete network (note that Theorem 1.2, (a) $\Rightarrow$ (c), implies that the identity map $(Z, \text{norm}) \to (Z, \text{weak})$ has the properties of the map f in the definition of a descriptive space). A similar statement holds for spaces of the type $C_p(K)$, and also for (Z^*, weak^*) where Z^* is a dual Banach space with RNP.

We will see below in Theorem 1.12 that descriptive, K-descriptive and Čech analytic are all equivalent for a Banach space when the weak topology is hereditarily weakly θ-refinable. We have already noted (Theorem 1.3, (a) $\Longleftrightarrow$ (d)) that a dual Banach space with RNP is weak* descriptive if, and only if, the weak* topology is hereditarily weakly θ-refinable. To see that there are dual Banach spaces with RNP whose weak* topology does not have these properties, recall that the ordinal space $[0, \omega_1]$, where ω_1 is the first uncountable ordinal, is a scattered compact space which is not hereditarily weakly θ-refinable (since $[0, \omega_1)$ is countably compact and any weakly θ-refinable countably compact space is compact [7, §9]). Since $[0, \omega_1]$ is scattered, $C([0, \omega_1])$ is an Asplund space, and so the dual space $C([0,\omega_1])^*$ has RNP [57, Theorem 18]; but $(C([0,\omega_1]),\mathrm{weak}^*)$ is not hereditarily weakly θ-refinable since it contains $[0,\omega_1]$.

As we will see, the following theorem shows that the class of descriptive Banach spaces is significantly wide.

Theorem 1.5. *Let Z be a Banach space, τ a locally convex topology on Z, and let $S=\{z\in Z:\|z\|=1\}$. If (S,τ) has a σ-relatively discrete network, then any subspace of (Z,τ) has a σ-relatively discrete network. In particular, if the norm and weak topologies agree on S, then Z is descriptive.*

The class of Banach spaces Z that have an equivalent norm $\|\cdot\|$ for which the weak and norm topologies coincide on $\{z\in Z:\|z\|=1\}$ is known to be quite extensive. Such a norm is said to be a *Kadec norm*. Properties such as (a), (b) and (e) of Theorem 1.4 are known to hold for Banach spaces having an equivalent Kadec norm (see [11, Theorem 1.1] and [12, §2]). The most common way of showing that a Banach space has such a norm is to produce a "long sequence" of projections and then apply the renorming technique of Troyanski [75]. In particular, this is true for the class of all countably determined (hence all K-analytic) Banach spaces [77], [71], as well as all function spaces of the type $C(K)$, where K is of the form $\{0,1\}^\Gamma$ for any index set Γ, or $K=[0,\tau]$ for any ordinal τ [12] (hence also for any continuous image of these compact spaces). It is also known that $C(K)$ will have an equivalent Kadec norm whenever K is a "Valdivia compact" (see the definition below) which includes Corson compacts and all "cubes" $[0,1]^\Gamma$ (see [10], [3] and also [76]). For each of these Banach spaces the weak topology will have a σ-relatively discrete network by Theorem 1.5.

Classes of dual Banach spaces which have an equivalent Kadec norm include all those having RNP [16] and all dual Banach spaces of the form $C(K)^*$ for any compact Hausdorff space K [75].

In all of the above cases the Banach space Z can be shown to have an equivalent *locally uniformly convex norm* $\|\cdot\|$, that is, for any sequence z_n and y in $S=\{z\in Z:\|z\|=1\}$, $z_n\to y$ in norm whenever $\|z_n+y\|\to 2$ as $n\to\infty$. It is easy to see that any locally uniformly convex norm is a Kadec norm.

A dual Banach space Z^* is said to have property $(**)$ if the weak* and norm topologies coincide on $\{z\in Z^*:\|z\|=1\}$ [57, p. 741], hence whenever Z^* has an equivalent locally uniformly convex dual norm. The space $\ell^1(\Gamma)=c_0(\Gamma)^*$, for example, is easily seen to have property $(**)$ for any set Γ. It is known that if Z^* has property $(**)$, then Z^* has the RNP ([57, Corollary 8], [5, Theorem 4.4.3]). It follows from Theorem 1.5 that (Z^*,weak^*) is descriptive whenever Z^* has property $(**)$.

The standard renorming techniques however will not permit us to use

Theorem 1.5 to conclude that $C_p(K)$ is descriptive when $C(K)$ has an equivalent Kadec norm. This is because the equivalent norm may not be lower semi-continuous with respect to the topology of pointwise convergence, hence we cannot conclude that the norm and *pointwise convergence* topologies coincide on the unit sphere associated with the Kadec norm. Equivalent locally uniformly convex norms satisfying this lower semi-continuity condition have recently been shown to exist for $C(K)$ for any "Valdivia compact" K [10]. A compact Hausdorff space K, which we may assume is embedded in some cube $[0,1]^\Gamma$, is called a *Valdivia compact* if $K \cap \Sigma(\Gamma)$ is dense in K where $\Sigma(\Gamma)$ is the set of points $x \in [0,1]^\Gamma$ such that $x(\gamma) \neq 0$ for at most countably many $\gamma \in \Gamma$. Any compact $K \subset \Sigma(\Gamma)$ (Corson compact) and any cube $[0,1]^\Gamma$ is a Valdivia compact, but the ordinal space $[0, \omega_2]$ is not [10]. From the results in [10] and Theorem 1.5 we obtain the following.

Theorem 1.6. *Let K be the continuous image of a Valdivia compact. Then $C_p(K)$ has a σ-relatively discrete network and hence is a descriptive space.*

There is another way to conclude that $C_p(K)$ will be descriptive when we know that $C(K)$ admits a certain type of continuous linear injection into a Banach space of the type $c_0(\Gamma)$. Recall that, for any infinite set Γ, $c_0(\Gamma)$ denotes the Banach space where

$$c_0(\Gamma) = \{x \in \mathbb{R}^\Gamma : \text{ for each } \varepsilon > 0 \text{ the set } \{\gamma \in \Gamma : |x(\gamma) > \varepsilon\} \text{ is finite}\},$$

and $\|x\| = \sup\{|x(\gamma)| : \gamma \in \Gamma\}$. In §7 we give a direct proof that the norm topology of $c_0(\Gamma)$ has a network that is σ-relatively discrete with respect to the topology of pointwise convergence.

A number of authors have described classes of compact Hausdorff spaces K for which there is a one-to-one bounded linear operator $T : C(K) \to c_0(\Gamma)$ which is also continuous when both Banach spaces are equipped with the topology of pointwise convergence [24], [3], [10]. These results generalize a deep result of Amir-Lindenstrauss [1], who origonally dealt with the case when K is a weakly compact subset of a Banach space, Further, Spahn [68] has shown that the linear operator T in these results also has the property that it takes norm discrete collections in $C(K)$ to norm σ-discretely decomposable collections in $c_0(\Gamma)$. Such a map T is said to be *index-σ-discrete* [52], and it is known that such maps will preserve Souslin sets [27] (these and related maps are discussed in more detail in §§2–3 below). These results enable us to prove the following.

Theorem 1.7. *Let K be a compact Hausdorff space for which there exist a set Γ and a one-to-one bounded linear operator $T : C(K) \to c_0(\Gamma)$ such that T*

is also continuous relative to the topologies of pointwise convergence and is norm index-σ-discrete. Then $C_p(K)$ has a σ-relatively discrete network and hence is a descriptive space.

It is considerably more difficult to find Banach spaces $C(K)$ such that $C_p(K)$ is not descriptive. In answer to a question of Edgar [11, Prob. 1.3], Talagrand [73] has shown that the weak and norm Borel sets of $\ell^\infty = C(\beta\mathbb{N})$ do not coincide, and that $(\ell^\infty, \text{weak})$ does not embed as a Borel set in $(\ell^{\infty**}, \text{weak}^*)$. Thus ℓ^∞ is not (weakly) descriptive (by (a) or (c) of Theorem 1.4). More generally, it has recently been shown in [39] that $(\ell^\infty, \text{weak})$ is not even Čech analytic, so $(\ell^\infty, \text{weak})$ does not embed as a Souslin set in $(\ell^{\infty**}, \text{weak}^*)$. It follows that $C_p(\beta\mathbb{N})$ is not descriptive, since this would imply that ℓ^∞ is descriptive by Theorem 1.2, (a)$\Rightarrow$(b).

The problem of delimiting the class of compact Hausdorff spaces K for which C_p(K) will be descriptive has, at present, the same status as the corresponding questions dealing with the existence of an equivalent Kadec norm for $C(K)$. Known counterexamples seem to be limited to those Banach spaces which contain an isomorphic copy of ℓ^∞. For example, we do not know if $C_p(K)$ is descriptive for every scattered compact space K, or even in the case when the derived set $K^{(\omega_1)} = \emptyset$ (cf. [10], Problem 3, where the corresponding question concerning the existence of an equivalent locally uniformly convex norm is raised). Note that in the case when K is scattered it is enough to show that $C(K)$ is (weakly) descriptive to conclude that $C_p(K)$ is, although in general we do not know if $C_p(K)$ will be descriptive whenever $C(K)$ is.

We now describe a second class of Hausdorff spaces obtained by weakening the property of a relatively discrete collection in the definition of a descriptive space. This class of spaces, and the type of network they give rise to, also relate well to the various topologies associated with a Banach space. The resulting class of "almost descriptive" Banach spaces has many properties similar to those described above for the class of descriptive Banach spaces, except Borel and Souslin sets are generally replaced by various types of sets having the Baire property. Unexpectedly, our almost descriptive Banach spaces turn out to be exactly equivalent to the Banach spaces, recently studied in [39], whose weak topology is "σ-fragmented" by the norm metric. In order to describe these spaces we need some additional terminology.

A collection of sets $\mathcal{E}$ in a topological space is said to be *scattered* if $\mathcal{E}$ is disjoint and there is a well-ordering $\leq$ of $\mathcal{E}$ such that, for each $E \in \mathcal{E}$, $\cup\{M \in \mathcal{E} : M \leq E\}$ is open relative to $\cup\mathcal{E}$. Note that if $\{U_\alpha : \alpha < \lambda\}$ is any ordinal-indexed family of open sets in a space, and

$$D_\alpha \subset U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta,$$

then $\{D_\alpha : \alpha < \lambda\}$ is scattered. Consequently, any open collection in a space has a scattered refinement. A collection is *σ-scattered* if it is a countable union of scattered collections. Note that, as with the notion of a relatively discrete collection, the concept of a scattered collection is intrinsic and therefore is independent of the particular space containing the given collection. Below we describe several properties equivalent to the definition of a scattered collection from which it will be clear that a topological space X is scattered in the usual sense (i.e., each non-empty subset has an isolated point) if, and only if, $\{\{x\} : x \in X\}$ is a scattered collection (see Lemma 2.1). It is easy to see that any relatively discrete collection is a scattered collection (Lemma 2.2), and we will show that in any hereditarily weakly θ-refinable space (hence, in any metric space) every scattered collection is σ-relatively discretely decomposable (Lemma 3.6).

Using the notion of a scattered collection as a substitute for relatively discrete collections in the definitions of a descriptive and K-descriptive space we obtain another interesting class of "analytic" spaces. However, we need to modify the definition slightly to circumvent an anomaly of scattered collections (see Example 2.8 and the discussion preceding it in §2).

We say that a Hausdorff space X is *almost descriptive* (resp. *almost K-descriptive*) if there is a complete metric space T and a continuous (resp. an upper semi-continuous compact valued) surjection $f : T \to X$ such that, whenever $\{E_a : a \in A\}$ is a scattered family in T, $\{f(E_a) : a \in A\}$ is point-countable and has a σ-scattered base. Recall that a collection of sets $\mathfrak{B}$ is said to be a base for a family $\mathfrak{A}$ if each member of $\mathfrak{A}$ is a union of sets from $\mathfrak{B}$.

If P denotes a property of families of sets in a topological space, it is easy to see that for any family of sets $\mathcal{E}$ we have

$$\sigma\text{-}P \text{ decomposable} \Rightarrow \text{has a } \sigma\text{-}P \text{ base} \Rightarrow \text{has a } \sigma\text{-}P \text{ refinement.}$$

A useful converse of the first implication is that any point-countable family having a σ-discrete base is σ-discretely decomposable [52, Lemma 3.2], and the same proof works for the corresponding statement involving relatively discrete collections. This, however, does not hold for scattered collections (see Example 2.8) and this necessitates the difference in the above definition from the corresponding one for descriptive spaces.

As noted above, we will see that in any hereditarily weakly-θ-refinable space all scattered collections are σ-relatively discretely decomposable, and the converse is also true (see Lemma 3.6). This leads to the following simple relationships between these classes of spaces.

Theorem 1.8. *Any Hausdorff space is descriptive if, and only if it is almost descriptive and hereditarily weakly θ-refinable. Any hereditarily weakly θ-refinable almost K-descriptive space is K-descriptive.*

We say that a Banach space Z is *almost descriptive* (or *almost K-descriptive*) whenever (Z, weak) has this property. In analogy with descriptive spaces, it is easy to see that any almost descriptive space has a σ-scattered network (Theorem 5.1), and that the converse is not true in general. However, for Banach spaces taken with the weak topology we will see that the existence of a σ-scattered network not only implies that the Banach space is almost descriptive, but also a number of other structural properties which are seemingly much stronger. Before discussing these results we describe yet another class of spaces recently investigated in [39] which, rather unexpectedly, turns out to lead to exactly the same class of Banach spaces.

Following Jayne, Namioka and Rogers [39], we say that a topological space (X, τ) is *σ-fragmented* by a metric ρ on the set X if, for each $\varepsilon > 0$, we can write $X = \cup\{X_n : n \in \mathbb{N}\}$ so that, for each n, each nonempty subset of X_n contains a nonempty relatively τ-open set with ρ-diameter less than ε. If, on the other hand, every nonempty subset of X contains nonempty relatively τ-open subsets of arbitrarily small ρ-diameter, then (X, τ) is said to be *fragmented* by the metric ρ. We used the terms *properly σ-fragmented* or *properly fragmented* if, in addition, the metric topology contains the given topology τ. The metric ρ is said to be lower semi-continuous if ρ is lower semi-continuous as a real-valued map on $(X, \tau) \times (X, \tau)$ (equivalently, $\{(x, y) : \rho(x, y) \leq r\}$ is closed in $(X, \tau) \times (X, \tau)$ for each real $r \geq 0$). The concept of a fragmentable space, introduced explicitly by Jayne and Rogers in [42], has proved useful in studying various topological properties of Banach spaces ([42], [37], [56], [62], [69]). The weaker notion of a σ-fragmentable space was recently introduced in [39] where, in particular, the following result was obtained.

Lemma 1.9 [39, Thm. 4.1]**.** *Let ρ be a lower semi-continuous metric on a Hausdorff space (X, τ). If (X, τ) is σ-fragmented by ρ, then each τ-compact subset of X is fragmented by ρ, and the converse holds whenever (X, τ) is Čech analytic.*

Namioka has shown that, for any compact Hausdorff space K, each compact subset of $C_p(K)$ is fragmented by the supremum norm metric, hence any weakly compact subset of a Banach space is fragmented by its norm metric ([56, Theorem 1.2]). Thus, in view of the above lemma, for any weakly Čech analytic Banach space Z, (Z, weak) is σ-fragmented by the norm metric of Z [39, Proposition 6.3]. We will see that this implies that (Z, norm) has a weakly σ-scattered network.

Dual Banach spaces with the Radon-Nikodým Property are characterized by the condition that each weak*-compact subset is fragmented by the norm metric ([57, Lemma 3], [56]). Consequently, a dual Banach space Z^* has the Radon-Nikodým Property if, and only if, (Z^*, weak^*) is σ-fragmented by the norm metric of Z^* [39, Theorem 6.2]. We show below that these properties are also equivalent to (Z^*, norm) having a weak*σ-scattered network (see Theorem 1.11 below).

The following theorem shows that there is a close analogy between the investigation described above, concerning the existence of certain types of networks for the topologies associated with a Banach space, and the corresponding situation for the topologies associated with a properly σ-fragmented space.

Theorem 1.10. *Let (X, τ) be a topological space which is properly σ-fragmented by some metric ρ for the set X. Let P denote the property of a family of sets being either relatively discrete or scattered for the space (X, τ). Then the following are equivalent.*

(a) *Any collection of τ-open sets has a σ-P refinement.*

(b) *(X, τ) has a σ-P network.*

(c) *Any ρ discrete collection has a τ σ-P refinement.*

(d) *(X, ρ) has a network that is σ-P with respect to (X, τ).*

Moreover, property (a), *and thus all four properties, will always hold when P = scattered; in the case when P = relatively discrete, property* (a) *is equivalent to (X, τ) being hereditarily weakly θ-refinable.*

Conversely, if ρ is any metric on X satisfying property (d) *and τ is contained in the metric topology, then (X, τ) is properly σ-fragmented by ρ.*

If in Theorem 1.10 X is a Banach space, ρ is the norm metric, τ is the weak topology and $P = \text{scattered}$, then properties (b), (c) and (d) can be shown to be equivalent irrespective of whether the norm metric is known to σ-fragments the weak topology (note that this is in analogy with Theorem 1.2). In this case, we can conclude from the converse part of Theorem 1.10 that the norm metric will σ-fragmented the weak topology whenever the latter has a σ-scattered network. In the other direction, if the Banach space is weakly Čech analytic, then Lemma 1.9 implies that the norm metric σ-fragments the weak topology and so properties (a)–(c) of Theorem 1.10 are now satisfied; in particular, the Banach space must be almost descriptive. These observations are summarized in the following general theorem.

Theorem 1.11. *Let X be a subset of a Banach space Z and let τ denote either the weak topology of Z, or the topology of pointwise convergence when $Z = C(K)$, or the weak* topology if Z is a dual Banach space with RNP. Then the following are equivalent, and always hold in the case of dual Banach spaces with RNP.*

(a) (X, τ) *has a σ-scattered network.*

(b) (X, norm) *has a network which, relative to the topology τ, is σ-scattered and consists of $\mathcal{F} \cap \mathcal{G}$ sets.*

(c) *Each norm discrete collection has a τ σ-scattered base.*

(d) (X, τ) *is σ-fragmented by the norm metric on X.*

Property (c) *implies that (X, τ) will be almost descriptive whenever X is a norm Souslin subset of Z. Moreover, the above properties hold whenever (X, τ) is Čech analytic.*

Conversely, if X is a dual Banach space satisfying any one of the properties (b)–(d)*, then X has RNP.*

Lemma 1.9 and Theorems 1.8 and 1.10 combine to give a number of equivalent conditions for a subset of a Banach space to be weakly descriptive.

Theorem 1.12. *Let X be a norm Souslin subset of a Banach space Z and let τ denote either the weak topology of Z, or the topology of pointwise convergence when $Z = C(K)$, or the weak* topology if Z is a dual Banach space with RNP. Then the following properties* (a)–(e) *are equivalent.*

(a) (X, τ) *is descriptive.*

(b) (X, τ) *is K-descriptive and hereditarily weakly θ-refinable.*

(c) (X, τ) *is Čech analytic and hereditarily weakly θ-refinable.*

(d) (X, τ) *is σ-fragmented by the norm metric of Z and hereditarily weakly θ-refinable.*

(e) (X, τ) *is almost K-descriptive and hereditarily weakly θ-refinable.*

Further, in the case of the weak topology on a dual Banach space Z with RNP, the above are equivalent to*

(f) (X, τ) *is hereditarily weakly θ-refinable.*

We have already indicated that it is an open question as to whether the weak topology of *every* Banach space is hereditarily weakly θ-refinable. In view of the equivalence (a) $\iff$ (e) in the above theorem, this would be answered in the negative if we could find an example of an almost K-descriptive Banach space which was not descriptive. But at present these remain very subtle questions to be resolved.

Theorem 1.5 has the following analog for σ-scattered networks.

Theorem 1.13. *Let Z be a Banach space, τ a locally convex topology on Z, and let $S = \{z \in Z : \|z\| = 1\}$. If (S, τ) has a σ-scattered network, then any subspace of (Z, τ) has a σ-scattered network.*

In order to describe certain properties of almost descriptive Banach spaces we need to recall the definitions of several additional concepts from descriptive topology. A set B in a topological space X is said to have the *Baire property* in X, or to be a *BP-set* in X, if, for some open set G in X, the symmetric difference $(G \setminus B) \cup (B \setminus G)$ is of the first category in X; equivalently, B is the union of a $\mathcal{G}_\delta$ set and a first category set in X (see [46, §11] or [60, Chapt. 4]). Since the family of all sets having the Baire property in X is closed under the Souslin operation [46, p. 94], every Souslin set in X is a BP-set. We say that $B \subset X$ is a *restricted BP-set* in X (in the terminology of [46, p. 92] B has the *Baire property in the restricted sense*) if, for each $E \subset X$, $B \cap E$ is a BP-set in the subspace E. Again, it is easy to see that all Souslin sets in X have this property. We will say that $B \subset X$ is a *strong BP-set* in X if $B = G \cup M$, where G is an open set and M is a set of the first category in X. *Strong restricted BP-sets* are defined analogously. Any $(\mathcal{F} \cap \mathcal{G})_\delta$ set in a space X is a strong restricted BP-set in X (see Lemma 2.6).

A fundamental property of scattered collections is that the union of any collection of (strong) BP-sets having a σ-scattered refinement will again be a (strong) BP-set, and a similar result holds for BP-sets in the restricted sense (see Lemmas 2.3 and 2.6).

Let $f : X \to Y$ be a map from a topological space X to a metric space Y. If Y is separable f is a Borel class 1 map (that is, $f^{-1}(U)$ is a $(\mathcal{F} \cap \mathcal{G})_\delta$ set in X for each open set $U \subset Y$), then it is well known that the set of points of discontinuity for f is a set of the first category in X. Whether this continues to hold when Y is non-separable is not known in general, even when X is metrizable. Similar remarks apply to the case when f is a general Borel map and we wish to conclude that there is a set M of the first category in X such that $f \restriction X \setminus M$ is continuous. The following theorem shows that we can give affirmative answers to these questions when X is an almost descriptive space. In particular, these results apply to any Čech analytic subset of a function space of the type $C_p(K)$, and for any norm Souslin subset of a dual Banach space with RNP in the weak* topology (by Theorem 1.11)

Theorem 1.14. *The following are true for any almost descriptive Hausdorff space X.*

(a) *X is a restricted BP-set in every topological embedding.*

(b) *Each point-finite Souslin-additive collection in X is σ-scattered decomposable.*

(c) *If $f : X \to Y$ is any Borel class 1 map into a metric space Y, then the points of discontinuity for f is a set of the first category in (X, τ).*

(d) *If $f : X \to Y$ is any Borel map into a metric space Y, then there is a set M of the first category in X such that $f \restriction X \setminus M$ is continuous.*

(e) *If $A \subset X$ is a Souslin set, then either A is a σ-scattered set or else it contains a compact perfect set.*

Our final application deals with the Namioka property concerning separate-to-norm continuity [55]. A compact space K is said to have the Namioka property if for every Baire topological space X and every continuous map $f : X \to C_p(K)$, there is a dense $\mathcal{G}_\delta$ set $G \subset X$ such that f is norm continuous, as a map from X to $C(K)$, at each point of G. It is not difficult to see that this is equivalent to the statement that if $f : X \times K \to \mathbb{R}$ is separately continuous, then f is jointly continuous at each point of $G \times K$. Deville and Godefroy [10] have recently shown that all Valdivia compacts (defined above) have the Namioka property, extending an earlier result of Debs [9] who had shown that all Corson compacts have this property. The class of compacts having the Namioka property is easily seen to be closed to continuous images, but it is not stable to closed subspaces, since all cubes $[0,1]^\Gamma$ are Valdivia compact [10] but not all compact spaces have the Namioka property [74].

In [10, Lemma II.1] it is shown, more generally, that if K is a compact Hausdorff space such that $C(K)$ has an equivalent locally uniformly convex norm which is τ_p-lower semi-continuous, then K has the Namioka property. By Theorem 1.5, such a space $C(K)$ will be descriptive, without assuming the norm is τ_p-lower semi-continuous. In view of this, the following theorem improves on the results in [10].

Theorem 1.15. *Let X be a Baire topological space, K a compact Hausdorff space and $f : X \to C_p(K)$ a continuous map. If $(f(X), \tau_p)$ has a σ-scattered* *network, then $f : X \to C(K)$ will be norm continuous at each point of some dense $\mathcal{G}_\delta$ subset of X. In particular, K has the Namioka property whenever $C_p(K)$ is almost descriptive.*

In order to avoid duplicating proofs for the two types of networks described above, and to possibly pave the way for other similar types of network, we deal with an abstract generalization of both which we call a *discreteness class* in §6. The proofs of the main theorems are given in §9 essentially by referring back to the appropriate lemmas and theorems in §§2–8.

2. Scattered collections and Baire property sets. In this section we give several equivalent descriptions of scattered collections of sets and use these to prove some permanence properties of sets with the Baire property.

Recall that a topological space X is said to be *scattered* if each nonempty subset has an isolated point, or, equivalently, X has no perfect subset. From the equivalence (a) $\iff$ (c) in the following lemma it follows that a space X is scattered if and only if $\{\{x\} : x \in X\}$ is a scattered collection.

Lemma 2.1. *For any collection $\mathcal{E}$ of disjoint subsets of a topological space the following are equivalent.*

(a) *$\mathcal{E}$ is scattered.*

(b) *There is a well-ordering $\le$ of $\mathcal{E}$ such that, for any $E \in \mathcal{E}$, the set $\cup\{M \in \mathcal{E} : M \le E\}$ is open relative to $\cup\mathcal{E}$.*

(c) *For any nonempty $\mathcal{H} \subset \mathcal{E}$, some $H \in \mathcal{H}$ is open relative to $\cup\mathcal{H}$.*

(d) *Each nonempty $H \subset \cup\mathcal{E}$ has a nonempty relatively open subset of the form $H \cap E$ for some $E \in \mathcal{E}$.*

Proof. The equivalence (a) $\iff$ (b) is simply the definition of a scattered collection (see §1), and the proof of (b) $\iff$ (d) is given in [53, Lemma 2.1].

To prove that (b) $\Rightarrow$ (c), suppose H is the first element of $\mathcal{H}$ and let

$$U = \cup\{E \in \mathcal{E} : E \le H\}.$$

Then U is open in $\cup\mathcal{E}$ and $U \cap (\cup\mathcal{H}) = H$, showing that H is open in $\cup\mathcal{H}$. Conversely, suppose $\mathcal{E}$ satisfies (c), and define E_0 to be any member of $\mathcal{E}$ which is open relative to $\cup\mathcal{E}$. Assuming we have defined $\{E_\alpha : \alpha < \lambda\} \subset \mathcal{E}$ with $\bigcup_{\beta \le \alpha} E_\alpha$ open relative to $\cup\mathcal{E}$ for each $\alpha < \lambda$, if

$$\mathcal{H} = \mathcal{E} \setminus \{E_\alpha : \alpha < \lambda\}$$

is nonempty, define E_λ to be any $H \in \mathcal{H}$ open relative to $\cup\mathcal{H}$, and note that $\bigcup_{\alpha\le\lambda} E_\alpha$ is open relative to $\cup\mathcal{E}$. This defines a well-ordering of $\mathcal{E}$ for which (b) is satisfied. □

A scattered cover of a space has also been called an "exhaustive cover" [53] (cf. also [65] where the term "relatively open" is used for a scattered partition). Our terminology follows [79]. Of the three properties characterizing scattered collections, property (d) of Lemma 2.1 is often the most useful to work with and is due to Michael [53, Lemma 2.1]. When a transfinite sequence $\mathcal{E} = \{E_\alpha : \alpha < \lambda\}$ is said to be scattered, it will always be understood that sets having different indices are disjoint and that $\cup\{E_\beta : \beta \le \alpha\}$ is open in $\cup\mathcal{E}$ for each $\alpha < \lambda$. Note that $\{E_\alpha : \alpha < \lambda\}$ is a scattered collection of subsets of a space X if, and only if, there is a collection $\{U_\alpha : \alpha < \lambda\}$ of open sets in X such that

$$E_\alpha \subset U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta$$

for each $\alpha < \lambda$.

Lemma 2.2. *Let $\Sigma = \Sigma(X)$ denote the family of all scattered collections of subsets of a space X. Then Σ has the following properties.*

(a) *If $\mathcal{E}$, $\mathcal{H} \in \Sigma$, then $\{E \cap H : E \in \mathcal{E}, H \in \mathcal{H}\} \in \Sigma$.*
(b) *If $\mathcal{E} \in \Sigma$ and $H_E \subset E$ for each $E \in \mathcal{E}$, then $\{H_E : E \in \mathcal{E}\} \in \Sigma$. In particular, any subcollection of a scattered collection is scattered.*
(c) *If $\mathcal{E} \in \Sigma$ and, for each $E \in \mathcal{E}$, $E = \cup\mathcal{H}_E$ with $\mathcal{H}_E \in \Sigma$, then $\mathcal{H} \equiv \{H : H \in \mathcal{H}_E, E \in \mathcal{E}\} \in \Sigma$.*
(d) *If X is a subspace of Y, then $\Sigma(X) \subset \Sigma(Y)$ and $\mathcal{E} \in \Sigma(Y)$ implies $\{E \cap X : E \in \mathcal{E}\} \in \Sigma(X)$.*
(e) *If $\mathcal{E}$ is a relatively discrete collection of subsets of X, then $\mathcal{E} \in \Sigma$.*

Proof. (a) Given any nonempty $A \subset (\cup\mathcal{E}) \cap (\cup\mathcal{H})$, by (d) of Lemma 2.1 there is some $E \in \mathcal{E}$ such that $A \cap E$ is nonempty and relatively open in A. Since $A \cap E \subset \cup\mathcal{H}$, a second application of Lemma 2.1 (d) implies there exists $H \in \mathcal{H}$ such that $A \cap E \cap H$ is nonempty and relatively open in $A \cap E$, hence also in A. It follows that $\{E \cap H : E \in \mathcal{E}, H \in \mathcal{H}\}$ by Lemma 2.1, (d) $\Rightarrow$ (a).

(b) Given any nonempty $A \subset \cup\{H_E : E \in \mathcal{E}\}$, by Lemma 2.1 (d) there is some $E \in \mathcal{E}$ such that $A \cap E$ is nonempty and relatively open in A, since $\mathcal{E} \in \Sigma$. But $A \cap E = A \cap H_E$ (since the members of $\mathcal{E}$ are disjoint), and so $\{H_E : E \in \mathcal{E}\} \in \Sigma$.

(c) Given any nonempty $A \subset \cup\mathcal{H} = \cup\mathcal{E}$, since $\mathcal{E}$ is scattered there is some $E \in \mathcal{E}$ such that $A \cap E$ is nonempty and relatively open in A. Since $A \cap E \subset E = \cup\mathcal{H}_E$ and $\mathcal{H}_E$ is scattered, there is some $H \in \mathcal{H}_E$ such that $A \cap E \cap H = A \cap H$ is nonempty and relatively one in $A \cap E$, hence also in A. It follows that $\mathcal{H}$ is scattered by Lemma 2.1, (d) $\Rightarrow$ (a).

(d) This is obvious from the definition.

(e) If $H \subset \cup\mathcal{E}$ is nonempty, then any nonempty set of the form $H \cap E$, $E \in \mathcal{E}$, will be a nonempty open set relative to H, since each E is open relative to $\cup\mathcal{E}$.

Lemma 2.3. *If $\{E_\alpha : \alpha < \lambda\}$ is a scattered collection of subsets of a topological space X, then there is a scattered collection $\{H_\alpha : \alpha < \lambda\}$ of $\mathcal{F} \cap \mathcal{G}$ sets in X such that $E_\alpha \subset H_\alpha$ for each $\alpha < \lambda$.*

Proof. We can choose open sets $\{U_\alpha : \alpha < \lambda\}$ in the space X such that $U_\alpha \cap (\cup\mathcal{E}) = \bigcup_{\beta \le \alpha} E_\beta$ for each $\alpha < \lambda$ (see the remarks after Lemma 2.1). If

$$H_\alpha = U_\alpha \setminus \bigcup_{\beta < \alpha} U_\beta,$$

then $E_\alpha \subset H_\alpha$ for each $\alpha < \lambda$, and $\{H_\alpha : \alpha < \lambda\}$ is a scattered collection of $\mathcal{F} \cap \mathcal{G}$ sets in X. □

Lemma 2.4. *Let P be a property of subsets of a topological space X closed to finite unions and such that, whenever a set $A \subset X$ is locally-P (i.e., each point of A has a neighborhood U such that $U \cap A$ has P), then A has P. Then the union of any scattered collection of subsets of X having property P will also have P.*

Proof. Let $\mathcal{E}$ be a scattered collection of subsets E of X each having property P. We may assume that $\mathcal{E} = \{E_\alpha : \alpha < \lambda\}$, where $\bigcup_{\beta \le \alpha} E_\beta$ is open in $\cup\mathcal{E}$, for each $\alpha < \lambda$, and λ is an infinite ordinal. Proceeding inductively on the order type of λ, we may assume that $\bigcup_{\beta \le \alpha} E_\beta$ has property P for each $\alpha < \lambda$. But then $\cup\mathcal{E}$ is clearly locally-P, and thus has property P by assumption. □

Corollary 2.5. *The union of a σ-scattered collection (hence also any collection with a σ-scattered refinement) consisting of sets of the first category in X, is of the first category in X.*

Proof. This follows easily from Lemma 2.4 and the Banach Category Theorem [60, Theorem 16.1]. □

The following result is the basis for many of the properties of almost descriptive spaces.

Lemma 2.6. (a) *Every $(\mathcal{F} \cap \mathcal{G})_\sigma$ set in a topological space X is a restricted strong BP-set in X.*

(b) *The union of any collection of BP-sets (resp. strong BP-sets) in X having a σ-scattered refinement, will again be a BP-set (resp. strong BP-set) in X, and the same results hold in the restricted case.*

Proof. (a) If H is a $(\mathcal{F} \cap \mathcal{G})_\delta$ in X and $E \subset X$ is arbitrary, then $H \cap E$ is a $(\mathcal{F} \cap \mathcal{G})_\sigma$ in E, so it suffices to show that H is a strong BP-set in X. Let $H = \bigcup_{n\in\mathbb{N}} F_n \cap G_n$, where each F_n is closed and each G_n is open in X, and let

$$G = \bigcup_{n\in\mathbb{N}} (\mathrm{int} F_n) \cap G_n \quad \text{and} \quad M = H \setminus G.$$

Since $M \subset \bigcup_{n\in\mathbb{N}} (F_n \setminus \mathrm{int} F_n)$, M is of the first category in X, whence $H = G \cup M$ is a strong BP-set in X.

(b) Let $\mathfrak{C}$ be a collection of restricted BP-sets in X having a σ-scattered refinement $\mathfrak{B}$. Then, for any $E \subset X$, $\{B \cap E : B \in \mathfrak{B}\}$ is a σ-scattered refinement of $\{C \cap E : C \in \mathfrak{C}\}$ and the latter is a collection of BP-sets in E. Thus it suffices to show that $\cup\mathfrak{B} = \cup\mathfrak{C}$ is a BP-set in X. Since the BP-sets are closed to countable unions we may assume that $\mathfrak{B}$ is scattered.

By Lemma 2.3, we can find a scattered collection $\{H_B : B \in \mathfrak{B}\}$ of $\mathcal{F} \cap \mathcal{G}$ sets in X such that $B \subset H_B$ for each $B \in \mathfrak{B}$. For each $B \in \mathfrak{B}$, let $C_B \in \mathfrak{C}$ be such that $B \subset C_B$. It follows that $\mathcal{H} = \{C_B \cap H_B : B \in \mathfrak{B}\}$ is a scattered collection of BP-sets in X, and $\cup\mathfrak{B} = \cup\mathcal{H}$.

For each $B \in \mathfrak{B}$ let

$$C_B \cap H_B = (W_B \setminus P_B) \cup Q_B$$

where W_B is open in X, P_B and Q_B are of the first category in X, and $P_B \subset C_B \cap H_B$. As the union of a scattered collection of sets of the first category in X, the sets

$$P = \cup\{P_B : B \in \mathfrak{B}\} \quad \text{and} \quad Q = \cup\{Q_B : B \in \mathfrak{B}\}$$

are of the first category in X by Corollary 2.5. Letting $W = \cup\{W_B : B \in \mathfrak{B}\}$, it follows that

$$\cup\{C_B \cap H_B : B \in \mathfrak{B}\} = (W \setminus P) \cup Q,$$

since the sets H_B are disjoint. Hence $\cup\mathcal{H}$ has the Baire property in X.
The proof for the strong Baire property follows the same line of reasoning. $\square$

We need to establish certain stability properties for families of sets that are point-countable and have a σ-scattered base. It will be convenient to let $\Omega(X)$ denote all such families for a given space X.

Lemma 2.7.

(a) *If* $\{E_a : a \in A\} \in \Omega(X)$ *and* $H_a \subset E_a$ *for each* $a \in A$, *then* $\{H_a : a \in A\} \in \Omega(X)$.

(b) *If* $\{E_a : a \in A\} \in \Omega(X)$ *and* $\{H_b : b \in B\} \in \Omega(X)$, *then* $\{E_a \cap H_b : a \in A, b \in B\} \in \Omega(X)$.

(c) *If* $\{E_a : a \in A\} \in \Omega(X)$ *and* $A = \cup\{A_p : p \in P\}$ *is a partition of* A, *then* $\{H_p : p \in P\} \in \Omega(X)$ *where* $H_p = \cup\{E_p : p \in A_p\}$ *for each* $p \in P$.

(d) *If* $\{E_a : a \in A\} \in \Omega(X)$, *then we can write* $E_a = \bigcup_{n\in\mathbb{N}} E_{an}$ *so that, for each* n, $\{E_{an} : a \in A\}$ *is disjoint and has a scattered base.*

Proof. (a) It is clear that $\{H_a : a \in A\}$ is point-countable. Let $\mathfrak{B} = \bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$ be a base for $\{E_a : a \in A\}$ where each $\mathfrak{B}_n$ is a scattered collection. For each $B \in \mathfrak{B} \setminus \{\emptyset\}$ let $N_B = \{a_1, a_2, \ldots\}$ be an enumeration of $\{a \in A : B \in E_a\}$ as a finite or infinite sequence, and define

$$\mathfrak{B}_{nm} = \{B \cap H_a : B \in \mathfrak{B}_n \text{ and } a = a_m \in N_B\}.$$

Then $\mathfrak{B}_{nm}$ is a scattered collection by Lemma 2.2 (b), and it is routine to verify that $\bigcup_{n,m\in\mathbb{N}} \mathfrak{B}_{nm}$ is a base for $\{H_a : a \in A\}$.

(b) The point-countability of $\{E_a \cap H_b : a \in A, b \in B\}$ is clear, and the existence of a σ-scattered base follows from Lemma 2.2 (a).

(c) Again, it is clear that $\{H_p : p \in P\}$ is point-countable since each $a \in A$ can belong to only one A_p. Moreover, any base for $\{E_a : a \in A\}$ will also be a base for $\{H_p : p \in P\}$.

(d) Let $\mathfrak{B} = \bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$ be a base for $\{E_a : a \in A\}$ where each $\mathfrak{B}_n$ is a scattered collection. For each $B \in \mathfrak{B}$ let $N_B = \{a_1, a_2, \ldots\}$ be an enumeration of $\{a \in A : B \subset E_a\}$ as a finite or infinite sequence, and define

$$E_{anm} = \cup\{B : B \in \mathfrak{B}_n \ \text{ and } \ a = a_m \in N_B\}.$$

Then $E_a = \bigcup_{n,m\in\mathbb{N}} E_{anm}$ for each $a \in A$, by the property of a base, and $\{E_{anm} : a \in A\}$ is disjoint and has a subcollection of $\mathfrak{B}_n$ as a base. $\square$

It will be convenient to have names for the maps used in the definitions of a descriptive and almost descriptive Hausdorff space. If P denotes a property of disjoint families of sets in a topological space, we say that a map (or set-valued map) $f : X \to Y$ is *index-σ-P* if, whenever $\{E_a : a \in A\}$ is a P family in X, then $\{f(E_a) : a \in A\}$ is (index) point-countable and has a σ-P base. Thus a Hausdorff space is descriptive (resp. almost descriptive) if, and only if, it is a continuous index-σ-relatively discrete (resp. index-σ-scattered) image of a complete metric space. For the case of descriptive spaces, this utilizes the fact that a family is σ-relatively discretely decomposable if, and only if, it is point-countable and has a σ-relatively discrete base [52, Lemma 3.2]. (Example 2.9 below shows that this relationship doesn't hold for the case of scattered families.) Note that when the domain space is metrizable, a map will take scattered families to point-countable families (always taken with the same indexing set) if, and only if, all point-inverses are separable, since scattered collections in separable metric spaces are at most countable.

Our final lemma of this section deals with the stability of scattered collections relative to taking inverse images under continuous maps, and to the stability of index-σ-scattered maps under compositions.

Lemma 2.8. (a) *If $f : X \to Y$ is a continuous map and $\mathcal{E} \in \Sigma(Y)$, then $f^{-1}(\mathcal{E}) \in \Sigma(X)$.*
(b) *The composition of two index-σ-scattered maps is also index-σ-scattered.*

Proof. (a) Let $\mathcal{E} = \{E_\alpha : \alpha < \lambda\}$ and let $\{U_\alpha : \alpha < \lambda\}$ be a collection of open sets in Y such that

$$E_\alpha \subset U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta$$

for each $\alpha < \lambda$. But then

$$f^{-1}(E_\alpha) \subset f^{-1}(U_\alpha) \setminus \bigcup_{\beta<\alpha} f^{-1}(U_\beta),$$

for each $\alpha < \lambda$, proving that $f^{-1}(\mathcal{E}) \in \Sigma(X)$, since the sets $f^{-1}(U_\alpha)$ are open in X.

(b) Let $f : X \to Y$ and $g : Y \to Z$ be index-σ-scattered, and suppose $\mathcal{E} = \{E_a : a \in A\}$ is a scattered collection of subsets of X. Then $f(\mathcal{E})$ has a σ-scattered base $\mathfrak{B}$, and $g(\mathfrak{B})$ has a σ-scattered base $\mathfrak{C}$. Since it is easy to see that $\mathfrak{C}$ will also be a base for $g \circ f(\mathcal{E})$, it remains only to show that the family

$$\{g \circ f(E_a) : a \in A\}$$

is point-countable. Suppose, on the contrary, there is some $z \in Z$ such that the set

$$I = \{a \in A : z \in g \circ f(E_a)\}$$

is uncountable. For each $a \in I$, choose $x_a \in X$ and $B_a \in \mathfrak{B}$ so that

$$x_a \in E_a \cap f^{-1}(g^{-1}(z)) \quad \text{and} \quad f(x_a) \in B_a \subset f(E_a).$$

Since $\{f(E_a) : a \in A\}$ is point-countable, we must have

$$\{B \in \mathfrak{B} : B = B_a \ \text{ for some } \ a \in I\}$$

uncountable. But this implies that $z = g(f(x_a)) \in g(B)$ for uncountably many $B \in \mathfrak{B}$, contradicting the point-countability of $\{g(B) : B \in \mathfrak{B}\}$. $\square$

We conclude this section with an example which shows that a point-countable (in fact, a disjoint) family with a σ-scattered base (or even a scattered base) need not be σ-scattered decomposable. The problem is related to the simple observation that, although the intervals $(1,2)$, $[2,3)$ and $[3,4)$ form a scattered partition of $(1,4) \subset \mathbb{R}$, the sets $[2,3)$ and $(1,2) \cup [3,4)$ do not.

Example 2.9. Let $X = [0, \omega_1)$ be the space of all countable ordinals with the order topology. Since X is a scattered space, every collection of subsets of X has a scattered base consisting of singletons. Let

$$\mathcal{E} = \{[\alpha, \omega_1) : \alpha < \omega_1\}.$$

The family $\mathcal{E}$ is clearly point-countable and so it suffices to show that $\mathcal{E}$ is not σ-scattered decomposable. Suppose, on the contrary, that $\mathcal{E}$ is σ-scattered decomposable, say

$$[\alpha, \omega_1) = \bigcup_{n \in \mathbb{N}} E_{\alpha n},$$

where $\{E_{\alpha n} : \alpha < \omega_1\}$ is a scattered collection for each $n \in \mathbb{N}$. By Lemma 2.3, we can find a scattered collection $\{H_{\alpha n} : \alpha < \omega_1\}$ of $\mathcal{F} \cap \mathcal{G}$ sets in X such that

$$E_{\alpha n} \subset H_{\alpha n} \subset [\alpha, \omega_1)$$

for each $\alpha < \omega_1$ and for each $n \in \mathbb{N}$.

We will make use of the following two facts from [25, p. 231, Ex. 10]:

(i) The class of all closed and unbounded (abbrev. c.u.b.) subsets of X is closed under countable intersections.

(ii) Each Borel set $B \subset X$ is such that either B or $X \setminus B$ contains a c.u.b. of X (but not both, of course).

Since $[\alpha, \omega_1)$ is a c.u.b. in X and is the union of the sequence of Borel sets $H_{\alpha n}$, it follows from (i) and (ii) that $H_{\alpha n}$ contains a c.u.b. for at least one $n \in \mathbb{N}$. But since $\mathcal{E}$ is uncountable, this implies that for some n we have disjoint sets $H_{\alpha n}$ and $H_{\beta n}$ with each containing a c.u.b. which is impossible (again by (i)). This shows that $\mathcal{E}$ is not σ-scattered decomposable.

To see that this implies the existence of a *disjoint* collection with the same properties, for each $\beta < \omega_1$ let $\{\beta_1, \beta_2, \ldots\}$ be an enumeration of the set $\{\alpha : \beta \in [\alpha, \omega_1)\}$ as a finite or infinite sequence. Defining

$$E_{\alpha n} = \{\beta \in [\alpha, \omega_1) : \alpha = \beta_n\},$$

we have $[\alpha, \omega_1) = \bigcup_{n \in \mathbb{N}} E_{\alpha n}$ for each $\alpha < \omega_1$, and $\{E_{\alpha n} : \alpha < \omega_1\}$ is a disjoint collection for each $n \in \mathbb{N}$. It follows that, for at least one $n \in \mathbb{N}$, $\{E_{\alpha n} : \alpha < \omega_1\}$ is not σ-scattered decomposable, for otherwise $\mathcal{E}$ would be.

3. Relatively discrete collections and Borel and Souslin sets. In this section we present the basic properties of relatively discrete collections leading up to the principal result that certain classes of Borel sets, as well as all Souslin sets, are stable under unions of σ-relatively discretely refinable collections.

Lemma 3.1. *Let $\Delta^\rho(X)$ denote the family of all relatively discrete collections of subsets of a space X. Then $\Delta^\rho(X)$ has the following properties.*

(a) *If $\mathcal{E}$, $\mathcal{H} \in \Delta^\rho(X)$, then $\{E \cap H : E \in \mathcal{E}, H \in \mathcal{H}\} \in \Delta^\rho(X)$.*
(b) *If $\mathcal{E} \in \Delta^\rho(X)$ and $H_E \subset E$ for each $E \in \mathcal{E}$, then $\{H_E : E \in \mathcal{E}\} \in \Delta^\rho(X)$.*
(c) *If $\mathcal{E} \in \Delta^\rho(X)$ and, for each $E \in \mathcal{E}$, $E = \cup \mathcal{H}_E$ with $\mathcal{H}_E \in \Delta^\rho(X)$, then $\{H : H \in \mathcal{H}_E, E \in \mathcal{E}\} \in \Delta^\rho(X)$.*
(d) *If X is a subspace of Y, then $\Delta^\rho(X) \subset \Delta^\rho(Y)$ and $\mathcal{E} \in \Delta^\rho(Y)$ implies $\{E \cap X : E \in \mathcal{E}\} \in \Delta^\rho(X)$.*

Proof. All of these properties are routine consequences of the definitions. □

Lemma 3.2. *If $\mathcal{E}$ is a relatively discrete collection of subsets of a space X, then there is a relatively discrete collection $\{H_E : E \in \mathcal{E}\}$ of $\mathcal{F} \cap \mathcal{G}$ sets in X such that $E \subset H_E$ for each $E \in \mathcal{E}$.*

Proof. Since $\mathcal{E}$ is relatively discrete, for each $E \in \mathcal{E}$ we can choose an open set U_E in X such that $E \subset U_E$ and $E' \cap U_E = \emptyset$ for all $E' \neq E$ in $\mathcal{E}$. It now suffices to take $H_E = \overline{E} \cap U_E$, where $\overline{E}$ denotes the closure of E in X, as one easily verifies. $\square$

For a general topological space X and for any countable ordinal $\alpha < \omega_1$, we define the Borel classes $\mathfrak{B}_\alpha$ as follows: $\mathfrak{B}_0 = \mathcal{F} \cap \mathcal{G}$, and if $\alpha = \beta + 1$, then $\mathfrak{B}_\alpha$ is the class $(\mathfrak{B}_\beta)_\sigma$ of all countable unions of sets in $\mathfrak{B}_\beta$, if α is odd, or $\mathfrak{B}_\alpha$ is the class $(\mathfrak{B}_\beta)_\delta$ of all countable intersections of sets in $\mathfrak{B}_\beta$, if α is even; if α is a limit ordinal, then we define $\mathfrak{B}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{B}_\beta$. It will also be convenient to define $\mathfrak{C}_\alpha = \{X \setminus B : B \in \mathfrak{B}_\alpha\}$, for each $\alpha < \omega_1$, so that $\mathfrak{C}_0 = \mathcal{F} \cup \mathcal{G}$, $\mathfrak{C}_1 = (\mathcal{F} \cup \mathcal{G})_\delta$, etc. It is clear that $\mathrm{Borel}(X) = \bigcup_{\alpha<\omega_1} \mathfrak{B}_\alpha = \bigcup_{\alpha<\omega_1} \mathfrak{C}_\alpha$.

Lemma 3.3. *Let $\{E_a : a \in A\}$ be a relatively discrete collection of subsets of a space X, and let E denote its union.*

(a) *If each $E_a \in \mathfrak{B}_\alpha$ (resp. $\mathfrak{C}_\alpha$) for some $\alpha < \omega_1$ (resp. $1 \le \alpha < \omega_1$), then $E \in \mathfrak{B}_\alpha$ (resp. $\mathfrak{C}_\alpha$).*

(b) *If each E_a is a Souslin set in X, then so is E.*

Proof. (a) Let $\{F_a \cap G_a : a \in A\}$ be a relatively discrete collection of $\mathcal{F} \cap \mathcal{G}$ sets in X, where F_a is closed and G_a is open in X. For each $a \in A$, we can choose an open set $U_a \subset G_a$ such that $F_a \cap G_a \subset U_a$ and $\overline{F_a \cap U_a} \cap U_b = \emptyset$ for all $b \neq a$. Since the sets $F_a \cap G_a$ form a discrete collection relative to the subspace $G = \bigcup_{b\in A} U_b$, the union of their closures in G will be a closed set in G, and so there is a closed set F in X such that

$$F \cap G = \bigcup_{a\in A} \overline{F_a \cap G_a} \cap G.$$

Clearly, $F \cap G$ contains $\bigcup_{a\in A} F_a \cap G_a$. Conversely, if $x \in F \cap G$, then $x \in \overline{F_a \cap G_a}$ and $x \in U_b$ for some $a, b \in A$. By the definition of U_b we must have $a = b$, and so

$$x \in \overline{F_a \cap G_a} \cap U_a \subset \overline{F_a} \cap G_a = F_a \cap G_a.$$

This proves the lemma for the class $\mathfrak{B}_0$. The proof for the class $\mathfrak{B}_1 = (\mathcal{F} \cap \mathcal{G})_\sigma$ follows easily from this. To prove it for the class $\mathfrak{C}_1$ of $(\mathcal{F} \cap \mathcal{G})_\delta$ sets, we first use Lemma 3.2 to expand $\{E_a : a \in A\}$ to a relatively discrete family $\{H_a : a \in A\}$ of $\mathcal{F} \cap \mathcal{G}$ sets in X. Since $\{H_a \setminus E_a : a \in A\}$ is then a relatively discrete collection of sets in $\mathfrak{B}_1$, and $H = \bigcup\{H_a : a \in A\} \in \mathfrak{B}_0 \subset \mathfrak{C}_1$, it follows that

$$\bigcup_{a\in A} E_a = H \setminus \bigcup_{a\in A} (H_a \setminus E_a) \in \mathfrak{C}_1.$$

Suppose the lemma is true for all classes $< \lambda$ and let $\{E_a : a \in A\}$ be a relatively discrete collection of sets in $\mathfrak{B}_\lambda$. Suppose $\lambda = \beta + 1$ and that λ is even, so that $E_a = \bigcap_{n\in\mathbb{N}} E_{an}$, where $E_{an} \in \mathfrak{B}_\beta$ for each a. By Lemma 3.2, there is a relatively discrete collection $\{H_a : a \in A\}$ of $\mathcal{F} \cap \mathcal{G}$ sets in X with $E_a \subset H_a$ for each $a \in A$. It follows from our inductive hypothesis that $\cup\{H_a \cap E_{an} : a \in A\}$ belongs to $\mathfrak{B}_\beta$ for each n. Since the sets H_a are also disjoint we have

$$E = \bigcup_{a\in A} E_a = \bigcup_{a\in A} \bigcap_{n=1}^{\infty} H_a \cap E_{an} = \bigcap_{n=1}^{\infty} \bigcup_{a\in A} H_a \cap E_{an},$$

hence E belongs to $\mathfrak{B}_\lambda$. The proofs of the other cases are similar to the case $\lambda = 1$ treated above.

(b) Now suppose each set E_a is a Souslin set in X, and let $\{H_a : a \in A\}$ be as in part (a). For each $a \in A$, let

$$E_a = \cup \left\{ \bigcap_{n=1}^{\infty} E^a_{t|n} : t \in \mathbb{N}^{\mathbb{N}} \right\},$$

where $E^a_{t|n}$ is either a closed or open set in X. For each $t|n$, $\{E^a_{t|n} \cap H_a : a \in A\}$ is a relatively discrete collection of $\mathcal{F} \cap \mathcal{G}$ sets in X. In particular, since these sets are disjoint, we have

$$E = \bigcup_{a\in A} \bigcup_{t\in\mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} E^a_{t|n} \cap H_a = \bigcup_{t\in\mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} \bigcup_{a\in A} E^a_{t|n} \cap H_a,$$

and it follows from this and part (a) that E is a Souslin set in X. $\square$

Lemma 3.4. *Let $\mathcal{E}$ be a family of subsets of a space X, and suppose $\mathcal{E}$ has a σ-relatively discretely refinement. If each member of $\mathcal{E}$ is a $(\mathcal{F} \cap \mathcal{G})_\sigma$ (resp. Souslin) set in X, then so is $\cup\mathcal{E}$. Moreover, $(\mathcal{F} \cap \mathcal{G})_\sigma$ can be replaced by any Borel class which is closed to countable unions.*

P r o o f. Let $\mathcal{E}$ be a family of $(\mathcal{F} \cap \mathcal{G})_\sigma$ sets in X, and let $\mathcal{R} = \bigcup_{n\in\mathbb{N}} \mathcal{R}_n$ be a refinement of $\mathcal{E}$ such that each collection $\mathcal{R}_n$ is relatively discrete. By Lemma 3.2, for each $n \in \mathbb{N}$ there exists a relatively discrete families $\{H_R : R \in \mathcal{R}_n\}$ of $\mathcal{F} \cap \mathcal{G}$ sets in X such that $R \subset H_R$ for each R. For each $R \in \mathcal{R}$ we fix some $E_R \in \mathcal{E}$ such that $R \subset E_R$. Then

$$\mathcal{H}_n = \{E_R \cap H_R : R \in \mathcal{R}_n\}$$

is a relatively discrete family of $(\mathcal{F} \cap \mathcal{G})_\sigma$ sets in X, and

$$\cup\mathcal{E} = \cup\{\cup\mathcal{H}_n : n \in \mathbb{N}\}.$$

It follows from part (a) of Lemma 3.3 that $\cup\mathcal{E}$ is a $(\mathcal{F}\cup\mathcal{G})_\sigma$ sets in X.

The proof for any countably additive Borel class and for Souslin sets is exactly the same in view of the other parts of Lemma 3.3. $\square$

The union of a relatively discrete collection of Borel sets in a space X is not necessarily a Borel set in X, even if X is a locally separable metric space. For if $X = \Sigma\{X_\alpha : \alpha < \omega_1\}$ is the (discrete) sum of ω_1 copies of the space of real numbers and $B_\alpha \subset X_\alpha$ is a Borel set of class not less than α, then $\cup\{B_\alpha : \alpha < \omega_1\}$ is not a Borel set in X (for it has no class).

One of the first major results in the nonseparable topological theory of descriptive sets was Montgomery's Lemma [54]: If $\{U_\alpha : \alpha < \lambda\}$ is a transfinite sequence of open sets in an arbitrary metric space X and E_α is an $\mathcal{F}_\sigma$ (resp. a $\mathcal{G}_\delta$) set in X for each $\alpha < \lambda$, then

$$\bigcup_{\alpha<\lambda} E_\alpha \cap \left(U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta\right)$$

is an $\mathcal{F}_\sigma$ (resp. a $\mathcal{G}_\delta$) set in X. The result also holds for higher Borel classes. (For applications of Montgomery's Lemma see [46, §30, X], [60, Chapt 16], and [37].) Here we give the following general version of this useful result.

Lemma 3.5. *Let X be a hereditarily weakly θ-refinable space and let $\{U_\alpha : \alpha < \lambda\}$ be any transfinite sequence of open sets in X. If $D_\alpha = U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta$ for each $\alpha < \lambda$, then $\{D_\alpha : \alpha < \lambda\}$ is σ-relatively discretely decomposable. Moreover, if $\mathfrak{B}$ denotes any one of the Borel classes $\mathfrak{B}_\gamma$ or $\mathfrak{C}_\gamma$ with $\gamma \geq 1$, and $\{H_\alpha : \alpha < \lambda\} \subset \mathfrak{B}$, then $\cup\{H_\alpha \cap D_\alpha : \alpha < \lambda\}$ belongs to $\mathfrak{B}$.*

Proof. It is quite routine to verify that if $\mathcal{E}$ is any σ-relatively discrete cover of $\bigcup_{\alpha<\lambda} D_\alpha$ such that $\{E \cap D_\alpha : \alpha < \lambda\}$ is σ-relatively discretely decomposable for each $E \in \mathcal{E}$, then $\{D_\alpha : \alpha < \lambda\}$ is σ-relatively discretely decomposable (see [26, p. 151] for the proof when the word "relatively" is omitted). Consequently, since the assumptions on X imply that every collection of open sets in X has a σ-relatively discrete refinement, it follows that any collection which is *locally* σ-relatively discretely decomposable in X will be σ-relatively discretely decomposable.

Proceeding to prove the lemma by induction on λ, assume that for each $\gamma < \lambda$ the collection $\{D_\alpha : \alpha < \gamma\}$ is σ-relatively discretely decomposable. We may also assume that λ is a limit ordinal. But this implies that $\{D_\alpha : \alpha < \lambda\}$ is locally σ-relatively discretely decomposable, since each point in the union of this family has some U_α as a neighborhood meeting at most sets with a smaller index.

Since $\{D_\alpha : \alpha < \gamma\}$ is σ-relatively discretely decomposable, the second part of the lemma has already been established for Borel classes which are closed to countable unions (see Lemma 3.4). Using this we can obtain the result for countably multiplicative classes via complementation, as we now illustrate in the case of $(\mathcal{F} \cup \mathcal{G})_\delta$ sets. If each H_α is a $(\mathcal{F} \cup \mathcal{G})_\delta$ set in X, then each complement H_α^c is a $(\mathcal{F} \cap \mathcal{G})_\sigma$ set in X and it follows that

$$\bigcup_{\alpha<\lambda} H_\alpha \cap D_\alpha = \bigcup_{\alpha<\lambda} U_\alpha \setminus \Big(\bigcup_{\alpha<\lambda} H_\alpha^c \cap D_\alpha \Big)$$

is a $(\mathcal{F} \cup \mathcal{G})_\delta$ set in X, since the set in parenthesis is a $(\mathcal{F} \cap \mathcal{G})_\sigma$ set in X by Lemma 3.4. $\square$

As a corollary to Lemma 3.5 we have the following basic relationship between relatively discrete and scattered collections.

Lemma 3.6. *A topological space X is hereditarily weakly θ-refinable if, and only if, each scattered collection in X is σ-relatively discretely decomposable.*

Proof. If X is hereditarily weakly θ-refinable and $\{H_\alpha : \alpha < \lambda\}$ is a scattered collection in X, then there is an open collection $\{U_\alpha : \alpha < \lambda\}$ in X such that $H_\alpha \subset U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta$. But the latter sets form a σ-relatively discretely decomposable collection by Lemma 3.5, hence $\{H_\alpha : \alpha < \lambda\}$ also has this property.

Conversely, suppose each scattered collection in X is σ-relatively discretely decomposable. If $\{U_\alpha : \alpha < \lambda\}$ is any collection of open sets in X, then $\Big\{U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta : \alpha < \lambda\Big\}$ is scattered, hence σ-relatively discretely decomposable. This easily implies that $\{U_\alpha : \sigma < \lambda\}$ has a σ-relatively discrete refinement. $\square$

A space X is *σ-relatively discrete* if it is a countable union of subsets each of which is discrete in its relative topology (equivalently, $\{\{x\} : x \in X\}$ is a σ-relatively discrete collection; cf. the notion of a scattered space). Lemma 3.6 immediately implies that in any hereditarily weakly θ-refinable space, a set is σ-scattered if, and only if, it is σ-relatively discrete. This also implies the following.

Corollary 3.7 [58, Theorem 3.4]**.** *A scattered space is hereditarily weakly θ-refinable if, and only if, it is σ-relatively discrete.*

We close this section with two lemmas on index-σ-relatively discrete maps. Since the proof of Lemma 3.8 differs in no essential way from that given for index-σ-scattered maps (Lemma 2.8), we omit it.

Lemma 3.8. (a) *If $f : X \to Y$ is a continuous map and $\mathcal{E} \in \Delta^\rho(Y)$, then $f^{-1}(\mathcal{E}) \in \Delta^\rho(X)$.*
(b) *The composition of two index-σ-relatively discrete maps is also index-σ-relatively discrete.*

Lemma 3.9. *Let $f : X \to Y$ be a map (or a set-valued map) between topological spaces. For the following conditions we have* (a) $\iff$ (b) $\Rightarrow$ (c)*, and all three properties are equivalent if X is metrizable.*

(a) *f is index-σ-relatively discrete.*
(b) *$\{f(E_a) : a \in A\}$ is a σ-relatively discretely decomposable family in Y, whenever $\{E_a : a \in A\}$ is a relatively discrete family in X.*
(c) *$\{f(E_a) : a \in A\}$ is a σ-relatively-discretely decomposable family in Y, whenever $\{E_a : a \in A\}$ is a discrete family in X.*

P r o o f. The proof of (a) $\iff$ (b) is essentially known, and not difficult to prove (see [52, Lemma 3.2]). The implication (b) $\Rightarrow$ (c) is immediate. Finally, if X is metrizable, then each relatively discrete family in X is σ-discretely decomposable by [26, §1.2], and it follows that (c) $\Rightarrow$ (b). □

4. Embedding properties of K-descriptive and almost K-descriptive spaces. Recall that a topological space X is a *Lindelöf* space if each open cover of X has a countable subcover (equivalently, a countable refinement), and X is *subparacompact* if each open cover has a σ-discrete refinement. Thus, Lindelöf implies subparacompact, and subparacompact implies weakly θ-refinable. It is known that each K-analytic space is a Lindelöf space and will be a Souslin($\mathcal{F}$) set in every Hausdorff embedding [64, 2.5.2]. Similarly, if X is the image of a complete metric space T (not necessarily separable) under an upper semi-continuous compact-valued map which is index-σ-discrete (this is always satisfied when T is separable), then it is known that X is a subparacompact space and is a Souslin($\mathcal{F}$) $\cap\, \mathcal{G}_\delta$ set in every Hausdorff embedding [35, Theorem 12]. In this section we prove analogous results for K-descriptive and almost K-descriptive spaces.

Theorem 4.1. *Every K-descriptive Hausdorf space X is weakly θ-refinable and is a Souslin set in any Hausdorff topological embedding. Consequently, X is Čech analytic whenever it is completely regular.*

Every almost K-descriptive space has the Baire property in the restricted sense in any Hausdorff topological embedding. Consequently, X will contain a *dense $\mathcal{G}_\delta$ Čech complete subspace whenever it is a completely regular Baire space.*

Proof. We will give the proof for K-descriptive spaces and cover the almost K-descriptive case by appending statements in brackets. Let Ψ be an upper semi-continuous compact-valued map from a complete metric space T to X such that $\Psi(T) = X$ and Ψ maps discrete (hence also σ-discrete) families in T to point-countable families having a σ-relatively discrete base [respectively, to point-countable families having a σ-scattered base].

Let $\mathcal{V}$ be an open cover of X, and let $\mathcal{W}$ denote the open cover obtained by taking all finite unions of sets from $\mathcal{V}$. Then $\{\Psi^{-1}(W) : W \in \mathcal{W}\}$ is an open cover of T, where

$$\Psi^{-1}(W) = \{t \in T : \Psi(t) \subset W\}.$$

Since T is paracompact, this cover has a σ-discrete refinement, say $\mathcal{E}$. It follows that $\{\Psi(E) : E \in \mathcal{E}\}$ has a σ-relatively discrete [resp. σ-scattered] base $\mathcal{D}$, and it is easy to see that $\mathcal{D}$ is also a refinement of $\mathcal{W}$. For each $D \in \mathcal{D}$ there is a finite collection $\mathcal{V}_D \subset \mathcal{V}$ such that $D \subset \cup\mathcal{V}_D$. It follows that $\{\mathcal{D} \cap V : V \in \mathcal{V}_D, D \in \mathcal{D}\}$ is a σ-relatively discrete refinement of $\mathcal{V}$, proving that X is weakly θ-refinable.

Now suppose X is a subspace of the Hausdorff space Y. Let $\mathcal{U}$ be a σ-discrete (open) base for the complete metric space T and put

$$\mathcal{U}_n = \{U \in \mathcal{U} : \operatorname{diam} U < 1/n\},$$

for each $n \in \mathbb{N}$. Since $\mathcal{U}_n$ is σ-discrete in T, we can write, for each $U \in \mathcal{U}_n$,

$$\Psi(U) = \bigcup_{m=1}^{\infty} W_{U_m},$$

such that $\{W_{U_m} : U \in \mathcal{U}_n\}$ is disjoint and has a relatively discrete [resp. scattered] base $\mathfrak{B}_m$ for each $m \in \mathbb{N}$ [see Lemmas 2.7 (c) and 3.9]. By Lemma 3.1 [resp. Lemma 2.3] we can find relatively discrete [resp. scattered] collections $\{H_B : B \in \mathfrak{B}_m\}$ of $\mathcal{F} \cap \mathcal{G}$ sets in Y such that $B \subset H_B \subset cl_Y B$, for each $B \in \mathfrak{B}_m$ and $m \in \mathbb{N}$. For each $U \in \mathcal{U}_n$, define

$$H_{U_m} = \cup\{H_B : B \in \mathfrak{B}_m \text{ and } B \subset W_{U_m}\},$$

and note that

$$W_{U_m} \subset H_{U_m} \subset cl_Y W_{U_m},$$

and $\{H_{U_m} : U \in \mathcal{U}_n\}$ is disjoint and has $\{H_B : B \in \mathfrak{B}_m\}$ as a relatively discrete [resp. scattered] base of $\mathcal{F} \cap \mathcal{G}$ sets in Y.

For each finite sequence $m_1, \ldots, m_n \in \mathbb{N}$ we define

$$(*)\quad H_{m_1\ldots m_n} = \cup \left\{ \bigcap_{p=1}^{n} H_{U_p m_p} : \exists U_p \in \mathcal{U}_p, p \leq n \text{ and } cl_T U_p \subset U_{p+1}, p \leq n-1 \right\}.$$

Then each $H_{m_1\ldots m_n}$, as the union of sets having a relatively discrete [resp. scattered] base of $\mathcal{F} \cap \mathcal{G}$ sets in Y, is a $\mathcal{F} \cap \mathcal{G}$ set [resp. a set having the restricted Baire property] in Y by Lemma 3.3 [resp. Lemma 2.6]. Hence, it suffices to show that

$$(**)\qquad X = \cup \left\{ \bigcap_{n=1}^{\infty} H_{m_1\ldots m_n} : (m_n) \in \mathbb{N}^{\mathbb{N}} \right\}$$

[since sets having the restricted Baire property are closed with respect to the Souslin operation [46, p. 95]].

That X is contained in the set on the right is clear from the fact that $\bigcup_{n=1}^{\infty} \mathcal{U}_n$ is a base for T. Thus suppose $y \in \bigcap_{n=1}^{\infty} H_{m_1\ldots m_n}$. We define inductively (nonempty) sets $U_n \in \mathcal{U}_n$ such that $y \in H_{U_n m_n}$ and $cl_T U_n \subset U_{n+1}$ for each $n \in \mathbb{N}$. Since $y \in H_{m_1}$, we have $y \in H_{U_1 m_1}$ for some $U_1 \in \mathcal{U}_1$ by $(*)$. Suppose we have found $U_1, \ldots, U_{n-1}$ with the above properties. Since $y \in H_{m_1\ldots m_n}$, by $(*)$ there exist $V_p \in \mathcal{U}_p$, for $p \leq n$, with $cl_T V_p \subset V_{p+1}$, for $p \leq n-1$, such that

$$y \in \bigcap_{p=1}^{n} H_{V_p m_p}.$$

Now, for each $p < n$, $\{H_{U m_p} : U \in \mathcal{U}_p\}$ is a disjoint family and y belongs to both $H_{U_p m_p}$ and $H_{V_p m_p}$. It follows that $U_p = V_p$ for all $p < n$, and so we can define $U_n = V_n$.

By the definition of $\mathcal{U}_n$ and since T is complete, there is a $t \in T$ such that $t \in U_n$ for every $n \in \mathbb{N}$. The proof will be complete if we can show that $y \in \Psi(t)$. If $y \notin \Psi(t)$, then, since $\Psi(t)$ is compact and Y is Hausdorff, there exist disjoint open sets V and W in Y such that $y \in V$ and $\Psi(t) \subset W$. Since $y \in H_{U_n m_n} \subset cl_Y W_{U_n m_n}$, V must intersect $W_{U_n m_n}$ for each n, hence there exist $t_n \in U_n$ and $y_n \in \Psi(t_n) \cap V$. Now $t_n \to t$ in T (since $\mathrm{diam} U_n \to 0$), so by the upper semi-continuity of Ψ the sequence y_n must cluster in $\Psi(t)$. But this implies that $cl_Y V$ intersects $\Psi(t)$ which contradicts our choice of V. This proves $(**)$.

It follows that, if X is a completely regular K-descriptive space, then X is a Souslin set in some compact Hausdorff space and hence is Čech analytic. On the other hand, if X is completely regular and is almost K-descriptive, then, as a subspace of its Stone-Čech compactification, X has the form $G \cup M$ where G is a $\mathcal{G}_\delta$ set and M is a set of the first category in βX. Since X is dense in βX,

M is also of the first category in X. Consequently, if X is a Baire space, then G must be a dense Čech complete subspace of X. $\square$

5. Topological properties of general descriptive and almost descriptive spaces. The aim of this section is not to attempt to give a complete and systematic study of the properties of descriptive and almost descriptive Hausdorff spaces. Here we concentrate instead on those results most relevant to the present work, and illustrate in subsequent sections their application in the study of the topological properties of Banach spaces. As we illustrate below, a number of the results of the theory of nonseparable absolutely analytic metric spaces, and the recent generalization of these to non-metrizable spaces (see, for example, [35, 36]), carry over to descriptive or even K-descriptive spaces.

Theorem 5.1. *Every descriptive Hausdorff space X has a σ-relatively discrete network of Souslin sets (in fact, $\mathcal{F} \cap \mathcal{G}$ sets when X is regular), and is a Souslin set in every Hausdorff topological embedding. Consequently, X will be a Čech analytic space whenever it is completely regular.*

Every almost descriptive Hausdorff space X has a σ-scattered network of sets having the restricted Baire property (in fact, $\mathcal{F} \cap \mathcal{G}$ sets when X is regular), and X will have the restricted Baire property in every Hausdorff topological embedding. Consequently, X will contain a dense $\mathcal{G}_\delta$ Čech complete subspace whenever it is a completely regular Baire space.

P r o o f. Since any descriptive space is K-descriptive, the embedding property and the stated consequence follow from Theorem 4.1, and similarly for almost descriptive spaces.

Suppose X is a descriptive space, and let us show that it has a network of the type described. Let $f : T \to X$ be a continuous, index-σ-relatively discrete map from a complete metric space T onto X, and let $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ be a network for T where each $\mathcal{F}_n$ is a discrete family of closed sets in T. Since each $F \in \mathcal{F}$ is a complete metric space, $f(F)$ is a Souslin set in X by Theorem 4.1. For each $n \in \mathbb{N}$, let $\bigcup_{m\in\mathbb{N}} \mathfrak{B}_{nm}$ be a σ-relatively discrete base for $f(\mathcal{F}_n)$, and let $\{H_B : B \in \mathfrak{B}_{nm}\}$ be a relatively discrete family of $\mathcal{F}\cap\mathcal{G}$ sets in X expanding $\mathfrak{B}_{nm}$, which is possible by Lemma 3.2. For each nonempty $B \in \bigcup_{m\in\mathbb{N}} \mathfrak{B}_{nm}$ the set

$$S_B = \cap\{f(F) : F \in \mathcal{F}_n \text{ and } B \subset f(F)\}$$

will be a Souslin set in X by the point-countability of the family $\{f(F) : F \in \mathcal{F}_n\}$. It follows that

$$\mathfrak{N}_{nm} = \{H_B \cap S_B : B \in \mathfrak{B}_{nm}\}$$

is a relatively discrete family of Souslin sets in X for each $n, m \in \mathbb{N}$, and it suffices to show that $\bigcup_{n,m\in\mathbb{N}} \mathfrak{N}_{nm}$ is a network for X.

If $x \in V$ for some open $V \subset X$, then for any $t \in f^{-1}(x) \subset f^{-1}(V)$ we can find $n \in \mathbb{N}$ and $F \in \mathcal{F}_n$ such that $t \in F \subset f^{-1}(V)$, by the property of a network. Since $x \in f(F) \subset V$, for some $m \in \mathbb{N}$ and $B \in \mathfrak{B}_{nm}$, we have $x \in B \subset f(F)$, by the definition of a base. It follows that,

$$x \in H_B \cap S_B \subset f(F) \subset V,$$

as required. If X is regular, it is routine to verify that the family

$$\{H_B \cap cl_X B : B \in \mathfrak{B}_{nm} \text{ and } n, m \in \mathbb{N}\}$$

is a σ-relatively discrete network of $\mathcal{F} \cap \mathcal{G}$ sets in X. The proof for almost descriptive spaces is identical except we replace "relatively discrete" by "scattered". □

Theorem 5.2. *A Hausdorff space is descriptive if, and only if, it is almost descriptive and hereditarily weakly θ-refinable. Any hereditarily weakly θ-refinable almost K-descriptive space is K-descriptive.*

Proof. Since any space with a σ-relatively discrete network is clearly hereditarily weakly θ-refinable, the implication from left to right in the first statement follows from Theorem 5.1 and the fact that each relatively discrete collection is scattered (Lemma 2.2). The reverse implication follows from Lemma 3.6. The second statement also follows directly from Lemma 3.6. □

The following result gives the basic stability properties of descriptive and almost descriptive spaces.

Theorem 5.3. *Any index-σ-relatively discrete, continuous Hausdorff image of a descriptive space is descriptive. Any Souslin subset of a descriptive space, and any countable product of descriptive spaces, is descriptive. The same properties hold for almost descriptive spaces except that we may weaken index-σ-relatively discrete to index-σ-scattered.*

We first prove a lemma which gives a single criterion for a map to be index-σ-relatively discrete (resp. index-σ-scattered).

Lemma. *Given $f : T \to X$, suppose T has a network $\mathfrak{N}$ such that the image $\{f(N) : N \in \mathfrak{N}\}$ is point-countable and has a σ-relatively discrete (resp. σ-scattered) base. Then f is index-σ-relatively discrete (resp. index-σ-scattered).*

Proof. Let P stand for the property of a collection being relatively discrete or scattered, and let $\{E_\alpha : \alpha < \lambda\}$ be a P family of subsets of T. Choose open sets $U_\alpha \subset X$ such that

$$U_\alpha \supset E_\alpha \quad \text{and} \quad U_\alpha \cap E_\beta = \emptyset \quad \text{for} \quad \beta \neq \alpha,$$

in the case when P means relatively discrete, or

$$U_\alpha \cap \Big(\bigcup_{\beta<\lambda} E_\beta \Big) = \bigcup_{\beta\le\alpha} E_\beta,$$

if P means scattered. We first show that $\{f(E_\alpha) : \alpha < \lambda\}$ is point-countable.

Suppose, on the contrary, for some $x \in X$ we have

$$t_\alpha \in E_\alpha \cap f^{-1}(x)$$

for uncountably many $\alpha < \lambda$. By the property of a network, there exist sets $N_\alpha \in \mathfrak{N}$ such that

$$t_\alpha \in N_\alpha \cap E_\alpha \subset U_\alpha.$$

By the definition of U_α we have

$$U_\beta \cap (N_\alpha \cap E_\alpha) = \emptyset \text{ whenever } \beta < \alpha,$$

hence $N_\alpha \neq N_\beta$ whenever $\alpha \neq \beta$. But this implies that $x \in f(N)$ for uncountably many $N \in \mathfrak{N}$, contradicting the point-countability of the family $f(\mathfrak{N})$.

Now let $\mathfrak{B}$ be a σ-P base for $f(\mathfrak{N})$, and let us show that

$$\mathfrak{C} \equiv \{f(E_\alpha) \cap B : \ \exists\, N \in \mathfrak{N} \ \text{ with } \ N \cap E_\alpha \neq \emptyset, N \subset U_\alpha, \ \text{ and } \ B \subset f(N)\}$$

is a σ-P base for $\{f(E_\alpha) : \alpha < \lambda\}$. As noted above, for each $N \in \mathfrak{N}$ there can be at most one α such that $N \subset U_\alpha$ and $N \cap E_\alpha \neq \emptyset$. Also, for each (nonempty) $B \in \mathfrak{B}$ we can have $B \subset f(N)$ for at most countably many $N \in \mathfrak{N}$, by the point-countability of $f(\mathfrak{N})$. It follows that $\mathfrak{C}$ is σ-P. To see that $\mathfrak{C}$ is also a base, suppose $x = f(t)$ for some $t \in E_\alpha$. We need to show that $x \in C \subset f(E_\alpha)$ for some $C \in \mathfrak{C}$. Since $\mathfrak{N}$ is a network for T, we can choose some $N \in \mathfrak{N}$ such that $t \in N \subset U_\alpha$. Then $x \in f(N)$ and so there is some $B \in \mathfrak{B}$ such that $x \in B \subset f(N)$, since $\mathfrak{B}$ is a base for $f(N)$. It follows that $x \in f(E_\alpha) \cap B \in \mathfrak{C}$ as required. □

Proof of Theorem 5.3. The property dealing with continuous images is clear since the composition of two index-σ-relatively discrete maps will again have this property (Lemma 3.8), and similarly for index-σ-scattered maps (Lemma 2.8).

Any Souslin subset of a complete metric spaces is an index-σ-discrete, continuous image of a complete metric space (see the proof of Theorem 4.1 of [27] where the map $g \restriction C$ has the required properties). Such spaces are thus descriptive spaces. Since any Souslin subset of a descriptive (resp. almost descriptive) space is clearly a continuous, index-σ-relatively discrete (resp. index-σ-scattered) image of a Souslin subset of a complete metric space, the second property follows from the first.

For convenience, we again let P denote the property of a collection being either relatively discrete or scattered. Suppose that, for each $n \in \mathbb{N}$, X_n is a Hausdorff space which is an index-σ-P and continuous image of a complete metric space T_n under a map $f_n : T_n \to X_n$. Let $T = \Pi T_n$ and $X = \Pi X_n$ denote the product spaces, and let $f : T \to X$ denote the product map associated with the maps f_n. Since T is a complete metric space and f is a continuous surjection, we need only show that f is index-σ-P to prove that (almost) descriptive spaces are closed to countable products. In view of the above Lemma, it suffices to find a network $\mathfrak{N}$ for T such that $f(\mathfrak{N})$ is point-countable (as a family indexed by $\mathfrak{N}$) and has a σ-P base.

For each $n \in \mathbb{N}$, let $\bigcup_{m\in\mathbb{N}} \mathfrak{B}_{nm}$ be a σ-discrete open base for the metric space T_n, and for each $(m_1, \dots, m_n) \in \mathbb{N}^n$, let $\mathfrak{C}_{m_1\dots m_n}$ denote the family of sets in T of the form

$$\pi_1^{-1}(B_1) \cap \dots \cap \pi_n^{-1}(B_n) \tag{1}$$

with $B_k \in \mathfrak{B}_{km_k}$ for $k = 1, \dots, n$, and where π_n denotes the n-th projection map. Then each family $\mathfrak{C}_{m_1\dots m_n}$ is discrete in T and together they form a σ-discrete base for the topology of T. Moreover, for each set of the form (1) we have

$$f[\pi_1^{-1}(B_1) \cap \dots \cap \pi_n^{-1}(B_n)] = \pi_1^{-1}(f_1(B_1)) \cap \dots \cap \pi_n^{-1}(f_n(B_n)).$$

We know that, for each $n, m \in \mathbb{N}$, $\{f_n(B) : B \in \mathfrak{B}_{nm}\}$ is point-countable and has a σ-P base, hence this is true of $\{\pi_n^{-1}(f_n(B)) : B \in \mathfrak{B}_{nm}\}$ also (see Lemmas 2.8 and 3.8). Since the pairwise intersections of members of collections having these properties will again have this property (Lemma 2.7 (c)), it follows that the image of $\mathfrak{C}_{m_1\dots m_n}$ under f is point-countable and has a σ-P base. The above Lemma now implies that f is index-σ-P. $\square$

A classical result due to Souslin states that any Souslin set in a Polish space is either countable or contains a compact perfect set (i.e., a nonempty compact set in which each point is a limit point). El'kin [13] has shown that this result holds for nonseparable absolutely analytic metric spaces provided we replace "countable" by "σ-discrete" (i.e., a countable union of closed discrete subsets). In a recent paper, Koumoullis [45] has extended this by showing that any Čech analytic space is either σ-scattered or contains a compact perfect set. Koumoullis' proof of this depends on a number of measure-theoretic results, and a purely topological proof was recently given in [39]. We deduce these and two related results from a more general theorem which has the additional merit of providing a more direct proof of the above results.

Theorem 5.4. *Let X be a Hausdorff space, T a Čech analytic space, and $f : T \to X$ a continuous surjection. If $\{f^{-1}(x) : x \in X\}$ has a selector $E \subset T$ which is not σ-scattered, then X has a compact perfect subset.*

Proof. We first prove the theorem in the case when T is Čech complete. Let E be a non-σ-scattered subset of T having exactly one point in common with each of the sets $f^{-1}(x)$, $x \in X$. Since E is not scattered it contains a nonempty dense-in-itself subset D. Since T is Čech complete, we may choose a decreasing sequence of open sets $\{G_n : n \in \mathbb{N}\}$ in some compactification βT of T such that $T = \bigcap_{n\in\mathbb{N}} G_n$. Using a standard argument we inductively construct, for each finite sequence $i|n = i_1, \ldots, i_n$ in $\{0, 1\}$, open sets $U(i|n)$ in βT such that

$(1)_n$ $U(i|n) \cap D \neq \emptyset$,

$(2)_n$ $cl_{\beta T}U(i|n) \subset U(i|n-1) \subset G_{n-1}$ (we take $U(i|0) = \beta T = G_0$),

$(3)_n$ $cl_{\beta T}U(i|n-1, 0) \cap cl_{\beta T}U(i|n-1, 1) = \emptyset$,

$(4)_n$ $f[T \cap cl_{\beta T}U(i|n-1, 0)] \cap f[T \cap cl_{\beta T}U(i|n-1, 1)] = \emptyset$.

Given that we have constructed the sets $U(i|n-1)$, for all sequences of length $n-1$, $(1)_{n-1}$ implies that $U(i|n-1) \cap D$ is infinite, so it contains two distinct points t and s. Since $D \subset E$, $f(t)$ and $f(s)$ are distinct. Hence, by the continuity of f, we can find neighborhoods $U(i|n-1, 0)$ and $U(i|n-1, 1)$ in βT, of t and s respectively, so that their closures in βT satisfy properties $(2)_n - (4)_n$ above.

Now let Φ be the set-valued map from $\{0, 1\}^{\mathbb{N}}$ to the non-empty compact subsets of βT defined by

$$\Phi(i) = \bigcap_{n=1}^{\infty} cl_{\beta T}U(i|n),$$

and let $C = \Phi(\{0, 1\}^{\mathbb{N}})$. To see that Φ is upper semi-continuous, let $G \supset \Phi(i)$ be open. Then $U(i|n) \subset G$ for some n, by the compactness of $\Phi(i)$, and so $W = \{j \in \{0, 1\}^{\mathbb{N}} : j|n = i|n\}$ is a neighborhood of i such that $\Phi(W) \subset G$, proving that Φ is upper semi-continuous. Also, by property $(3)_n$, we have $\Phi(i) \cap \Phi(j) = \emptyset$ whenever $i \neq j$. Hence the map $g : C \to \{0, 1\}^{\mathbb{N}}$ given by $g(\Phi(i)) = i$ is well-defined. Moreover, since Φ takes compact sets to compact sets, g is continuous, hence a perfect map from C onto $\{0, 1\}^{\mathbb{N}}$. Let P be a closed subset of C minimal with respect to the property $g(P) = \{0,1\}^{\mathbb{N}}$. Then P is a compact perfect subset of T. For if $\{x\}$ were open in P and $x \in \Phi(i)$, then by the minimality of P it follows that $\{x\} = \Phi(i)$, hence

$$\{i\} = \{0,1\}^{\mathbb{N}} \setminus g(P \setminus \{x\})$$

is isolated in $\{0,1\}^{\mathbb{N}}$ which is not possible.

Finally, since $f \restriction C$ is one-to-one by property (4), $f \restriction P$ is a homeomorphism, proving that X has a compact perfect subset.

Now suppose T is Čech analytic. By one of the equivalent properties characterizing Čech analytic spaces [17], T is the projection on βT of some Čech complete subspace S of $\beta T \times \mathbb{N}^{\mathbb{N}}$ (cf. also [34, §5]). By a result we prove below (see Lemma 7.1), any projection map parallel to a space with a countable network preserves scattered σ-decomposable collections (thus, in particular, takes scattered sets to σ-scattered sets). It follows that, if $p : S \to T$ denotes the above projection, then p preserves σ-scattered sets (since scattered sets in S are also scattered in βT).

To complete the proof, let $E \subset T$ be a non-σ-scattered selector for $\{f^{-1}(x) : x \in X\}$. Choosing any set $H \subset S$ having a single point in common with $p^{-1}(t)$, for $t \in E$, it follows that H is non-σ-scattered. But this implies that any selector for $\{p^{-1}(f^{-1}(x)) : x \in X\}$ containing H is non-σ-scattered. The first part of the proof now applies and shows that X has a compact perfect subset. □

Part (a) of the following corollary improves on a recent result of Stegall [70] where T is assumed to be Čech complete and is a base for the topology of T.

Corollary 5.5. *Let X be a Hausdorff space, T a Čech analytic space, and $f : T \to X$ a continuous surjection.*

(a) *If T has a network $\mathfrak{N}$ such that* Card $\mathfrak{N} <$ Card X *and* Card X *is uncountable, then X has a compact perfect subset.*

(b) *If f maps scattered sets to σ-scattered sets, then X is either σ-scattered or contains a compact perfect set.*

(c) *If T is hereditarily weakly θ-refinable and f maps relatively discrete sets to σ-relatively discrete sets, then X is either σ-relatively discrete or contains a compact perfect set.*

Proof. Suppose $\mathfrak{N}$ is a network for T such that Card $\mathfrak{N} <$ Card X where Card X is uncountable. In view of Theorem 5.4 it is enough to show that any scattered subset of T has cardinality at most Card $\mathfrak{N}$, since this will imply that any selector for $\{f^{-1}(x) : x \in X\}$ must be non-σ-scattered.

Suppose $S \subset T$ is scattered and has cardinality λ. Then we can write $S = \{t_\alpha : \alpha < \lambda\}$ and find open sets U_α in T such that $t_\alpha \in U_\alpha$ and $t_\alpha \notin U_\beta$ for all $\beta < \alpha$, for each $\alpha < \lambda$. Since $\mathfrak{N}$ is a network for T, we can find $N_\alpha \in \mathfrak{N}$ such that $t_\alpha \in N_\alpha \subset U_\alpha$ for each α. Since $\beta < \alpha$ implies that $t_\alpha \notin N_\beta$, the correspondence $t_\alpha \mapsto N_\alpha$ is one-to-one, hence Card $S \leq$ Card $\mathfrak{N}$.

To prove (b), note that since $f : T \to X$ preserves σ-scattered sets, if X is not σ-scattered, then any selector for $\{f^{-1}(x) : x \in X\}$ must be non-σ-scattered. Hence the theorem applies.

The proof for part (c) is exactly the same, except we replace "scattered" throughout by "relatively discrete", noting that scattered sets in T are now σ-relatively discrete (by Lemma 3.6), and that Lemma 7.1 shows that the projection map in Theorem 5.4 will also preserve σ-relatively discrete sets. $\square$

The following corollary is an immediate consequence of Theorem 5.4 and the definitions.

Corollary 5.6.

(a) *Any Čech analytic hereditarily weakly θ-refinable space (in particular, any descriptive space) is either σ-relatively discrete or contains a compact perfect subset.*

(b) (Koumoullis [45]) *A Čech analytic space is either σ-scattered or contains a compact perfect subset.*

(c) *An almost descriptive space is either σ-scattered or contains a compact perfect subset.*

Lemma 3.4 shows that if a family of Souslin sets in arbitrary topological space X has a σ-relatively discrete base, then the union of any subcollection will be a Souslin set in X. The following theorem gives a useful converse to this. The proof illustrates how results that are known to hold for Borel and Souslin sets in complete (or absolutely analytic) metric spaces can often be carried over to give corresponding results in descriptive and almost descriptive spaces. (Unfortunately, the situation is not as clear in the case of K-descriptive spaces as we will indicate below).

Theorem 5.7. *The following properties hold for any descriptive (resp. almost descriptive) Hausdorff space X.*

(a) *Every point-finite Souslin-additive family in X has a σ-relatively discrete (resp. σ-scattered) base.*

(b) *Every point-countable* $(\mathcal{F} \cap \mathcal{G})_\sigma$ *set cover of* X *either has a* σ*-relatively discrete (resp.* σ*-scattered) refinement, or* X *contains a compact subset meeting uncountably many members of the cover.*

Proof. (a) Let T be a complete metric space and let $f : T \to X$ be a continuous index-σ-relatively discrete (resp. index-σ-scattered) surjection. If $\mathcal{S}$ is a point-finite Souslin-additive collection in X, then $f^{-1}(\mathcal{S}) = \{f^{-1}(S) : S \in \mathcal{S}\}$ is a point-finite Souslin($\mathcal{F}$)-additive collection in T, hence $f^{-1}(\mathcal{S})$ is σ-discretely decomposable in T [44]. In particular, $f^{-1}(\mathcal{S})$ has a σ-discrete base $\mathfrak{B}$. Taking forward images it follows that $f(\mathfrak{B})$, and hence also $\mathcal{S}$, has a σ-relatively discrete (resp. σ-scattered) base.

(b) Let $f : T \to X$ have the same meaning as in part (a). If $\mathcal{H}$ is a point-countable cover of X by $(\mathcal{F} \cap \mathcal{G})_\sigma$ sets, then $f^{-1}(\mathcal{H})$ is a point-countable cover of T by $\mathcal{F}_\sigma$ sets. By [28, Theorem 2.1], $f^{-1}(\mathcal{H})$ either has a σ-discrete refinement or else some compact subset of T meets uncountably many members of $f^{-1}(\mathcal{H})$. The properties of f easily imply that $\mathcal{H}$ has the desired properties. $\square$

Frolík and Holický [19] have shown that point-finite Souslin($\mathcal{F}$)-additive collections are σ-discretely decomposable in general (not necessarily Lindelöf) K-analytic spaces. The same proof in [19] shows that this will hold for point-finite Souslin($\mathcal{F}$)-additive collections in K-descriptive spaces provided the conclusion reads "σ-relatively discretely decomposable" (cf. [18, Theorem 3]). What would be of greater interest in studying the topological properties of Banach spaces is whether Theorem 5.7 holds in K-descriptive spaces. (For a positive answer based on additional set-theoretic assumptions see the paper by Holický [38].)

By a *function base* for a map $f : X \to Y$ we mean a family $\mathfrak{B}$ of subsets of X which is a base for $\{f^{-1}(V) : V \text{ open in } Y\}$. Note that if $\mathfrak{N}$ is a network for Y, then $\{f^{-1}(N) : N \in \mathfrak{N}\}$ will be a function base for f. To motivate the need for introducing this concept, we consider several problems in nonseparable descriptive topology that are still not completely resolved, but which have nice solutions when certain types of function bases are assumed. For example, if $f : T \to X$ is a Borel measurable map for metric spaces T and X, must f have a class in the sense that there is a fixed countable ordinal α such $f^{-1}(U)$ is of Borel class α for each open set $U \subset X$? The answer is yes if f has a σ-discrete function base (hence if X is separable or T is absolutely analytic [26]), or if additional set theoretic axioms are assumed [20]. As a second example, suppose $f : T \to X$ is a class 1 map, for metric spaces T and X, and suppose T is a Baire space. Must f be continuous at each point of a dense $\mathcal{G}_\delta$ subset of T? Again, the answer is yes (even for a non-metrizable T) if f has a σ-discrete function base [31, Theorem

4]. As a final example, suppose $f : T \to X$ is a Borel class 1 map from a metric spaces T to a normed linear space X. Then a necessary and sufficient condition for f to be a pointwise limit of a sequence of continuous maps is that f have a σ-discrete function base. The sufficiency is given in [33, Lemma 7] (where such maps are said to be "σ-discrete") and the necessity follows from [30, Lemma 3.2]. This result was recently applied in [42] and [37] to obtain Baire class 1 selectors for upper semi-continuous set-valued maps.

The point we wish to make is that function properties that are known to hold for maps into separable metric space can often be extended to maps having a σ-discrete function base taking values in a non-separable metric space (see [26], [30], and [31]). The following lemma, which is needed for the proof of Theorem 1.14 of the introduction, is in the same spirit. Recall that a map has the *Baire property* if the inverse image of each open set is a BP-set in the domain (see [60, p. 36] or [46, p. 399]). Similarly, we say that a map has the *strong Baire property* if the inverse image of each open set is a strong BP-set in the domain. Note that all class 1 Borel maps (i.e., inverse images of open sets are $(\mathcal{F} \cap \mathcal{G})_\sigma$ sets) have the strong property of Baire by Theorem 2.6 (a).

Lemma 5.8. *Suppose $f : X \to Y$ is a map of topological spaces having a σ-scattered function base $\mathfrak{B}$.*

(a) *If each set in $\mathfrak{B}$ has the Baire property, then there is a set M of the first category in X such that $f \restriction X \setminus M$ is continuous.*

(b) *If each set in $\mathfrak{B}$ has the strong Baire property (in particular, if $\mathfrak{B}$ consists of $(\mathcal{F} \cap \mathcal{G})_\sigma$ sets in X), then the set of points of discontinuity of f is a set of the first category in X.*

Proof. (a) For each $B \in \mathfrak{B}$ we have $B = (W_B \setminus P_B) \cup Q_B$ where W_B is open in X, P_B and Q_B are of the first category in X, and we may assume that $P_B \subset B$. Then

$$M = \cup\{P_B \cup Q_B : B \in \mathfrak{B}\}$$

is of the first category in X by Corollary 2.5. To prove that

$$g = f \restriction X \setminus M$$

is continuous, let $U \subset Y$ be a given open set. By the property of a function base,

$$g^{-1}(U) = \cup\{W_B \setminus M : B \in \mathfrak{B} \text{ and } B \subset f^{-1}(U)\},$$

showing that $g^{-1}(U)$ is open in $X \setminus M$.

(b) For each $B \in \mathfrak{B}$ we have $B = W_B \cup Q_B$ where W_B is open and Q_B is of the first category in X. Let x be any point not in the set

$$M = \cup\{Q_B : B \in \mathfrak{B}\},$$

a set which is of the first category in X by Corollary 2.5. If $f(x) \in U$, for some open set $U \subset Y$, then we have $x \in B \subset f^{-1}(U)$, for some $B \in \mathfrak{B}$, by the property of a function base. Then it follows that $x \in W_B$ (since $x \notin M$) and $f(W_B) \subset U$, proving that f is continuous at x. $\square$

The following lemma gives a sufficient condition for maps to has a σ-relatively discrete or σ-scattered function base.

Lemma 5.9. *Let $f : T \to X$ be Borel measurable where X is a descriptive space.*

(a) *If T is descriptive, then f has a σ-relatively discrete function base of Souslin sets.*

(b) *If T is almost descriptive, then f has a σ-scattered function base of restricted Baire property sets.*

P r o o f. (a) Since X is descriptive, it has a network $\underset{n\in\mathbb{N}}{\cup} \mathfrak{N}_n$ where each $\mathfrak{N}_n$ is a relatively discrete collection of Souslin sets in X by Theorem 5.1. Since the union of each subcollection of $\mathfrak{N}_n$ is a Souslin set in Y by Lemma 3.3, it follows that $f^{-1}(\mathfrak{N}_n)$ is a disjoint Souslin-additive family in T. Hence, by Theorem 5.7, $f^{-1}(\mathfrak{N}_n)$ has a σ-relatively discrete base $\mathfrak{B}_n$, which we may assume consists of Souslin sets in view of Lemma 3.2 (just as in the proof of Theorem 5.1). It follows that $\underset{n\in\mathbb{N}}{\cup} \mathfrak{B}_n$ is the desired function base for f. The proof of (b) is identical using Theorem 5.7 and Lemma 2.3. $\square$

If $\mathfrak{M}$ is a collection of subsets of a set T containing the empty set and closed to the operations of finite intersections and countable unions, then we call $(T, \mathfrak{M})$ a *measurable space*. Recall that, if X is a topological space, then a map $f : T \to X$ is said to be $\mathfrak{M}$-*measurable* if $f^{-1}(U) \in \mathfrak{M}$ whenever U is an open set in X. It is often difficult, if not impossible, to establish useful stability properties for $\mathfrak{M}$-measurable maps in such a general setting. For example, if $f : T \to X$ and $g : T \to Y$ are $\mathfrak{M}$-measurable for metric spaces X and Y, must the diagonal map $t \mapsto (f(t), g(t))$ be $\mathfrak{M}$-measurable? The answer isn't known in general, even when $\mathfrak{M}$ is the Borel σ-algebra of a metric space (see, however, [31] for a proof in the case when both maps are of bounded class). Affirmative answers to this and related stability questions can usually be found given the existence of certain function bases.

A collection $\mathcal{E} \subset \mathfrak{M}$ is said to be $\mathfrak{M}$-*hereditarily additive* if, whenever $M_E \in \mathfrak{M}$ for each $E \in \mathcal{E}$, then

$$\cup\{M_E \cap E : E \in \mathcal{E}\} \in \mathfrak{M}.$$

By Lemma 2.6, if T is a topological space and $\mathfrak{M}$ is the family of restricted BP-sets in T, then any collection in $\mathfrak{M}$ having a σ-scattered base is $\mathfrak{M}$-hereditarily additive. Similarly, any collection of Souslin sets having a σ-relatively discrete base is Souslin-hereditarily additive by Lemma 3.4. (Conversely, if T is descriptive, then any point-finite Souslin-additive family in T will have a σ-relatively discrete base by Lemma 5.7.) Note that any map $f : T \to X$ which has a $\mathfrak{M}$-hereditarily additive function base is necessarily $\mathfrak{M}$-measurable. Stability properties of maps having $\mathfrak{M}$-hereditarily additive function bases can usually be established as Lemma 5.10 below illustrates (cf. also [31, Theorem 7]). We say that a map $f : T \to X$ is $\mathfrak{M}$-*descriptive* if it has a function base which is a countable union of disjoint $\mathfrak{M}$-hereditarily additive families.

We make use of the following lemma proved in [32, Lemma 2.2].

Lemma 5.10. *Let $(T, \mathfrak{M})$ be a measurable space. If $\mathcal{E}$ and $\mathcal{H}$ are $\mathfrak{M}$-hereditarily additive, then so is $\{E \cap H : E \in \mathcal{E}, H \in \mathcal{H}\}$.*

Lemma 5.11. *Let $(T, \mathfrak{M})$ be a measurable space. Let $X = \Pi\{X_a : a \in A\}$ be a product space with projection maps $\pi_a : X \to X_a$ where A is at most countable. If $f : T \to X$ is such that $\pi_a \circ f$ is $\mathfrak{M}$-descriptive for each $a \in A$, then f is $\mathfrak{M}$-descriptive.*

Proof. Let $\bigcup_{n\in\mathbb{N}} \mathcal{H}_{an}$ be a function base for $\pi_a \circ f$ for each $a \in A$, where each $\mathcal{H}_{an}$ is disjoint and $\mathfrak{M}$-hereditarily additive. Then, for each $m \in \mathbb{N}$ and finite sets $F = \{a_1, \ldots, a_m\} \subset A$ and $\{n_1, \ldots, n_m\} \in \mathbb{N}$,

$$\mathfrak{B}_F(n_1, \ldots, n_m) = \{H_1 \cap \ldots \cap H_m : H_i \in \mathcal{H}_{a_i n_i} \quad \text{for} \quad i = 1, \ldots, m\}$$

is also disjoint and $\mathfrak{M}$-hereditarily additive by Lemma 5.9. Now note that any base for the sets

$$f^{-1}[\pi_{a_1}^{-1}(U_1) \cap \ldots \cap \pi_{a_m}^{-1}(U_m)],$$

as $\{a_1, \ldots, a_m\}$ runs over all finite subsets of A and where each U_i is open in X_{a_i}, will be a function base for the map f, in view of the definition of the product topology. But clearly, the countably many families of the type $\mathfrak{B}_F(n_1, \ldots, n_m)$ described above will be such a base. It follows that f is $\mathfrak{M}$-descriptive. $\square$

Corollary 5.12. *Let $(T, \mathfrak{M})$ be a measurable space and let X be a linear topological space. If $f, g : T \to X$ are $\mathfrak{M}$-descriptive, then so is the vector sum $f + g$.*

Proof. By Lemma 5.10 the complex map $(f, g) : T \to X \times X$ is $\mathfrak{M}$-descriptive, hence the composition of this map followed by any continuous map is $\mathfrak{M}$-descriptive. But $f + g$ is the composition of (f, g) followed by the continuous map $+ : X \times X \to X$. $\square$

Our final topic in this section deals with the concept of a Radon space. By a *measure* on a Hausdorff space X we will mean a non-negative, countably additive and finite real-valued set function defined on the Borel sets of X. A measure μ on X is said to be a *Radon measure* if

$$\mu(B) = \sup\{\mu(K) : K \subset B \text{ compact}\}$$

for each Borel set B. The space X is said to be a *Radon space* if each measure on X is a Radon measure. An important result in the work of Schwartz [67, p. 122-6] due to P. A. Meyer is that every Souslin space is a Radon space. If a space contains uncountable discrete subspaces, then in order to be a Radon space one needs to impose some cardinality restriction on these subspaces. Recall that a cardinal κ is said to have *measure zero* if every finite measure defined on the power set $\mathcal{P}(\kappa)$ and assigning measure zero to each singleton is identically zero (such cardinals are also said to be *not real-valued measurable*). From Ulam's work [76] it follows that one can consistently assume that all cardinals are of measure zero.

Consider the following condition for a topological space X:

$(*)$ *the cardinality of each discrete subset of X has measure zero.*

Marczewski and Sikorski [47] has shown that if a metric space X has $(*)$, then every measure on X has closed separable support. Hence any absolutely analytic metric space (not necessarily separable) satisfying $(*)$ is a Radon space (using the above result of Meyer's for example). Schachermayer [66] has shown that, for any weakly compact subset K of a Banach space, (K, weak) will be a Radon space whenever it satisfies condition $(*)$. He later noted that, for any Banach space having an equivalent norm for which the norm and weak topologies coincide on the unit sphere, the same conclusion holds for any weak Borel set with the weak topology (see [11, p. 676]).

There have been a number of generalizations of the Marczewski-Sikorski theorem leading to sufficient conditions for a space to be Radon. The most recent of these, due to Gardner and Pfeffer, is well-suited for our work:

Lemma 5.13 [22, Theorem 9]. *Any completely regular Hausdorff space X satisfying the following properties is a Radon space.*

(1) *X is universally Radon measurable,*
(2) *X is hereditarily weakly θ-refinable,*
(3) *X satisfies condition $(*)$.*

A completely regular Hausdorff space X is *universally Radon measurable* if it is measurable relative to the completion of every Radon measure on βX, its Stone-Čech compactification. Since every Souslin set in βX has this property [63, Theorem 26], any Čech analytic space is universally Radon measurable.

Theorem 5.14. *Any completely regular descriptive space satisfying condition $(*)$ above is a Radon space.*

Proof. This follows immediately from Lemma 5.13 and Theorem 5.1 □

6. Networks for the various topologies of a Banach space. The topological relationships between the norm and weak topologies of descriptive and almost descriptive Banach spaces follow from the existence of a network for the norm topology having certain discreteness properties relative to the weak topology. In this section we show how such networks are obtained. We establish general results of this type which enable us to prove similar theorems for the norm and pointwise convergence topologies on function spaces of the type $C(K)$, and also for the norm and weak* topologies on a dual Banach space. In order to cover simultaneously the cases of a σ-relatively discrete and a σ-scattered network, it will be convenient to work with an abstraction of both properties.

By a *discreteness class* associated with topological spaces we mean a correspondence $\Delta : X \mapsto \Delta(X)$ which associates with each topological space X a family $\Delta(X)$ whose members are disjoint collections of subsets of X having the following properties:

(i) $\{E\} \in \Delta(X)$ for any $E \subset X$.
(ii) If $\mathcal{E}$ and $\mathcal{H}$ belong to $\Delta(X)$, then so does $\{E \cap H : E \in \mathcal{E}, H \in \mathcal{H}\}$.
(iii) If $\mathcal{E} \in \Delta(X)$ and $D_E \subset E$ for each $E \in \mathcal{E}$, then $\{D_E : E \in \mathcal{E}\} \subset \Delta(X)$.
(iv) If X is a subspace of Y, then $\Delta(X) \subset \Delta(Y)$ and $\mathcal{E} \in \Delta(Y)$ implies $\{E \cap X : E \in \mathcal{E}\} \subset \Delta(X)$.

The members of $\Delta(X)$ are said to be $\Delta(X)$-*discrete.* It will also be convenient to let $\Delta_\sigma(X)$ denote the set of all families that are countable unions of $\Delta(X)$-discrete collections. Note that $\Delta_\sigma(X)$ will also satisfy the properties (i)–(iv). It is easy to verify using Lemmas 2.2 and 3.1 that the correspondences $X \mapsto \Sigma(X)$ (scattered collections) and $X \mapsto \Delta^\rho(X)$ (relatively discrete collections) define discreteness classes for each topological space X. When the space X is fixed we will often write Δ in place of $\Delta(X)$, and say that Δ defines a discreteness class for X.

If Δ defines a discreteness class for X, a map $f : X \to Y$ is said to be Δ_σ*-simple* if X has a partition $\mathcal{E} \in \Delta_\sigma$ such that $f \restriction E$ is constant for each $E \in \mathcal{E}$. Note that this property is independent of any topology on Y.

Lemma 6.1. *Let Δ define a discreteness class for X and let Y be a vector space. If $f, g : X \to Y$ are Δ_σ-simple maps, then so is any linear combination of f and g.*

Proof. Any scalar multiple of a Δ_σ-simple map is clearly Δ_σ-simple. Also, if $\mathcal{E}$ and $\mathcal{H}$ are partitions of X which belong to Δ_σ and are such that $f \restriction E$ and $g \restriction H$ are constant for each $E \in \mathcal{E}$ and $H \in \mathcal{H}$, then $\{E \cap H : E \in \mathcal{E}, H \in \mathcal{H}\}$ belongs to Δ_σ (as easily follows from the definitions) and is a partition of X on each member of which $f + g$ takes a constant value. $\square$

The following result is central to our work.

Theorem 6.2. *Let Δ define a discreteness class for X, and let Y be a Banach space. Suppose $f : X \to Y$ is a weak pointwise limit of a sequence of Δ_σ-simple maps $f_n : X \to Y$. Then f is a norm uniform limit of a sequence of Δ_σ-simple maps $g_n : X \to Y$. Moreover, f has a norm function base $\mathfrak{B} \in \Delta_\sigma(X)$.*

Proof. Let $\{h_n\}_{n \geq 1}$ be a sequence representing all rational (finite) linear combinations of the maps f_n. By Lemma 6.1, each h_n is Δ_σ-simple, so there is a $\mathcal{H}_n \in \Delta_\sigma$ which partitions X and is such that $h_n \restriction H$ takes a constant value, say $y_H^{(n)}$, for each $H \in \mathcal{H}_n$.

Given $\varepsilon > 0$, for each $n \in \mathbb{N}$ and $H \in \mathcal{H}_n$ define

$$N_H(n, \varepsilon) = \{x \in X : \|f(x) - y_H^{(n)}\| < \varepsilon\}$$

and let

$$M(n, \varepsilon) = \cup\{H \cap N_H(n, \varepsilon) : H \in \mathcal{H}_n\}.$$

To see that $X = \cup\{M(n, \varepsilon) : n \in \mathbb{N}\}$, let $x \in X$. Since the norm and weak closures of the convex hull of $\{f_n(x) : n \geq 1\}$ coincide, it follows that $f(x)$ will be a norm cluster point of the sequence $\{h_n(x)\}_{n \geq 1}$, and so we have $\|f(x) - h_n(x)\| < \varepsilon$ for some n. Let $H \in \mathcal{H}_n$ be such that $x \in H$. Then $h_n(x) = y_H^{(n)}$, and it follows that

$$x \in H \cap N_H(n, \varepsilon) \subset M(n, \varepsilon).$$

We now choose any disjoint sets $D(n,\varepsilon) \subset M(n,\varepsilon)$ such that

$$X = \cup\{D(n,\varepsilon) : n \in \mathbb{N}\}.$$

Doing the above with $\varepsilon = 1/m$ for each $m \in \mathbb{N}$, we define maps $g_m : X \to Y$ by the rule

$$g_m(x) = y_H^{(n)} \quad \text{if and only if} \quad x \in H \cap D(n, 1/m),\ H \in \mathcal{H}_n,\ n \in \mathbb{N}.$$

Since the family $\{H \cap D(n,1/m) : H \in \mathcal{H}_n, n \in \mathbb{N}\}$ belongs to Δ_σ and partitions X, each g_m is a Δ_σ-simple map. Moreover, since

$$H \cap D(n,1/m) \subset H \cap M(n,1/m) \subset N_H(n,1/m)$$

it follows that $\|f(x) - g_m(x)\| < 1/m$ for all $x \in X$. Whence the sequence $\{g_m\}_{m\geq 1}$ converges norm uniformly to f.

Finally, let us show that the Δ_σ family

$$\mathfrak{B} = \{H \cap D(n,1/m) : H \in \mathcal{H}_n, n \in \mathbb{N}\}$$

is a norm function base for f. Let $x_0 \in f^{-1}(U)$ for some norm open set $U \subset Y$. Choose $m \in \mathbb{N}$ so that the ball about $f(x_0)$ of radius $1/m$ is contained in U, and let $n \in \mathbb{N}$ be such that $x_0 \in D(n,1/2m)$. Finally, choose $H \in \mathcal{H}_n$ such that $x_0 \in H$. Then for every $x \in H \cap D(n,1/2m)$ we have

$$\|f(x) - f(x_0)\| \leq \|f(x) - y_H^{(n)}\| + \|y_H^{(n)} - f(x_0)\| \leq 1/m.$$

It follows that $x_0 \in H \cap D(n,1/2m) \subset f^{-1}(U)$. $\square$

If Δ defines a discreteness class for X and X has associated with it several topologies, then $\Delta(\tau)$ will denote the Δ-discrete families in X relative to the topology τ.

Theorem 6.3. *Let X be a subset of a Banach space Y and let Δ define a discreteness class for X.*

(a) *If (X, weak) has a network $\mathfrak{N} \in \Delta_\sigma(\text{weak})$, then there is some $\mathfrak{N}^* \in \Delta_\sigma(\text{weak})$ that is also a network for (X, norm).*

(b) *If $Y = C(K)$, for some compact Hausdorff space K, and (X, τ_p) has a network $\mathfrak{N} \in \Delta_\sigma(\tau_p)$, then there is some $\mathfrak{N}^* \in \Delta_\sigma(\tau_p)$ that is also a network for (X, norm).*

Proof. (a) Let $\mathfrak{N} = \bigcup_{n\in\mathbb{N}} \mathfrak{N}_n$ where each $\mathfrak{N}_n \in \Delta(\text{weak})$. Let

$$\mathfrak{B}_n = \{N_1 \cap \ldots \cap N_n : N_k \in \mathfrak{N}_k \cup \{X \setminus \cup\mathfrak{N}_k\} \text{ for } k = 1, \ldots, n\}.$$

Then $\mathfrak{B}_n$ is a partition of X and is easily seen to belong to $\Delta_\sigma(\text{weak})$ using properties (i) and (ii) of a discreteness class.

We fix a point in each member of $\mathfrak{B}_n$ and let $f_n : X \to X$ be the map that is constant on each member of $\mathfrak{B}_n$ and assumes the fixed point of that member. Then each f_n is a Δ_σ-simple map, and we claim that the sequence $\{f_n\}$ will converge pointwise relative to the weak topology to the identity map on X.

Given $x \in X$ let B_n be the member of $\mathfrak{B}_n$ that contains x, and suppose $f_n \restriction B_n$ takes the constant value $x_n \in B_n$. If U is any neighborhood of the point x relative to the weak topology, we can find $n_0 \in \mathbb{N}$ and $N_{n_0} \in \mathfrak{N}_{n_0}$ such that $x \in N_{n_0} \subset U$ by the property of a network. Since $\{B_n\}_{n\geq 1}$ is necessarily decreasing and $B_{n_0} \subset N_{n_0}$ (by the definition of $\mathfrak{B}_{n_0}$), it follows that $x_n \in B_n \subset U$ for all $n \geq n_0$. This proves that $\{f_n(x)\}_{n\geq 1}$ converges weakly to x.

Letting h denotes the identity map on X, it follows from Theorem 6.2 that there is a $\mathfrak{B} \in \Delta_\sigma(\text{weak})$ which is a base for the sets $h^{-1}(U) = U$, $U \subset X$ norm open, and thus $\mathfrak{N}^* = \mathfrak{B}$ is the desired network for the norm topology on X.

(b) Note first that, for any discreteness class Δ, if a space is a countable union of subspaces each having a Δ_σ-network, then the union of these networks will be a Δ_σ-network for the whole space (as follows easily from property (iv) of a discreteness class and the definition of a network). Consequently, it suffices to show that (b) holds when (X, τ_p) is further assumed to be a bounded subset of $C(K)$. Also, the assumptions of (b) imply the existence of a network in $\Delta_\sigma(\tau_p)$ for any subspace of X. If we proceed exactly as in the proof of part (a) and adopt the notation of that proof, it follows that the sequence $\{f_n(x)\}_{n\geq 1}$ now converge to x relative to the topology of pointwise convergence on X, for each $x \in X$. However, by a theorem of Grothendieck [21, Theorem 5], it follows that the sequence $\{f_n(x)\}_{n\geq 1}$ must also converge to x relative to the weak topology on X. Hence the rest of the proof of part (a) applies verbatim. □

For the case of a dual Banach space we need the following theorem. For the definition of a space being (properly) σ-fragmented by a metric see the discussion leading up to Theorem 1.10 in §1.

Theorem 6.4. *Let (X, τ) be a topological space which is properly σ-fragmented by some metric ρ on the set X. Let Δ denote either of the discreteness classes Σ or Δ^ρ. Then the following are equivalent.*

(a) *Any collection of τ-open subsets of X has a $\Delta_\sigma(\tau)$ refinement.*

(b) *(X,τ) has a network $\mathfrak{N} \in \Delta_\sigma(\tau)$.*
(c) *Any ρ-discrete collection has a $\Delta_\sigma(\tau)$ refinement.*
(d) *(X,ρ) has a network $\mathfrak{N} \in \Delta_\sigma(\tau)$.*
Moreover, any topological space will satisfy condition (a) *when $\Delta = \Sigma$, hence (X,ρ) always has a network which is σ-scattered relative to the τ topology.*

Conversely, if ρ is a metric on X such that $\tau_\rho \supset \tau$ and (X,ρ) has a network which is σ-scattered relative to τ, then (X,τ) is σ-fragmented by ρ.

Proof. For each $m \in \mathbb{N}$, let $X = \cup\{X_{mn} : n \in \mathbb{N}\}$ satisfy the definition of σ-fragmentability for $\varepsilon = 1/m$ (see §1). For each $m,n \in \mathbb{N}$ we define inductively a τ-open cover $\{U_\lambda : \lambda < \Lambda_{mn}\}$ of X_{mn} such that

$$\rho - \operatorname{diam} D_\lambda < 1/m, \quad \text{where} \quad D_\lambda = (U_\lambda \setminus \bigcup_{\xi<\lambda} U_\xi) \cap X_{mn}.$$

This is always possible, for if the relatively τ-closed set

$$F = X_{mn} \setminus \bigcup_{\xi<\lambda} U_\xi$$

is nonempty, it must contain a nonempty relatively τ-open set W of diameter less than $1/m$, and so we may take U_λ to be any τ-open set in X such that $U_\lambda \cap F = W$.

Note that if $\Delta_\sigma(\tau)$ is the class of all σ-relatively discrete collections in (X,τ), then (a) is just the condition that (X,τ) is hereditarily weakly θ-refinable. If, on the other hand, $\Delta_\sigma(\tau)$ is the class of all σ-scattered collections, then (a) is satisfied by any space (X,τ).

(a) $\Rightarrow$ (c) If (X,τ) is hereditarily weakly θ-refinable, then each of the families $\{D_\lambda : \lambda < \Lambda_{mn}\}$ is σ-relatively discretely decomposable by Lemma 3.5. In either case, $\{D_\lambda : \lambda < \Lambda_{mn}\}$ has a $\Delta_\sigma(\tau)$ base $\mathfrak{N}_{mn}$, and it suffices to show that $\mathfrak{N} = \bigcup_{m,n\in\mathbb{N}} \mathfrak{N}_{mn}$ is a network for (X,ρ). For this it is enough to show that if B is the open ρ-ball about $x \in X$ of radius $1/m$, then there is some $N \in \mathfrak{N}$ such that $x \in N \subset B$. Let n be such that $x \in X_{mn}$, and thus $x \in D_\lambda \subset U_\lambda$ for some $\lambda < \Lambda_{mn}$. Since $\mathfrak{N}_{mn}$ is a base for $\{D_\lambda : \lambda < \Lambda_{mn}\}$, there is some $N \in \mathfrak{N}_{mn}$ such that $x \in N \subset D_\lambda \subset B$ as required.

The implication (c) $\Rightarrow$ (b) follows since $\tau \subset \tau_\rho$. Since (b) $\Rightarrow$ (a) is clear the proof of the first part is complete.

Conversely, let ρ be a metric on X such that $\tau_\rho \supset \tau$, and suppose (X,ρ) has a network $\mathfrak{N} = \bigcup_{n\in\mathbb{N}} \mathfrak{N}_n$ where each collection $\mathfrak{N}_n$ is scattered relative to τ. Given $\varepsilon > 0$, define

$$\mathfrak{N}_n(\varepsilon) = \{N \in \mathfrak{N}_n : \rho - \operatorname{diam}(N) < \varepsilon\},$$

and put $X_n = \underset{n\in\mathbb{N}}{\cup} \mathfrak{N}_n(\varepsilon)$, for each $n \in \mathbb{N}$. Note that $X = \underset{n\in\mathbb{N}}{\cup} X_n$ since $\mathfrak{N}$ is a network for (X, ρ). Since $\mathfrak{N}_n(\varepsilon)$ is τ-scattered, we can write $\mathfrak{N}_n(\varepsilon) = \{N_\alpha : \alpha < \lambda_n\}$ and find τ-open sets $U_\alpha \subset X$ so that

$$\cup\{N_\xi : \xi \leq \alpha\} = U_\alpha \cap X_m$$

for each $\alpha < \lambda_n$ and $n \in= \mathbb{N}$. Given any nonempty set $A \subset X_n$, if

$$\xi = \min\{\alpha < \lambda_n : A \cap N_\alpha \neq \emptyset\},$$

then $A \cap U_\xi = A \cap N_\xi$ is a nonempty relatively τ-open subset of A such that

$$\rho - \mathrm{diam}(A \cap U_\xi) \leq \rho - \mathrm{diam}(N_\xi) < \varepsilon.$$

This proves that (X, τ) is σ-fragmented by the metric ρ. $\square$

Theorem 6.5. *Let Δ denote either of the discreteness classes Σ or Δ^ρ, let Z be a dual Banach space with the Radon-Nikodým Property (RNP), and let X be any subset of Z. Then the following are equivalent.*
(a) (X, weak^*) *has a network* $\mathfrak{N} \in \Delta_\sigma(\mathrm{weak}^*)$.
(b) (X, norm) *has a network* $\mathfrak{N} \in \Delta_\sigma(\mathrm{weak}^*)$.
(c) *Each norm discrete collection has a* $\Delta_\sigma(\mathrm{weak}^*)$ *refinement.*
(d) *Any collection of weak* open subsets of X has a* $\Delta_\sigma(\mathrm{weak}^*)$ *refinement.*
Moreover, (a) *always holds when $\Delta = \Sigma$, hence the norm topology of any dual Banach space with RNP always has a weak* σ-scattered network.*

Conversely, if the norm topology of a dual Banach space Z^* has a network that is σ-relatively discrete for the weak topology, then Z^* has RNP.*

P r o o f. Since Z has the Radon-Nikodým Property, (Z, weak^*) is σ-fragmented by the norm of Z [39, Theorem 6.2]. Since this property is inherited by every topological subspace, the theorem follows from Lemma 6.4.

To prove the converse, recall that Z will have the Radon-Nikodým Property if every nonempty weak* compact subset K of Z contains weak* relatively open nonempty subsets of arbitrarily small norm diameter or, equivalently, $(K, \mathrm{weak}^*) \to (K, \mathrm{norm})$ has at least one point of continuity (see [57, Lemma 3] or [5, Theorem 4.2.13]). Since K is a Baire space, this follows from Theorem 6.6, (a) $\Rightarrow$ (e), below. $\square$

We now describe some of the relationships between the norm and weak (or τ_p or weak*) topologies of a Banach space which result when the norm topology has a network of the type under consideration. In order to cover all the cases simultaneously the theorem is presented in a general setting.

Theorem 6.6. *Let X be a set with two topologies τ and μ such that, whenever $x \in U$ with $U \in \mu$, there exists $V \in \mu$ such that $x \in V$ and $cl_\tau V \subset U$. Let Δ denote either of the discreteness classes Σ or Δ^ρ. Then* (a) $\iff$ (b) *and* (b) *implies each of the properties* (c) *through* (e). *If (X, μ) has a network $\mathfrak{N} \in \Delta_\sigma(\mu)$ (for example, if μ is metrizable), then* (c) $\Rightarrow$ (a).

- (a) *(X, μ) has a network $\mathfrak{N} \in \Delta_\sigma(\tau)$.*
- (b) *(X, μ) has a network $\mathfrak{N}^* \in \Delta_\sigma(\tau)$ consisting of $\mathcal{F} \cap \mathcal{G}$ sets with respect to τ.*
- (c) *The identity map $(X, \mu) \to (X, \tau)$ is index-σ-Δ (equivalently, any $\Delta(\mu)$ family has a $\Delta_\sigma(\tau)$ base).*
- (d) *For $\Delta = \Delta^\rho$, each open set in (X, μ) is a $(\mathcal{F} \cap \mathcal{G})_\sigma$ set in (X, τ).*
- (d)$'$ *For $\Delta = \Sigma$, each open set in (X, μ) is a strong restricted BP-set in (X, τ).*
- (e) *For any $A \subset X$, the set of points of discontinuity of the identity map $(A, \tau) \to (A, \mu)$ is a set of the first category in (A, τ). In particular, if (A, τ) is a Baire space, then the τ and μ topologies coincide on a τ dense $\mathcal{G}_\delta$ subset of A.*

Proof. (a) $\Rightarrow$ (b) Let $\mathfrak{N} = \bigcup_{n\in\mathbb{N}} \mathfrak{N}_n$ be a network for (X, μ) where each $\mathfrak{N}_n$ belongs to $\Delta(\tau)$. By Lemmas 2.3 and 3.2, each of the families $\mathfrak{N}_n$ can be expanded to a family $\mathfrak{N}_n^* = \{H_N : N \in \mathfrak{N}_n\} \in \Delta(\tau)$ consisting of $\mathcal{F} \cap \mathcal{G}$ relative to the τ topology, and such that $N \subset H_N \subset cl_\tau N$ for each N. Hence it suffices to show that $\mathfrak{N}^* = \bigcup_{n\in\mathbb{N}} \mathfrak{N}_n^*$ is a network for (X, μ). Let $x \in U$ with $U \in \mu$, and chose $V \in \mu$ such that $x \in V \subset cl_\tau V \subset U$. Since $\mathfrak{N}$ is a network for (X, μ), there are $n \in \mathbb{N}$ and $N \in \mathfrak{N}_n$ such that $x \in N \subset V$, and so

$$x \in H_N =\subset cl_\tau N \subset cl_\tau V \subset U.$$

This proves (b).

(b) $\Rightarrow$ (a) This is obvious.

(b) $\Rightarrow$ (c) Let $\mathfrak{N} = \bigcup_{n\in\mathbb{N}} \mathfrak{N}_n$ be a network for (X, μ) where each $\mathfrak{N}_n$ belongs to $\Delta(\tau)$. Let $\mathcal{E} = \{E_\alpha : \alpha < \lambda\} \in \Delta(\mu)$. It is enough to show that $\mathcal{E}$ has a base in $\Delta_\sigma(\tau)$. Choose μ-open sets $U_\alpha \supset E_\alpha$ such that, in the case when $\Delta = \Delta^\rho$, $U_\alpha \cap E_\beta = \emptyset$ whenever $\beta \neq \alpha$, or $U_\alpha \cap (\cup\mathcal{E}) = \bigcup_{\beta\leq\alpha} E_\beta$ in the case when $\Delta = \Sigma$. Now define

$$\mathfrak{B}_n = \{E_\alpha \cap N : \alpha < \lambda,\ N \in \mathfrak{N}_n,\ N \subset U_\alpha, \text{ and } E_\alpha \cap N \neq \emptyset\}$$

for each $n \in \mathbb{N}$. Then $\bigcup_{n\in\mathbb{N}} \mathfrak{B}_n \in \Delta_\sigma(\tau)$ and it is easy to verify that it is a base for $\mathcal{E}$.

(b) $\Rightarrow$ (d) and (b) $\Rightarrow$ (d)$'$ Since each open set in (X,μ) is a union of sets from the network $\mathfrak{N}^*$, property (d) follows from Lemma 3.4, and (d)$'$ follows from Lemma 2.6 (b).

(b) $\Rightarrow$ (e) This follows immediately from Lemma 5.8 (b) since $\{N \cap A : N \in \mathfrak{N}^*\}$ will be a σ-scattered function base for the map $(A,\tau) \to (A,\mu)$ and each μ-open set in A is a strong BP-set by (d)$'$.

Finally, suppose $\mathfrak{N} = \bigcup_{n\in\mathbb{N}} \mathfrak{N}_n$ is a network for (X,μ) where each $\mathfrak{N}_n$ belongs to $\Delta(\mu)$, and let us show that (c) $\Rightarrow$ (a). Assuming (c), it follows that each $\mathfrak{N}_n$ has a base of the form $\bigcup_{m\in\mathbb{N}} \mathfrak{N}_{nm}$ with $\mathfrak{N}_{nm} \in \Delta(\tau)$ for each m. But then $\bigcup_{n,m\in\mathbb{N}} \mathfrak{N}_{nm}$ is clearly a $\Delta_\sigma(\tau)$ network for (X,μ). $\square$

7. Banach spaces which are descriptive. The main objective of this section is to describe classes of Banach spaces Z for which Z is descriptive, and similarly for the cases when $Z = C(K)$ or when Z is a dual Banach space. Our first lemma is at the heart of the proof of Theorem 1.5 and has a number of interesting precursors (see [27, 3.11], [44, Lemma 2], [29, Theorem 5] and [18, Lemma 1]).

Lemma 7.1. *If Y has a countable network, then for any space X the projection map $\pi : X \times Y \to X$ is index-σ-relatively discrete and index-σ-scattered.*

Proof. For the first part, it suffices to prove that π maps relatively discrete collections in $X \times Y$ to σ-relatively discretely decomposable collections in X. Let $\{E_\alpha : \alpha \in A\}$ be a relatively discrete collection of subsets of $X \times Y$. We need to show that

$$\pi(E_\alpha) = \bigcup_{n=1}^{\infty} E_{\alpha n}$$

where $\{E_{\alpha n} : \alpha \in A\}$ is a relatively discrete family for each n.

For each α, let U_α be open in $X \times Y$ and satisfy $E_\alpha \subset U_\alpha$ and $E_\beta \cap U_\alpha = \emptyset$ for all $\beta \neq \alpha$. Let $\{N_n : n \in \mathbb{N}\}$ be a countable network for Y, and define

$$E_{\alpha n} = \pi(\{(x,y) \in E_\alpha : (x,y) \in U \times N_n \subset U_\alpha \text{ for some open set } U \subset X\}$$

$$U_{\alpha n} = \cup\{U : U \text{ is open in } X \text{ and } U \times N_n \subset U_\alpha\}$$

for each $\alpha \in A$ and $n \in \mathbb{N}$. By the definition of a network, it is easy to verify that the above defined sets satisfy

$$\pi(E_\alpha) = \bigcup_{n=1}^{\infty} E_{\alpha n} \text{ and } E_{\alpha n} \subset U_{\alpha n}$$

for each $\alpha \in A$ and $n \in \mathbb{N}$. Moreover, if for some β we have $x \in E_{\beta n} \cap U_{\alpha n}$, then, for some $y \in N_n$ and for some open neighborhood U of x, we have

$$(x, y) \in E_\beta \quad \text{and} \quad U \times N_n \subset U_\alpha.$$

But this leads to the contradiction that $E_\beta \cap U_\alpha \neq \emptyset$. It follows that the collection $\{E_{\alpha n} : \alpha \in A\}$ is relatively discrete for each n.

Now suppose $\{E_\alpha : \alpha < \lambda\}$ is scattered in $X \times Y$ and let $\{U_\alpha : \alpha < \lambda\}$ be a collection of open sets in $X \times Y$ such that

$$E_\alpha \subset U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta$$

for each $\alpha < \lambda$. Let $E_{\alpha n}$ and $U_{\alpha n}$ be defined exactly as above, and let us show that

$$E_{\alpha n} \subset U_{\alpha n} \setminus \bigcup_{\beta<\alpha} U_{\beta n}.$$

If $x \in E_{\alpha n}$, then for some open set $U \subset X$ and for some $y \in Y$ we have $(x, y) \in U \times N_n \subset U_\alpha$, and it follows that $x \in U \subset U_{\alpha n}$. Were $x \in U_{\beta n}$ for some $\beta < \alpha$, then for some open $U^* \subset X$ we would have $x \in U^*$ and $U^* \times N_n \subset U_\beta$, and this would imply that $(x, y) \in U_\beta$, which is impossible since $(x, y) \in E_\alpha$. □

The proof of the following theorem employs a very useful technique due to W. Schackermeyer (see [12, §2] or [5, p. 362]).

Theorem 7.2. *Let Z be a Banach space, τ a locally convex topology on Z, and let $S = \{z \in Z : \|z\| = 1\}$. If (S, τ) has a σ-relatively discrete [resp. σ-scattered] network, then any subspace of (Z, τ) has a σ-relatively discrete [resp. σ-scattered] network. In particular, if the norm and weak topologies agree on S, then Z (hence also any norm Souslin subset of Z) is weakly descriptive.*

Proof. It is enough to show that (Z, τ) has a σ-relatively discrete network, since this property is hereditary [and similarly for the σ-scattered case]. With the τ topology on both S and $Z \setminus \{0\}$, the map $(t, z) \mapsto (t, tz)$ is a homeomorphism from the product space $(0, \infty) \times S$ onto some subset M of the product space $(0, \infty) \times Z \setminus \{0\}$. (Note that the inverse map is the "diagonal" of the continuous maps $(t, tz) \mapsto t$ and $(t, tz) \mapsto (t^{-1}, tz) \mapsto t^{-1}(tz) = z$.) Moreover, the space M projects onto all of $Z \setminus \{0\}$. Since $(0, \infty)$ has a countable network and (S, τ) has a σ-relatively discrete network, it is easy to verify that the product space $(0, \infty) \times S$ has a σ-relatively discrete network. From this it follows that the space

$$M \subset (0, \infty) \times Z \setminus \{0\}$$

also has a σ-relatively discrete network. By Lemma 7.1, the image of this network under the projection map $\pi : (0, \infty) \times Z \setminus \{0\} \to Z \setminus \{0\}$ has a σ-relatively discrete base; hence, this will be a σ-relatively discrete network for $Z \setminus \{0\}$. (Note that, if $\mathfrak{N}$ is a network for the domain of a continuous map f, then any base for $f(\mathfrak{N})$ will be a network for the topology of the range.) Adjoining to this network the singleton $\{0\}$ yields a σ-relatively discrete network for (Z, τ). The proof for the σ-scattered case is identical. $\square$

Any Banach space which admits an equivalent Kadec norm satisfies the assumptions in Theorem 7.2 with τ taken to be the weak topology. As noted in the introduction, this class is relatively large and apparently only those spaces containing an isomorphic copy of ℓ^∞ are known to be excluded.

As noted in §1, the following theorem is an immediate consequence of Theorem 7.2.

Theorem 7.3. *If Z^* is a dual Banach space with property $(**)$, then Z^* is a descriptive space.*

Since the Banach space $c_0(\Gamma)$ is locally uniformly convexifiable [75], it admits an equivalent Kadec norm, and it follows that $c_0(\Gamma)$ is descriptive by Theorem 7.2. Since the topology of pointwise convergence and the weak topology on $c_0(\Gamma)$ coincide for bounded subsets, it follows that $(c_0(\Gamma), \tau_p)$ is also descriptive (since any space which is a countable union of subspaces each having a σ-relatively discrete network will itself have such a network). It would seem instructive to give a more direct proof of this result (not dependent on the existence of a certain equivalent norm), which we presently do. It is then shown how this in turn implies that $C_p(K)$ will be a descriptive space whenever there exists a certain type of linear injection from $C(K)$ into $c_0(\Gamma)$ (Theorem 1.7 of §1). We first prove a lemma.

Lemma 7.4. *Any point-finite collection of open sets in an arbitrary topological space is σ-relatively discretely decomposable.*

P r o o f. Let $\mathcal{U}$ be a point-finite collection of open sets in the space X. Since relative discreteness is independent of the containing space, we may assume that $\mathcal{U}$ is a cover of X. For each $x \in X$ and $n \in \mathbb{N}$ let

$$\mathcal{U}_x = \{U \in \mathcal{U} : x \in U\} \quad \text{and} \quad C_n = \{x \in X : \operatorname{card}(\mathcal{U}_x) = n\}.$$

Now put

$$\mathcal{U}(n) = \{\cap\mathcal{V} : \mathcal{V} \subset \mathcal{U} \quad \text{and} \quad \operatorname{card}(\mathcal{V}) = n\},$$

and note that if $U \in \mathcal{U}$ and $V = \cap\mathcal{V} \in \mathcal{U}(n)$, then either $V \cap C_n \subset U$ (when $U \in \mathcal{V}$) or $V \cap C_n \cap U = \emptyset$. It follows that $\{V \cap C_n : V \in \mathcal{U}(n)\}$ is relatively discrete for each $n \in \mathbb{N}$. Moreover, if for each $U \in \mathcal{U}$ and $n \in \mathbb{N}$ we let

$$U_n = \cup\{V \cap C_n : V \in \mathcal{U}(n) \quad \text{and} \quad V \cap C_n \subset U\},$$

then $U = \cup\{U_n : n \in \mathbb{N}\}$ and $\{U_n : U \in \mathcal{U}\}$ is relatively discrete for each n. That completes the proof. $\square$

Theorem 7.5. *The norm topology of $c_0(\Gamma)$ has a network $\mathfrak{N}$ that is σ-relatively discrete with respect to the topology of pointwise convergence.*

Proof. Let $\mathfrak{B} = \{B_n : n \in \mathbb{N}\}$ be a sequence of bounded open intervals of $\mathbb{R}$ such that, for each n there is an $\varepsilon > 0$ for which $B_n \subset (-\infty, -\varepsilon)$ or $B_n \subset (\varepsilon, \infty)$, and each nonempty open subset of $\mathbb{R} \setminus \{0\}$ is a union of some subcollection of $\{B_n : n \in \mathbb{N}\}$.

For $n \in \mathbb{N}$ let

$$\Gamma(n) = \{\Lambda \in \Gamma : \operatorname{card}(\Lambda) \le n\},$$

$$M(n) = \{f : f : \Lambda \to \{B_1, \dots, B_n\}, \Lambda \in \Gamma(n)\}.$$

For each $n, m \in \mathbb{N}$ and for each $\Lambda \in \Gamma(n)$ and $f \in M(n)$ define

$$R(\Lambda, f) = c_0(\Gamma) \cap \prod_{\lambda < \Gamma} R_\lambda \text{ where } R_\lambda = \begin{cases} f(\lambda) & \text{if } \lambda \in \Lambda \\ \mathbb{R} & \text{otherwise} \end{cases}$$

and

$$R_m(\Lambda, f) = c_0(\Gamma) \cap \prod_{\lambda < \Gamma} R_\lambda \text{ where } R_\lambda = \begin{cases} f(\lambda) & \text{if } \lambda \in \Lambda \\ \left(-\dfrac{1}{m}, \dfrac{1}{m}\right) & \text{otherwise} \end{cases}.$$

We first show that, for each fixed $n \in \mathbb{N}$,

$$\{R(\Lambda, f) : \Lambda \in \Gamma(n), f \in M(n)\}$$

is a point-finite collection of τ_p-open subsets of $c_0(\Gamma)$. Since the sets are clearly τ_p-open, we need only show that each $x \in c_0(\Gamma)$ belongs to $R(\Lambda, f)$ for at most a finite number of distinct pairs (Λ, f). But if $x \in R(\Lambda_m, f_m)$ with $\{(\Lambda_m, f_m) : m \in \mathbb{N}\}$ infinite, then $\bigcup_{m\geq 1} \Lambda_m$ must be infinite since each f_m takes values in $\{B_1, \ldots, B_n\}$ and n is fixed. Hence for some infinite set $\Psi \subset \bigcup_{m\geq 1} \Lambda_m$ and for some B_j, $1 \leq j \leq n$, we would have $x(\psi) \in B_j$ for each $\psi \in \Psi$; but this contradicts the fact that $x \in c_0(\Gamma)$ since B_j is bounded away from 0.

It follows from Lemma 7.4 that each of the families $\{R(\Lambda, f) : \Lambda \in \Gamma(n), f \in M(n)\}$ is σ-relatively-discretely decomposable for the topology τ_p. Since $R_m(\Lambda, f) \subset R(\Lambda, f)$ for each $m \in \mathbb{N}$, it also follows that the family $\mathcal{R}_{nm} = \{R_m(\Lambda, f) : \Lambda \in \Gamma(n), f \in M(n)\}$ is also σ-relatively discretely decomposable for the topology τ_p for each n and $m \in \mathbb{N}$.

To see that the sets $\bigcup_{n,m\geq 1} \mathcal{R}_{nm}$ form an open base for the norm topology on $c_0(\Gamma)$, let $x \in c_0(\Gamma)$ and let $B(x, 1/m)$ denote the open ball about x of radius $1/m$. If $\|x\| < 1/m$, let $\Lambda = \emptyset$, otherwise let

$$\Lambda = \{\lambda_1, \ldots, \lambda_k\} = \{\lambda \in \Gamma : |x(\lambda)| \geq 1/m\},$$

and choose sets $B_{n_i} \in \mathfrak{B}$ for $i = 1, \ldots, k$ such that

$$x(\lambda_i) \in B_{n_i} \subset \left(x(\lambda_i) - \frac{1}{m},\ x(\lambda_i) + \frac{1}{m}\right).$$

Let $n = \max\{k, n_1, \ldots, n_k\}$ and define $f : \Lambda \to \{B_1, \ldots, B_n\}$ so that $f(\lambda_i) = B_{n_i}$ for $i = 1, \ldots, k$. Then $\Lambda \in \Gamma(n)$, $f \in M(n)$ and

$$x \in R_m(\Lambda, f) \subset B(x, 1/m)$$

(note that $\|x\| = x(\lambda)$ for some $\lambda \in \Gamma$ for $x \in c_0(\Gamma)$).

It now follows that the sets making up a σ-relatively-discrete decomposition of $\mathcal{R}_{nm}$ (in the topology τ_p) will, when taken together, constitute a network for the norm topology of $c_0(\Gamma)$, and will of course be a σ-relatively-discrete family with respect to the topology τ_p. $\square$

Theorem 7.5 can be used to deduce similar results for any space that admits a certain type of continuous mapping into $c_0(\Gamma)$. Although the conditions in the following lemma might at first seem artificial, we will see that they are satisfied by a wide class of Banach spaces.

Lemma 7.6. *Let X be a set with two topologies τ and μ, and suppose that (X, μ) has a σ-relatively discrete network. Suppose there exists a map f :*

$X \to c_0(\Gamma)$ that is μ-to-norm index-σ-relatively discrete and τ-to-τ_p continuous. Then (X,μ) has a network that is σ-relatively discrete with respect to τ.

Proof. Let $\mathfrak{N} = \underset{n\in\mathbb{N}}{\cup} \mathfrak{N}_n$ be a network for (X,μ) with each $\mathfrak{N}_n$ μ-relatively discrete. Since f is μ-to-norm index-σ-relatively discrete, by Lemma 3.9 we can write $f(N) = \underset{n\in\mathbb{N}}{\cup} M_{N_m}$ so that $\{M_{N_m} : N \in \mathfrak{N}_n\}$ is a norm relatively discrete collection in $c_0(\Gamma)$ for each $n \in \mathbb{N}$. By Theorem 7.5, and since relatively discrete implies σ-discretely decomposable in metric spaces, we can further write

$$M_{N_m} = \underset{k\in\mathbb{N}}{\cup} M_{N_{mk}}$$

where $\{M_{N_{mk}} : N \in \mathfrak{N}_n\}$ is τ_p-relatively discrete for each fixed m and k. Since f is τ-to-τ_p continuous, it follows that $\{f^{-1}(M_{N_{mk}}) : N \in \mathfrak{N}_n\}$ is τ-relatively discrete for each fixed m and n. Letting

$$N_{mk} = N \cap f^{-1}(M_{N_{mk}})$$

for each $N \in \mathfrak{N}_n$, it is clear that we have

$$N = \cup\{N_{mk} : m,k \in \mathbb{N}\},$$

where $\{N_{mk} : N \in \mathfrak{N}_n\}$ is τ-relatively discrete and

$$\{N_m k : N \in \mathfrak{N}_n,\ m \in \mathbb{N},\ k \in \mathbb{N}\}$$

is a network for (X,μ). $\square$

Recall that the *weight* of a topological space X, denoted w(X), is the least cardinal of an open base for X. By a *retractional resolution of the identity* for a compact Hausdorff space K we mean a "long sequence" $\{r_\alpha : \omega_0 \le \alpha \le \mu\}$ of continuous retractions $r_\alpha : K \to K_\alpha$ (i.e., r_α is a continuous extension of the identity map on $K_\alpha \subset K$), where μ is the least ordinal such that $|\mu| = w(K)$, and which satisfies:

(i) $r_\alpha \circ r_\beta = r_\beta \circ r_\alpha = r_\alpha$ if $\omega_0 \le \alpha \le \beta \le \mu$,

(ii) $w(K_\alpha) \le |\alpha| \ \ \forall\alpha$,

(iii) $\overline{\cup\{K_{\beta+1} | \beta < \alpha\}} = K_\alpha \ \ \forall\alpha$,

(iv) $\lim\limits_{\alpha<\lambda} r_\alpha(x) = r_\lambda(x) \ \ \forall x \in K$, $\forall$ limit ordinals $\lambda \le \mu$.

(v) r_μ is the identity map on K,

(vi) $\{\alpha < \mu : \|f \circ r_\alpha - f \circ r_{\alpha+1}\| > \varepsilon\}$ is finite $\forall\ f \in C(K)$, $\forall\ \varepsilon > 0$.

Property (vi) can be shown to follow from the other conditions whenever K has a dense relatively sequentially compact subset, hence when K itself is sequentially compact. Any Valdivia compact has a retractional resolution of the identity [3, Lemma 1.3], [10, Lemma II.2]. Letting P_α denote the corresponding projection on $C(K)$ defined by

$$P_\alpha(f) = f \circ r_\alpha \ \ \forall\, C(K),$$

we obtain a "long sequence" of projections $\{P_\alpha : \omega_0 \le \alpha \le \mu\}$ which defines a projectional resolution of the identity on the Banach space $C(K)$ [8].

Theorem. (a) (Gul'ko [24]) *If K is a compact Hausdorff having a retractional resolution of the identity, then there exist a set Γ and a one-to-one bounded linear operator $T : C(K) \to c_0(\Gamma)$ which is also continuous when both Banach spaces are equipped with the topology of pointwise convergence (cf. also* [3, *Theorem* 1.4]*).*

(b) (Spahn [68, §3]) *The linear operator T in part* (a) *is also norm-to-norm index-σ-discrete.*

Theorem 7.7. *If K is a compact Hausdorff space having a retractional resolution of the identity, then $C_p(K)$ is descriptive.*

Proof. This follows from the preceding Theorem and Theorems 7.5 and 7.6. (Note also the fact that for metric spaces index-σ-relatively discrete and index-σ-discrete coincide [26, §1.2].) □

8. An embedding property of descriptive Banach spaces. Talagrand [71] has shown that if Z is a subspace of a weakly compactly generated Banach space, then (Z, weak) will be a $\mathcal{K}_{\sigma\delta}$ in (Z^{**}, weak^*), where $\mathcal{K}$ is the family of all weak* compact sets in the bidual Z^{**}. Also, a Banach space Z is a (weakly) Lindelöf K-analytic space if, and only if, (Z, weak) is a Souslin($\mathcal{K}$) set in (Z^{**}, weak^*) (see [71, Theorem 3.2 and Proposition 3.3]). Whether (Z, weak) will in fact be a $\mathcal{K}_{\sigma\delta}$ in (Z^{**}, weak^*) whenever Z is a Lindelöf K-analytic space remains an open question (cf. [72]). However, the corresponding question for descriptive Banach spaces has the following affirmative answer.

Theorem 8.1. *If Z is a descriptive Banach space, then (Z, weak) is a $(\mathcal{F} \cap \mathcal{G})_{\sigma\delta}$ set in (Z^{**}, weak^*). If Z is an almost descriptive Banach space, then (Z, weak) is a restricted Baire property set in (Z^{**}, weak^*).*

Proof. Since Z is descriptive, the norm topology has a weakly σ-relatively discrete network $\mathfrak{N}$ by Theorem 6.3. For each $n \in \mathbb{N}$, let

$$\mathfrak{N}_n = \{U \in \mathfrak{N} : \text{norm-diam}\, U < 1/n\},$$

and note that $\mathfrak{N}_n$ will be a cover of Z since $\mathfrak{N}$ is a network for the norm topology. Since each family $\mathfrak{N}_n$ is σ-relatively discrete in (Z^{**}, weak^*), by Lemma 3.2 we can find a σ-relatively discrete family $\mathcal{H}_n = \{H_U : U \in \mathfrak{N}_n\}$ of $\mathcal{F} \cap \mathcal{G}$ in (Z^{**}, weak^*) such that $U \subset H_U$ for each $U \in \mathfrak{N}_n$. Then

$$H_n = \cup\{H_U \cap cl^*[U] : U \in \mathfrak{N}_n\},$$

where cl^* denote the closure in the weak* topology, is a $(\mathcal{F}\cap\mathcal{G})_\sigma$ set in (Z^{**}, weak^*) by Lemma 3.4. Now let us show that $Z = \bigcap_{n\in\mathbb{N}} H_n$.
Suppose $z \in H_n$ for each $n \in \mathbb{N}$, so that $z \in cl^*[U_n]$ for some $U_n \in \mathfrak{N}_n$. Since norm-diam $U_n < 1/n$ and the dual norm is lower semi-continuous relative to the weak* topology, we have

$$\text{norm-diam}\, cl^*[U_n] \leq 1/n.$$

It follows that the norm distance from z to Z is zero, hence $z \in Z$ as Z is norm closed in Z^{**}. Since the reverse inclusion is obvious, that completes the proof. The exact same proof, with Lemma 2.6 in place of Lemma 3.4, proves the second statement of the theorem. □

9. Summary of proofs of the main theorems. In this section we supply the proofs of the theorems stated in the introduction. This is done primarily by citing the appropriate results from §§2–8.

9.1. Proof of Theorems 1.2. Let $X \subset C(K)$, for some compact Hausdorff space K. Theorem 6.6 can be applied to the space X with μ taken to be the norm topology on X and $\tau = \tau_p$ (note that sets of the form $\{x \in X : \|x - a\| \leq \varepsilon\}$ are τ_p-closed in X for the supremum norm). Since the norm topology on X is metrizable, properties (a), (b) and (c) of Theorem 6.6 are equivalent. With Δ taken to be the class of all τ_p relatively discrete families of subsets of X, it is easy to see that properties (b) and (c) of Theorem 6.6 are precisely properties (b) and (c) of Theorem 1.2. It remains to show that (a) $\iff$ (b) in Theorem 1.2. But (a) $\Rightarrow$ (b) follows immediately from Theorem 6.3 (b) and Theorem 6.6, (a) $\Rightarrow$ (b). Since the reverse implication is trivial, the proof of the equivalence is complete.

If Z is a Banach space and K denotes the closed unit ball of Z^* taken with the weak* topology, then by a standard result Z is isometrically isomorphic to a subspace of $C(K)$, and the topology τ_p coincides with the weak topology of

Z. Hence, Theorem 1.2 applies when X is a subset of any Banach space and τ_p is replaced by the weak topology on X. $\square$

9.2. Proof of Theorem 1.3. This is precisely Theorem 6.5 with Δ taken to be the class of all weak* relatively discrete families of subsets of X. The reference to $\mathcal{F} \cap \mathcal{G}$ sets in (b) of Theorem 1.3 is covered by Theorem 6.6, (a) $\Rightarrow$ (b). $\square$

9.3. Proof of Theorem 1.4. (a) In each of the three cases, the assumption that (X, τ) has a σ-relatively discrete network implies that (X, norm) has a network which is also σ-relatively discrete for the τ topology by Theorems 6.3 and 6.5. Property (a) now follows from Theorem 6.6, (a) $\Rightarrow$ (d).

(b) This is covered by Theorem 8.1.

(c) If X is a norm Souslin subset of the Banach space Z, then (X, norm) is an analytic metric space and hence descriptive. Since property (a) of Theorems 1.2 and 1.3 hold by assumption, property (c) implies that the identity map $(X, \mathrm{norm}) \to (X, \tau)$ is index-σ-relatively discrete, hence (X, τ) will be descriptive by Theorem 5.3. In particular, it follows that (X, τ) is Čech analytic by Theorem 4.1. The part concerning Souslin-additive families follows from Theorem 5.7 (a).

(d) As in part (c) above, the assumptions imply that (X, τ) is a descriptive space, hence the conclusion follows from Corollary 5.6 (a).

(e) This follows from Theorem 5.14 since (X, τ) is a completely regular descriptive space. $\square$

9.4. Proof of Theorem 1.5. This is covered by Theorem 7.2. $\square$

9.5. Proof of Theorem 1.6. If $g : K \to H$ is a continuous surjection, then $f \mapsto f \circ g$ is a τ_p-continuous isomorphism from $C(H)$ into $C(K)$. It thus suffices to prove the result for a Valdivia compact K. By [10, Lemma II.3] $C(K)$ has an equivalent locally uniformly convex norm $\|\cdot\|$ which is lower semi-continuous for the τ_p topology. But this easily implies that the norm and τ_p topologies coincide on $\{x : \|x\| = 1\}$ (see, for example, the proof of [10, Lemma II.2]). Theorem 7.2 now implies that $C_p(K)$ has a σ-relatively discrete network. $\square$

9.6. Proof of Theorem 1.7. This is covered by Theorem 7.7. $\square$

9.7. Proof of Theorem 1.8. This is covered by Theorem 5.2. $\square$

9.8. Proof of Theorem 1.10. This is covered by Theorem 6.4. □

9.9. Proof of Theorem 1.11. The proof of the equivalence of properties (a), (b) and (c) is exactly the same as that given for Theorems 1.2 and 1.3 in 9.1 and 9.2 above, except reference is made to the discreteness class of scattered families instead of relatively discrete families.

To prove that (d) $\iff$ (a) in Theorem 1.11, note that under the assumption of (d), all of the properties (a)–(d) of Theorem 6.4 hold (since we are dealing with the case when $\Delta = \Sigma$). In particular, property (b) of Theorem 6.4 implies that (X,τ) has a σ-scattered network. Conversely, if (a) holds, then the converse part of Theorem 6.4 implies that (X,τ) will be σ-fragmented by ρ.

If X is a norm Souslin subset of the Banach space Z, then (X,norm) is an analytic metric space and hence almost descriptive. Since property (c) implies that the identity map $(X,\text{norm}) \to (X,\tau)$ is index-σ-scattered, it follows that (X,τ) will be almost descriptive by Theorem 5.3.

If (X,τ) is Čech analytic, then X is a τ Souslin subset, hence also a norm Souslin subset, of Z and Lemma 1.9 implies that (X,τ) is σ-fragmented by the norm metric. Thus all of the properties apply in this case.

That any one of the properties (a)–(d) implies that X has RNP when X is a dual Banach space follows from Theorem 6.5. □

9.10. Proof of Theorem 1.12. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) follow from Theorem 5.1, Theorem 4.1 and Lemma 1.9, respectively. Since X is a norm Souslin subset of Z, (d) $\Rightarrow$ (e) follows from Theorem 1.11. Theorem 5.2 covers (e) $\Rightarrow$ (b). Since X is a norm Souslin subset of Z, (d) implies that (X,τ) is almost descriptive by Theorem 1.11, and hence descriptive by Theorem 5.2; that is, (d) $\Rightarrow$ (a). Finally, when Z is a dual Banach space with RNP, it is easy to see that Theorems 6.4 and 6.5 imply (d) $\iff$ (f). □

9.11. Proof of Theorem 1.13. This is covered by Theorem 7.2. □

9.12. Proof of Theorem 1.14. (a) This holds more generally for any almost K-descriptive Hausdorff space by Theorem 4.1.

(b) This is covered by Theorem 5.7 (a).

(c) This follows from Lemmas 5.9 (b) and 5.8 (b).

(d) This follows from Lemmas 5.9 (b) and 5.8 (a).

(e) This follows from Theorem 5.3 and Corollary 5.6 (c). □

9.13. Proof of Theorem 1.15. Let X be a Baire space, $f : X \to C_p(K)$ a continuous map and suppose (Y, τ_p) has a σ-scattered network where $Y = f(X)$. Then, by Theorem 1.11 (a) $\Rightarrow$ (b), (Y, norm) has a network $\mathfrak{N}$ which is, relative to the topology τ_p, σ-scattered and consists of $\mathcal{F} \cap \mathcal{G}$ sets. By the continuity of f and Lemma 2.8, $f^{-1}(\mathfrak{N})$ is a σ-scattered family of $\mathcal{F} \cap \mathcal{G}$ sets in X. Moreover, since $\mathfrak{N}$ is a network for (Y, norm), it is easy to check that $f^{-1}(\mathfrak{N})$ will be a function base for $f : X \to (Y, \mathrm{norm})$. Lemma 5.8 (b) now applies and shows that the points D of discontinuity of $f : X \to (Y, \mathrm{norm})$ is a set of first category in X. Since X is a Baire space, this implies that $X \setminus D$ is a dense $\mathcal{G}_\delta$ in X.

Standard arguments apply to show that the above result is also true when f is replaced by any minimal upper semi-continuous compact valued map (see, for example, [49, Theorem 1]).

10. Concluding remarks. Our work was motivated in part by the preprint [39] where the concept of a σ-fragmented Banach space first appeared (subsequently published in revised form in [40] and [41]). In [40] the ostensibly stronger property of the weak topology of a Banach space having a countable cover by sets of small norm-diameter was introduced – such Banach spaces are now said to have the JNR property. Similar to our proof showing that σ-fragmented and almost descriptive Banach spaces coincide (Theorem 6.4), it can be shown that Banach spaces with the JNR property and descriptive Banach spaces coincide. This was first established by Oncina (see [59]). A proof of this as well as other useful equivalents is given in the seminal paper of Molto, Orihuela, Troyanski, and Valdivia [48]. The results and proof techniques of the latter paper and recent results of Raja [61] seem to underscore the feeling that the network approach highlighted in the present paper is a fruitful one in the study of renorming theory of Banach spaces.

As was shown in Theorem 1.8, an almost descriptive Banach space will be descriptive if and only if the weak topology has the property that every collection of open sets has a σ-relatively discrete refinement (that is, the space is hereditarily weakly θ-refinable in the parlance of general topological spaces). It is not known whether every almost descriptive (equivalently, σ-fragmented) Banach space has this property. Our Theorem 6.4 shows that if the weak topology of a Banach space X is σ-fragmented by any metric on X such that the metric topology contains the weak topology, then the weak topology has a σ-scattered network. Hence the Banach space is almost descriptive and thus σ-fragmentable. From this we can now deduce the main result of a recent paper by Kenderov and Moors [43] who utilize game-theoretic techniques in proving the above. One of the main results of the present paper is Theorem 1.15 which states that a compact Hausdorff space K has the Namioka property whenever the function space $C(K)$ has a σ-scattered network relative to the topology of pointwise convergence. A similar result was published later in [40] in the guise of σ-fragmented Banach spaces.

As to the question raised concerning Theorem 5.7 (see the remarks following the proof of that theorem), for a positive answer based on additional set-theoretic assumptions the reader is referred to the paper of Holicky [38].

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